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CONDUCTED BY
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AND
WILLIAM FRANCIS, F.L.S.

"Nec araneorum sane textus ideo melior quia ex se fila gignunt, nec noster
vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit. lib. i. cap. 1. Not.*

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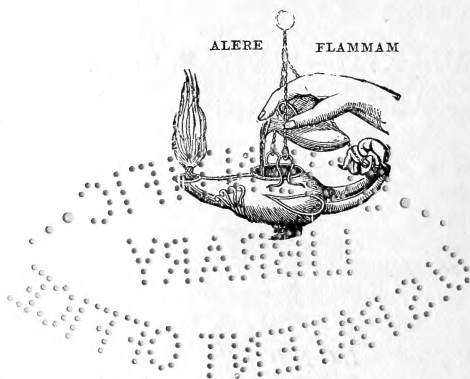
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“Meditationis est perscrutari occulta; contemplationis est admirari perspicua Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem.”—*Hugo de S. Victore.*

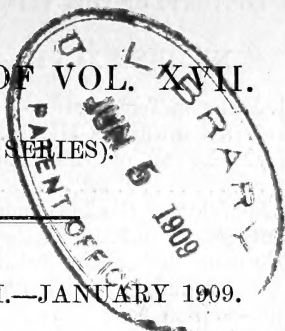
—“Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condât,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu.”

J. B. Pinelli ad Mazonium.



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[SIXTH SERIES.]

JANUARY 1909.

I. *On the Law of Probability for a System of Correlated Variables.* By S. H. BURBURY, F.R.S.*

1. **I**N the works of Karl Pearson, Yule, and other writers on the theory of evolution, the term correlation has acquired a special meaning. For the purposes of this paper I define it as follows:—Let x and y be two quantities, each of which varies continuously between its own limits, and let them be independent variables, in the general sense, that if either be given, the other is not thereby determined, but may vary continuously through a finite range of values. That is, the system of x and y has two degrees of freedom.

The chance that x shall lie between x' and $x' + dx$, that is the number of cases in which, out of a very great number of cases, it so lies, is a function of x' , say $f_1(x')dx$. The corresponding chance for y lying between y' and $y' + dy$ shall be $f_2(y')dy$. It may be that, notwithstanding the system having two degrees of freedom, $f_1(x')$ is a function of y as well as of x' , and $f_2(y')$ is a function of x as well as of y' , so that $\frac{d}{dy}f_1(x) \neq 0$, and $\frac{d}{dx}f_2(y) \neq 0$. If this is the case, x and y are correlated. The chance that simultaneously x shall lie between x_1 and $x_1 + dx$, and y between y_1 and $y_1 + dy$ is of the general form $\phi(x_1y_1)dx dy$. If

$$\frac{d}{dy}f_1(x) = 0, \text{ and } \frac{d}{dx}f_2(y) = 0, \phi(xy) = f_1(x)f_2(y),$$

and x and y are not correlated. If they are correlated,

* Communicated by the Author.

$\phi(xy)$ is not expressible as the product of two functions, one of x only, the other of y only.

2. It follows from this definition, that if x and y are both functions of any the same variables, x and y are correlated. If for instance (1) $f(x, t) = 0$, and (2) $\phi(y, t) = 0$, so that x and y are both functions of t , $f_1(x)$ of art. 1 is by (1) a function of t as well as of x , and therefore by (2) a function of y as well as of x ; and therefore x and y are, by the definition, correlated. Generally, if n variables $x_1 \dots x_n$ are connected by equations of condition less than n in number, correlation exists between them by virtue of those equations. It follows also that if x and y are correlated, any functions of x and y , as $f(x)$ and $\phi(y)$, are correlated with each other.

3. *Of the mean product of two correlated variables.*—The most general definition of mean product is this: There being N values of x , and N values of y , we assign to every x some one of the N values of y as a companion factor, by this means forming N products each of an x and one y . Let them be

$x_1y_1, x_2y_2, \dots x_Ny_N$. Then I define $\overline{xy} = \frac{x_1y_1 + \dots + x_Ny_N}{N^2}$.

This selection of products might be effected in any one of $[N]$ different ways. Practically, if x and y are both functions of a third variable t , we might take every x with the value of y for the same t , so that $\overline{xy} = \int xy dt$. Similarly, if x and y are functions of two other variables z and t , we should define $\overline{xy} = \iint xy dz dt$, and so on.

4. The square of the mean product, so defined, of two correlated variables, cannot be greater than the product of their mean squares. For, taking the general definition of mean product above given, there are, as there stated, $[N]$ different ways in which N x 's and N y 's may be arranged to form N products. Of these there must be some one way for which \overline{xy} is not less than for any other. And for this one, and therefore for every other way,

$$\overline{x^2} = \frac{\sum x^2}{N}, \quad \overline{y^2} = \frac{\sum y^2}{N}, \quad \overline{xy} = \frac{\sum xy}{N},$$

and

$$\overline{x^2} \cdot \overline{y^2} - (\overline{xy})^2 = \frac{1}{N^2} \{ (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + \&c. \\ + (x_py_q - x_qy_p)^2 \},$$

which is ≥ 0 .

5. Generally, if two variables are correlated, their mean product differs from zero, even though the means of each separately, or of one of them, are zero. For the mean product is $\overline{xy} = \iint \phi(xy)xy \, dx \, dy$, and if x and y are correlated, so that $\phi(xy)$ cannot be expressed as $f_1(x)f_2(y)$, this expression for \overline{xy} is not in general zero. If $\phi(xy) = f_1(x)f_2(y)$, then $\overline{xy} = \iint \phi(xy)xy \, dx \, dy = \int f_1(x)x \, dx \cdot \int f_2(y)y \, dy$, and since $\bar{x} = \int f_1(x)x \, dx$, and $\bar{y} = \int f_2(y)y \, dy$, the mean product is the product of the means, and is zero if either \bar{x} or \bar{y} is zero. As an example of this theorem let x and y be two vibrators, having the same period, but different phases, so that we may have $x = A \sin nt$, $y = B \sin (nt + \alpha)$. Then $\bar{x} = 0$, $\bar{y} = 0$, but $\overline{xy} = \frac{1}{2}AB \cos \alpha$. Also in this case $\overline{x^2} = \frac{1}{2}A^2$, $\overline{y^2} = \frac{1}{2}B^2$. And \overline{xy} may change, while $\overline{x^2}$ and $\overline{y^2}$ remain unchanged. Similarly, any two variables x and y may, with given numerical values, be more, or may be less, likely to have the same sign, than, with the same numerical values, to have opposite signs. If the chance of their having the same sign be $\phi_1(xy)$, and the chance of their having opposite signs be $\phi_2(xy)$, then

$$\overline{xy} = \iint (\phi_1(xy) - \phi_2(xy))xy \, dx \, dy.$$

But the mean squares $\overline{x^2}$, $\overline{y^2}$ are independent of the difference $\phi_1(xy) - \phi_2(xy)$, and may therefore be constant while \overline{xy} changes.

Of very small Correlations.

6. It may be that $f(x')$ of art. 1, although it is a function of y as well as of x' , yet is very little affected by change in y ; that is $\frac{df(x')}{dy}$ may be very small or negligible. Similarly $\frac{df_2(y')}{dx}$ may be very small or negligible. In the same case $\phi(xy)$ may differ inappreciably from a product of the form $f_1(x)f_2(y)$, so that for some purposes we may without appreciable error treat x and y as not correlated. For instance, in the kinetic theory of gases, if $x_1 \dots x_n$ are the vector velocities, $m_1 \dots m_n$ the masses, of n molecules, n being a very great number, $x_1 \dots x_n$ are, strictly speaking, correlated by virtue of the relation $\sum m x^2 = 2T$ where T is the kinetic energy, supposed constant. For if x_n be given we alter the limits of integration for $x_1 \dots x_{n-1}$, so that the chance of x_1 having a

given value depends on x_n as well as on x_1 . But n being very great, these correlations, e.g. $\frac{df(x_1)}{dx_n}$, &c., are inappreciable, and in the ordinary theory are generally treated as non-existent.

7. The definition above given of correlation is purely algebraic. Generally, if x and y be two of the variables which determine the state of a material system the parts of which mutually act on each other, the time differential coefficients, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, are correlated by virtue of that mutual action.

For $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are functions of the same variables, and therefore correlated. And if the system be defined by n generalized coordinates $q_1 \dots q_n$, and their corresponding velocities $\dot{q}_1 \dots \dot{q}_n$, the products $\dot{q}_1 \dot{q}_2$ &c. appear in the expression for the kinetic energy, and therefore the mean product $\overline{\dot{q}_1 \dot{q}_2}$ is not generally zero, and whether or not the means $\overline{\dot{q}_1}$ and $\overline{\dot{q}_2}$ are separately zero, \dot{q}_1 and \dot{q}_2 are correlated.

8. Since correlations may, or may not, be negligible we may suppose that $x_1 \dots x_n$, or the things to which they relate, have at every instant positions in space, and that the correlation between any two of them, as x_p and x_q , is or is not negligible, according to the distance which at the instant separates them from each other. But this localization is not essential to the general statement that correlations may or may not be negligible.

A General Problem stated.

9. Let $s_1 \dots s_n$ be n quantities, which, until otherwise stated, shall be each of zero dimensions, each of which varies continuously between assigned limits. I assume them to be in general correlated with each other, but that such correlations may as regards any s , as s_p , be negligible for some or for most of the others. The chance that they shall respectively lie

s_1 between r_1 and $r_1 + dr_1$,

s_2 „ „ r_2 „ „ $r_2 + dr_2$,

.

s_n between r_n and $r_n + dr_n$,

shall be denoted by $\phi(r_1 \dots r_n) dr_1 \dots dr_n$, so that

$$\iint \dots \phi(r_1 \dots r_n) dr_1 \dots dr_n = 1$$

between the given limits; and I assume the function $\phi(s_1 \dots s_n)$ to be finite and continuous for all values of $s_1 \dots s_n$ within the given limits.

It may be the case that a certain function of $s_1 \dots s_n$, say $\psi(s_1 \dots s_n)$, is necessarily constant, and that the variations of $s_1 \dots s_n$ are subject to that condition. Or there may be more than one such constant function. If so, the chance $\phi(s_1 \dots s_n) ds_1 \dots ds_n$ must be understood as subject to the constancy of ψ , or other such functions as aforesaid.

10. From this we can express the mean values of $s_1 \dots s_n$, of their squares, or binary products, or other functions, but it will not be necessary to go beyond powers and products of the second degree. Such mean values will be denoted as usual by a bar. Thus

$$\left. \begin{array}{l} \text{Also} \quad \bar{s}_1 = \iint \dots \phi(s_1 \dots s_n) s_1 ds_1 \dots ds_n, \text{ \&c.} \\ \bar{s}_1^2 = \iint \dots \phi(s_1 \dots s_n) s_1^2 ds_1 \dots ds_n, \text{ \&c.,} \\ \text{and} \quad \bar{s_p s_q} = \iint \dots \phi(s_1 \dots s_n) s_p s_q ds_1 \dots ds_n, \text{ \&c.} \end{array} \right\} \quad (1)$$

11. The object of the first part of this paper is to prove that, $s_1 \dots s_n$ being as above stated correlated *inter se*, then for very great values of n , $\phi(s_1 \dots s_n)$ necessarily has the form $\phi(s_1 \dots s_n) = A e^{-Q}$, in which A is a constant, and Q is a homogeneous quadratic function of $s_1 \dots s_n$, involving both their squares, and (as a consequence of the correlation) their products, namely

$$Q = \frac{1}{2} \alpha_1 s_1^2 + \beta_{12} s_1 s_2 + \frac{1}{2} \alpha_2 s_2^2 + \dots + \frac{1}{2} \alpha_n s_n^2. \quad (2)$$

and the coefficients $\alpha \beta$ are functions of the mean values $\bar{s}_1 \dots \bar{s}_n$, of the mean squares $\bar{s}_1^2 \dots \bar{s}_n^2$, and of the mean products $\bar{s_1 s_2} \dots \bar{s_p s_q}$, &c.

This is a well known result for the case in which the quadratic function Q is incapable of becoming negative. It is necessary, however, shortly to give the proof, which is done in Part I. In Part II. I propose to show *inter alia* why it is necessary for Q to be positive, and I shall then apply the theorem to certain physical problems.

PART I.

Proposition I.

12. Introducing $\theta_1, \theta_2, \dots \theta_n$ as auxiliary variables,

$$\phi(r_1 \dots r_n) = \iint_{-\infty}^{+\infty} \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\sum \overline{s-r} \theta \sqrt{-1}}. \quad (3)$$

multiplied by a constant independent of $r_1 \dots r_n$. Here

$$\sum \overline{s-r} \theta = \overline{s_1 - r_1} \theta_1 + \overline{s_2 - r_2} \theta_2 + \dots + \overline{s_n - r_n} \theta_n,$$

and the limits of integration for each θ are $\pm \infty$. Let the right-hand member of the equation (3) be denoted by S. And in S substitute for $e^{\overline{s_n - r_n} \theta_n \sqrt{-1}}$ its equivalent

$$\cos \overline{s_n - r_n} \theta_n + \sqrt{-1} \sin \overline{s_n - r_n} \theta_n,$$

and integrate according to θ_n from $\theta_n = A$ to $\theta_n = -A$. That reduces S to

$$\iint \dots d\theta_1 d\theta_2 \dots d\theta_{n-1} \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n \\ \times e^{(\overline{s_1 - r_1} \theta_1 + \dots + \overline{s_{n-1} - r_{n-1}} \theta_{n-1}) \sqrt{-1}} \dots 2 \frac{\sin \overline{s_n - r_n} A}{\overline{s_n - r_n}}, \quad (4)$$

the imaginary part evidently disappearing. This has to be integrated now for s_n . Let

$$(\overline{s_n - r_n}) A = x, \text{ or } s_n = \frac{x}{A} + r_n, \text{ and } \frac{ds_n}{\overline{s_n - r_n}} = \frac{dx}{x}.$$

Then make A infinite. The limits of x thus become $\pm \infty$, and S =

$$\iint \dots d\theta_1 \dots d\theta_{n-1} \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_{n-1} \\ \times e^{(\overline{s_1 - r_1} \theta_1 + \dots + \overline{s_{n-1} - r_{n-1}} \theta_{n-1}) \sqrt{-1}} \int_{-\infty}^{\infty} 2 \frac{\sin x}{x} dx, \quad (5)$$

in which $\phi(s_1 \dots s_n)$ is now by virtue of the equation $(\overline{s_n - r_n}) A = x$ a function of $s_1 \dots s_{n-1}, x$.

13. Now, $\int_0^{\infty} \frac{\sin x}{x} dx$ is evidently positive, and $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ is evidently negative. Therefore there exists an angle z between zero and π such that $\int_z^{\infty} \frac{\sin x}{x} dx = 0$, and similarly $\int_{-\infty}^{-z} \frac{\sin x}{x} dx = 0$. But $\int_{-z}^z \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

is a determinate positive quantity, which shall be denoted by Z .

Again, $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ is negative, and $\int_{2\pi}^{\infty} \frac{\sin x}{x} dx$ is positive. Therefore there exists an angle z_1 between π and 2π , such that $\int_{z_1}^{\infty} \frac{\sin x}{x} dx = 0$, and therefore also $\int_z^{z_1} \frac{\sin x}{x} dx = 0$. Similarly there is an angle z_2 between 2π and 3π such that $\int_{z_2}^{\infty} \frac{\sin x}{x} dx = 0$. And the range of integration from z to ∞ , or from $-z$ to $-\infty$, may thus be divided into parts $z_1 - z$, $z_2 - z_1$, &c., such that $\int_{z_{q-1}}^{z_q} \frac{\sin x}{x} dx = 0$ for all integral values of q greater than unity.

14. It can now be shown that

$$\int_{z_{q-1}}^{z_q} \frac{\sin x}{x} \phi(s_1 \dots s_n) dx = 0,$$

$\phi(s_1 \dots s_n)$ being by virtue of the equation $(s_n - r_n)A = x$ a function of x .

For we may suppose x to increase from $-\infty$ to $+\infty$ by successive increments each not greater than 2π . Then for any such increment of x , the increment of s_n is $\cong \frac{2\pi}{A}$, that is, it is infinitesimal. Also since $\phi(s_1 \dots s_n)$ is finite and continuous for all possible values of $s_1 \dots s_n$, $\frac{d\phi(s_1 \dots s_n)}{ds_n}$ cannot be infinite. It follows that corresponding to the

increment ($\leq 2\pi$) of x , the increment of $\phi(s_1 \dots s_n)$ is infinitesimal. Therefore in the integral

$$\int_{z_{q-1}}^{z_q} \phi(s_1 \dots s_n) \frac{\sin x}{x} dx,$$

$z_q - z_{q-1}$ being less than 2π , $\phi(s_1 \dots s_n)$ may be regarded as constant, and therefore

$$\int_{z_{q-1}}^{z_q} \phi(s_1 \dots s_n) \frac{\sin x}{x} dx = 0$$

for each integral value of q greater than unity. Therefore

$$\int_{-\infty}^{\infty} \phi(s_1 \dots s_n) \frac{\sin x}{x} dx = \int_{-z}^z \phi(s_1 \dots s_n) \frac{\sin x}{x} dx.$$

But in this integral, since z is less than π , the range of x is less than 2π , and $\phi(s_1 \dots s_n)$ may be treated as constant, having the value which it has when $x=0$, that is when

$$(s_n - r_n)A=0, \text{ or } s_n=r_n, \text{ and } \phi(s_1 \dots s_n)=\phi(s_1 \dots s_{n-1}r_n).$$

We have then

$$\begin{aligned} & \iint \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma_1^n \overline{s-r} \theta \sqrt{-1}} \\ &= Z \iint \dots d\theta_1 \dots d\theta_{n-1} \iint \dots \phi(s_1 \dots s_{n-1}r_n) ds_1 \dots ds_{n-1} e^{\Sigma_1^{n-1} \overline{s-r} \theta \sqrt{-1}}. \end{aligned} \quad (6)$$

15. As the result of the two integrations for θ_n and s_n , θ_n has disappeared, and s_n has been replaced by r_n . And by successive double integrations in this way

$$\iint \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma \overline{s-r} \theta \sqrt{-1}}$$

is reduced to $Z^n \phi(r_1 \dots r_n)$; or

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma \overline{s-r} \theta \sqrt{-1}}, \quad (7)$$

and $\frac{1}{Z^n}$ is a constant independent of $r_1 \dots r_n$.

Proposition I. is thus proved.

16. It comes next in order to prove that (n being very great) we may in evaluating the above integral neglect powers and products of $\theta_1 \dots \theta_n$ above the second degree.

Let
$$X = \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) e^{\Sigma s \theta \sqrt{-1}},$$
 so that

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n X e^{-\Sigma r \theta \sqrt{-1}}. \quad (8)$$

Then

$$X = \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\cos \Sigma s \theta + \sqrt{-1} \sin \Sigma s \theta) = \alpha + \beta \sqrt{-1},$$

if α denote the real, and $\beta \sqrt{-1}$ the imaginary part of the last expression for X . Since $\iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n = 1$, evidently if every $\theta = 0$, $\alpha = 1$, and $\beta = 0$, and $\alpha^2 + \beta^2 = 1$. And we can now prove

Proposition II.

That if any θ differs from zero, $\alpha^2 + \beta^2$ contains the product of n factors, each of which, unless $\theta = 0$, is numerically, and generally in a finite ratio, less than unity.

For

$$\alpha^2 + \beta^2 = \left\{ \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \cos(s_1 \theta_1 + \dots + s_n \theta_n) \right\}^2 + \left\{ \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \sin(s_1 \theta_1 + \dots + s_n \theta_n) \right\}^2, \quad (9)$$

that is, the sum of the squares of the integrals. Or replacing the square of each integral by the product of two similar integrals between the same limits, $\alpha^2 + \beta^2$

$$= \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \cos(\Sigma s \theta) \times \iint \dots ds'_1 \dots ds'_n \phi(s'_1 \dots s'_n) \cos(\Sigma s' \theta) + \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \sin(\Sigma s \theta) \times \iint \dots ds'_1 \dots ds'_n \phi(s'_1 \dots s'_n) \sin \Sigma s' \theta. \quad (10)$$

$$= \iint \dots ds_1 \dots ds_n ds'_1 \dots ds'_n \phi(s_1 \dots s_n) \phi(s'_1 \dots s'_n) \cos(\overline{s_1 - s'_1} \theta_1 + \overline{s_2 - s'_2} \theta_2 + \dots + \overline{s_n - s'_n} \theta_n) \quad (11)$$

17. Again,

$$\cos(\overline{s_1 - s'_1} \theta_1 + \overline{s_2 - s'_2} \theta_2 + \dots + \overline{s_n - s'_n} \theta_n)$$

consists, when expanded, of the term

$$\cos \overline{s_1 - s'_1} \theta_1 \cdot \cos \overline{s_2 - s'_2} \theta_2 \dots \cos \overline{s_n - s'_n} \theta_n,$$

and other terms each of which contains one or more of the factors $\sin \overline{s_1 - s'_1} \theta_1$, $\sin \overline{s_2 - s'_2} \theta_2$, &c., each in the first degree.

Each of these factors, *e. g.* $\sin(s_p - s'_p)\theta_p$, when integrated between the same limits for s_p and for s'_p , gives the result zero. Therefore every term vanishes in the integration for $s \dots s'$ except the product of the cosines, and

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \iint \dots ds'_1 \dots ds'_n \phi(s_1 \dots s_n) \phi(s'_1 \dots s'_n) \cos(\overline{s_1 - s'_1} \theta_1 + \\ & \qquad \qquad \qquad \dots + \overline{s_n - s'_n} \theta_n) \\ = & \iint \dots ds_1 \dots ds_n \iint \dots ds'_1 \dots ds'_n \phi(s_1 \dots s_n) \phi(s'_1 \dots s'_n) \cos \overline{s'_1 - s'_1} \theta_1 \cos \overline{s_2 - s'_2} \theta_2 \\ & \qquad \qquad \qquad \dots \cos \overline{s_n - s'_n} \theta_n. \end{aligned}$$

As n becomes indefinitely great, this product becomes indefinitely small, unless each of the factors $\cos \overline{s - s'} \theta$ is equal to or nearly equal to unity. That condition is satisfied for $\theta = 0$, but not if θ is any multiple of π , because if $\theta = 0$, $\cos \overline{s - s'} \theta = 1$ for all values of $s - s'$. But if θ is a multiple of π , only for a particular value of $\overline{s - s'}$.

18. If this condition be not satisfied, $\alpha^2 + \beta^2$, since it contains the product of n cosines, which are *not* in general nearly equal to unity, is, n being very great, indefinitely small. Therefore α and β , and therefore X , are indefinitely small. We might fix limits between which $\theta_1 \dots \theta_n$ respectively must lie, say

$$\begin{array}{lll} \theta_1 & \text{between } q_1 \text{ and } -q_1 \\ \theta_2 & \text{,, } q_2 \text{ ,, } -q_2 \\ \&c. \quad \theta_n & \text{,, } q_n \text{ ,, } -q_n, \end{array}$$

in which $q_1 \dots q_n$ are so small that all powers and products of them above the second degree may be neglected. Then unless $\theta_1 \dots \theta_n$ lie within these limits, X is indefinitely small, and therefore also $X e^{-\Sigma r \theta \sqrt{-1}}$ indefinitely small. If $\theta_1 \dots \theta_n$ do lie within these limits, we may in evaluating X , and therefore in evaluating

$$X e^{-\Sigma r \theta \sqrt{-1}}, \text{ or } \iint \dots \phi(s_1 \dots s_n) ds_1 \dots ds_n e^{\Sigma \overline{s - r} \theta \sqrt{-1}},$$

neglect powers and products of $\theta_1 \dots \theta_n$ above the second degree.

19. In the expression

$$\begin{aligned} \phi(r_1 \dots r_n) &= \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n X e^{-\Sigma r \theta \sqrt{-1}} \\ &= \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n \iint ds_1 \dots ds_n \phi(s_1 \dots s_n) e^{\Sigma \overline{s - r} \theta \sqrt{-1}}, \quad (12) \end{aligned}$$

expand the exponential, neglecting in accordance with the last article powers and products of $\theta_1 \dots \theta_n$ above the second degree. Then $e^{\Sigma \overline{s-r\theta} \sqrt{-1}}$ is replaced by

$$1 + \Sigma \overline{s-r\theta} \sqrt{-1} + \frac{1}{2} (\Sigma \overline{s-r\theta} \sqrt{-1})^2. \quad \dots \quad (13)$$

To this expression we may now apply art. 10, and so obtain

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) e^{\Sigma \overline{s-r\theta} \sqrt{-1}} \\ &= \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \\ & \quad + \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \Sigma_1^n \overline{s-r\theta} \sqrt{-1} \\ & \quad - \frac{1}{2} \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\Sigma_1^n \overline{s-r\theta})^2 \\ &= 1 + \Sigma_1^n (\overline{s-r}) \theta \sqrt{-1} \\ & \quad - \frac{1}{2} \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\Sigma_1^n \overline{s-r\theta})^2. \quad \dots \quad (14) \end{aligned}$$

20. Now,

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (\Sigma_1^n \overline{s-r\theta})^2 \\ &= \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \Sigma_{p=1}^{p=n} (\overline{s_p^2 - 2s_p r_p + r_p^2}) \theta_p^2 \end{aligned}$$

for all positive integral values of p

$$+ 2 \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) \Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (\overline{s_p s_q + r_p r_q - s_p r_q - s_q r_p}) \theta_p \theta_q,$$

the last term including all pairs of unequal integral values of p and q

$$\begin{aligned} &= \Sigma_{p=1}^{p=n} (\overline{s_p^2 - 2s_p r_p + r_p^2}) \theta_p^2 \\ & \quad + 2 \Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (\overline{s_p s_q + r_p r_q - s_p r_q - s_q r_p}) \theta_p \theta_q. \quad \dots \quad (15) \end{aligned}$$

by art. 10.

$$\text{Also} \quad (\Sigma (\overline{s-r}) \theta)^2$$

$$\begin{aligned} &= \Sigma_{p=1}^{p=n} (\overline{s_p^2 - 2s_p r_p + r_p^2}) \theta_p^2 \\ & \quad + 2 \Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (\overline{s_p s_q + r_p r_q - s_p r_q - s_q r_p}) \theta_p \theta_q. \quad \dots \quad (16) \end{aligned}$$

which differs from (15) only in $\overline{s_p^2}$ being replaced by $\frac{2}{s_p}$ for every p , and $\overline{s_p s_q}$ by $\overline{s_p s_q}$ for every p and q . Whence

$$\begin{aligned} & \iint \dots ds_1 \dots ds_n \phi(s_1 \dots s_n) (1 + \Sigma \overline{s} - r\theta \sqrt{-1} - \frac{1}{2}(\Sigma \overline{s} - r\theta)^2) \\ &= 1 + \Sigma \overline{s} - r\theta \sqrt{-1} - \frac{1}{2}(\Sigma \overline{s} - r\theta)^2 \\ & \quad - \frac{1}{2} \Sigma_{p=1}^{p=n} (\overline{s_p^2} - \frac{2}{s_p}) \theta_p^2 - \Sigma_{p=1}^{p=n} \Sigma_{q=1}^{q=n} (\overline{s_p s_q} - \overline{s_p s_q}) \theta_p \theta_q, \end{aligned}$$

and therefore finally

$$\begin{aligned} \phi(r_1 \dots r_n) &= \frac{1}{Z^n} \iint \dots d\theta_1 \dots d\theta_n e^{\Sigma (\overline{s} - r) \theta \sqrt{-1}} \\ & \quad \times e^{-\frac{1}{2} \Sigma (\overline{s^2} - \frac{2}{s}) \theta^2 + \Sigma \Sigma (\overline{s s'} - \overline{s s'}) \theta \theta'}, \dots \dots (17) \end{aligned}$$

because in restoring the exponential we may again neglect powers and products of $\theta_1 \dots \theta_n$ above the second degree.

21. I now simplify this expression by assuming every \overline{s} to be zero, and will point out in art. 37 how this becomes important. Our equation then becomes

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \int \dots \int_{-\infty}^{+\infty} d\theta_1 \dots d\theta_n e^{-(\frac{1}{2} \Sigma \overline{s^2} \theta^2 + \Sigma \Sigma \overline{s s'} \theta \theta')} e^{-\Sigma r \theta \sqrt{-1}}.$$

Let us now write $\overline{s_1^2} = a_1,$
 $\overline{s_2^2} = a_2,$
 &c.,

and $\overline{s_p s_q} = b_{pq} = b_{qp}$ for every p and q ; and let the index so obtained, or

$$\frac{1}{2} a_1 \theta_1^2 + b_{12} \theta_1 \theta_2 + \frac{1}{2} a_2 \theta_2^2 + \&c. + \frac{1}{2} a_n \theta_n^2 = Q_\theta,$$

so that

$$\phi(r_1 \dots r_n) = \frac{1}{Z^n} \int \dots \int_{-\infty}^{+\infty} d\theta_1 \dots d\theta_n e^{-Q_\theta} e^{-\Sigma r \theta \sqrt{-1}}. \quad (18)$$

22. Before integrating for $\theta_1 \dots \theta_n$, it is necessary to introduce the conjugate functions $u_1 \dots u_n$.

The coefficients a, b are a property of the given system, being determinate if $s_1 \dots s_n$ are given in form. Let us then

write, using $r_1 \dots r_n$ as any values of $s_1 \dots s_n$,

$$\begin{aligned} r_1 &= a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c., \\ r_2 &= b_{21} u_1 + a_2 u_2 + b_{23} u_3 + \&c., \\ \&c. &= \&c., \end{aligned}$$

from which follow algebraically

$$\left. \begin{aligned} u_1 &= \frac{D_{11}}{D} r_1 + \frac{D_{12}}{D} r_2 + \frac{D_{13}}{D} r_3 + \&c., \\ u_2 &= \frac{D_{21}}{D} r_1 + \frac{D_{22}}{D} r_2 + \&c., \end{aligned} \right\} \quad (19)$$

$$\&c. = \&c.,$$

in which D is the determinant of the quadratic function Q_θ , namely,

$$D = \begin{vmatrix} a_1 & b_{12} & b_{13} & \dots \\ b_{21} & a_2 & b_{23} & \dots \\ b_{31} & b_{32} & a_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad \text{or} \quad D = \Sigma \pm \frac{dr_1}{du_1} \frac{dr_1}{du_2} \dots,$$

and D_{11} , D_{22} , &c. are its coaxial minors obtained by erasing the first, respective second, or n th row and column. Also D_{pq} is the anaxial minor obtained by erasing the p th row and q th column, or *vice versa*. Since, by definition, $b_{pq} = b_{qp}$ for every p and q , it follows that $D_{pq} = D_{qp}$ for every p and q . The signs of the anaxial minors are so taken that

$$D = a_1 D_{11} + b_{12} D_{12} + b_{13} D_{13} + \&c.$$

23. Again, let

$$\frac{D_{11}}{D} = \alpha_1, \quad \frac{D_{22}}{D} = \alpha_2, \quad \&c.,$$

and

$$\frac{D_{12}}{D} = \beta_{12} \dots \dots \frac{D_{pq}}{D} = \beta_{pq}.$$

The r 's and u 's are then connected by the symmetrical systems

$$\left. \begin{aligned} r_1 &= a_1 u_1 + b_{12} u_2 + \&c., \\ r_2 &= b_{21} u_1 + a_2 u_2 + \&c., \\ \&c. &= \&c., \end{aligned} \right\} \quad \dots \quad (19a)$$

and

$$\left. \begin{aligned} u_1 &= \alpha_1 r_1 + \beta_{12} r_2 + \&c., \\ u_2 &= \beta_{21} r_1 + \alpha_2 r_2 + \&c., \\ \&c. &= \&c. \end{aligned} \right\} \quad \dots \quad (19b)$$

Also if Δ denote the determinant

$$\Delta = \begin{vmatrix} \alpha_1 & \beta_{12} & \beta_{13} & \dots \\ \beta_{21} & \alpha_2 & \beta_{23} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad \dots \quad (20)$$

$$\text{or} \quad \Delta = \Sigma \pm \frac{du_1}{dr_1} \frac{du_1}{dr_2} \dots,$$

then by a known proposition $\Delta = \frac{1}{D}$. And as

$$\alpha_1 = \frac{D_{11}}{D}, \quad \beta_{12} = \frac{D_{12}}{D}, \quad \&c.,$$

so reciprocally

$$a_1 = \frac{\Delta_{11}}{\Delta}, \quad b_{12} = \frac{\Delta_{12}}{\Delta}, \quad \&c.$$

These results may be confirmed as follows:—If we integrate $r_1^2 e^{-Q}$ for all the variables r except r_1 between limits $\pm \infty$, we obtain $\overline{r_1^2} = \frac{\Delta_{11}}{\Delta}$. But $\overline{r_1^2} = a_1$ by definition. Therefore

$a_1 = \frac{\Delta_{11}}{\Delta}$. Similarly $\overline{r_1 r_2} = b_{12} = \frac{\Delta_{12}}{\Delta}$, and so on. In the same way, expressing Q in terms of the u 's,

$$\iint \dots e^{-Q} u_1^2 du_2 \dots du_n = \frac{D_{11}}{D} = a_1 = \overline{u_1^2},$$

$$\iint \dots e^{-Q} u_1 u_2 du_3 \dots du_n = \frac{D_{12}}{D} = \beta_{12} = \overline{u_1 u_2} \quad \&c.$$

24. The integration of $e^{-Q_\theta} e^{-\Sigma r_\theta \sqrt{-1}}$ for $\theta_1 \dots \theta_n$ between limits $\pm \infty$ is given by Todhunter (Cambridge Phil. Trans. vol. xi. 1871, p. 219) on the assumption that the coefficients in Q_θ are such as to make Q_θ necessarily positive. The necessary and sufficient condition for which is that D and every coaxial minor of D is positive.

Let

$$\iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \cdot e^{-\Sigma r_\theta \sqrt{-1}} = U.$$

Then replacing $e^{-\Sigma r_\theta \sqrt{-1}}$ by $\cos \Sigma r_\theta - \sqrt{-1} \sin \Sigma r_\theta$, we have, since the imaginary term disappears in the integration between $+\infty$ and $-\infty$,

$$U = \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \cos \Sigma r_\theta.$$

Then

$$\begin{aligned}\frac{dU}{du_1} &= - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \frac{d}{du_1}(r_1\theta_1 + \dots + r_n\theta_n), \\ &= - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \left(\theta_1 \frac{dr_1}{du_1} + \dots + \theta_n \frac{dr_n}{du_1} \right), \\ &= - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) (a_1\theta_1 + b_{12}\theta_2 \dots \&c.),\end{aligned}$$

or

$$\frac{dU}{du_1} = - \iint \dots d\theta_1 \dots d\theta_n e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \frac{dQ_\theta}{d\theta_1} \dots \dots \dots (21)$$

Again integrating $e^{-Q_\theta} \cos(r_1\theta_1 + \dots + r_n\theta_n)$ for θ_1 by parts, remembering that Q_θ being positive the integrated term vanishes at both limits when θ_1 is infinite, we have

$$\begin{aligned}\int_{-\infty}^{\infty} d\theta_1 e^{-Q_\theta} \cos(r_1\theta_1 + \dots + r_n\theta_n) \\ = \frac{1}{r_1} \int_{-\infty}^{\infty} d\theta_1 e^{-Q_\theta} \sin(r_1\theta_1 + \dots + r_n\theta_n) \frac{dQ_\theta}{d\theta_1}.\end{aligned} \quad (22)$$

Integrate both sides of this for $\theta_2 \dots \theta_n$. That gives by (21) and (22)

$$U = - \frac{1}{r_1} \frac{dU}{du_1} \quad \text{or} \quad \frac{dU}{du_1} = -r_1 U.$$

Similarly

$$\frac{dU}{du_2} = -r_2 U, \&c.,$$

and therefore, since

$$u_1 \frac{dr_1}{du_1} + u_2 \frac{dr_2}{du_1} + \&c. = r_1,$$

$$U = A e^{-\frac{1}{2}(r_1 u_1 + r_2 u_2 + \dots + r_n u_n)} = A e^{-\frac{1}{2} \Sigma r u}, \quad \dots \quad (23)$$

where A is constant.

Let

$$Q = \frac{1}{2} \Sigma r u,$$

or

$$\left. \begin{aligned}Q_u &= \frac{1}{2} a_1 u_1^2 + b_{12} u_1 u_2 + \dots + \frac{1}{2} a_2 u_2^2 + \&c., \\ Q_r &= \frac{1}{2} \alpha_1 r_1^2 + \beta_{12} r_1 r_2 + \dots + \frac{1}{2} \alpha_2 r_2^2 + \&c.\end{aligned} \right\} \quad (24)$$

We have thus three equivalent forms of the index Q . Now Q_u is the same function of $u_1 \dots u_n$ that Q_θ is of $\theta_1 \dots \theta_n$. This is Todhunter's result modified only by a change in notation, he using a and $2b$ where I have used $\frac{1}{2}a$ and b . The proposition stated in art. 11 is thus proved.

25. It is convenient at this point to consider the relations between the coefficients a , b and α , β , and between them and

the r 's and u 's, as regards dimensions. In e^{-Q} , Q must have zero dimensions, that is $r_1 u_1$ has zero dimensions, or those of u are the reciprocal of those of r . Therefore α, β must have dimensions the reciprocal of those of r^2 . If, therefore, for instance r has dimensions $\frac{L}{T}$, u has $\frac{T}{L}$, α, β &c. have $\frac{L^2}{T^2}$, or if r has $\frac{T}{L}$, u has $\frac{L}{T}$. Also if r has $\frac{L}{T}$, D has $\left(\frac{L^2}{T^2}\right)^n$, and $\frac{D_{11}}{D}$ &c., or α, β , have $\frac{T^2}{L^2}$.

PART II.

26. We have thus proved that $\phi(r_1 \dots r_n)$, which represents the chance of the variables $s_1 \dots s_n$ having the values $r_1 \dots r_n$, or values infinitely near thereto, has the form $\phi(r_1 \dots r_n) = C e^{-Q}$, and Q is expressed as a quadratic function of $r_1 \dots r_n$, containing products as well as squares of the variables, and with coefficients as above stated. Also expressing $\phi(r_1 \dots r_n)$ in terms of $u_1 \dots u_n$, we may write

$$\begin{aligned} \phi(r_1 \dots r_n) dr_1 \dots dr_n &= \psi(u_1 \dots u_n) du_1 \dots du_n \Sigma \pm \frac{dr_1}{du_1} \frac{dr_1}{du_2} \\ &= \psi(u_1 \dots u_n) du_1 \dots du_n D, \end{aligned}$$

or since $D = \frac{1}{\Delta}$,

$$\psi(u_1 \dots u_n) du_1 \dots du_n = \phi(r_1 \dots r_n) dr_1 \dots dr_n \Delta.$$

Whether we use $r_1 \dots r_n$ or $u_1 \dots u_n$ for independent variable is a question of convenience.

Since e^{-Q} does not contain the time explicitly, we are in effect assuming that it is independent of the time, and therefore that if the system represented by $r_1 \dots r_n$ is in motion, such motion is stationary. And $\phi(r_1 \dots r_n) dr_1 \dots dr_n$, or $\psi(u_1 \dots u_n) du_1 \dots du_n$, represents the time during which, on average of any sufficiently long time in that stationary motion, the variables lie within the limits $r_1 \dots r_1 + dr_1$, &c., or $u_1 \dots u_1 + du_1$, &c.

27. Since Q contains products of the variables, the law of distribution of the values of $s_1 \dots s_n$ is not generally of the form $e^{-(m_1 s_1^2 + m_2 s_2^2 + \dots + m_n s_n^2)}$, with $m_1 \dots m_n$ constants, and cannot possibly have that form, unless firstly every $\bar{s} = 0$, and secondly there is no correlation, for if $Q = \Sigma m s^2$, e^{-Q} is the product of n factors each containing only one of the variables $s_1 \dots s_n$. Therefore, by definition, there is no correlation.

Every proof of the theorem in its ordinary form, *i. e.* with $Q = \sum m s^2$, either is based necessarily on the assumption, express or implied, that there is no correlation, or else it would prove the absence of correlation as a necessary fact, that is, that no physical system in which the motions of the parts are correlated can exist in stationary motion.

28. The theorem as above stated fails, or the proof fails, if Q can become negative; and therefore the proof fails if D , or as the case may be Δ , becomes negative. It fails also, and this is important, if one of the variables on which Q depends becomes discontinuous, which may happen by the variation of external conditions.

The Law of Maximum Probability.

29. Since Q contains products as well as squares of the variables $r_1 \dots r_n$ or $u_1 \dots u_n$, we can effectively make e^{-Q} maximum, subject to the constants of the system, and the kinetic energy may be one of such constants. By making e^{-Q} maximum we obtain the most probable, or normal state of the system, subject to the constants.

30. Whether we should use $r_1 \dots r_n$, or $u_1 \dots u_n$ for independent variables is, as above stated, a question of convenience.

The constant kinetic energy has dimensions $\frac{ML^2}{T^2}$. It is therefore convenient to use the r 's, if r has dimensions $\frac{L}{T}$, and u has $\frac{T}{L}$; and the u 's, if u has dimensions $\frac{L}{T}$. I will assume then that the kinetic energy, or $2nE$,

$$= m_1 u_1^2 + m_2 u_2^2 + \dots + m_n u_n^2$$

and

$$Q = \frac{1}{2} a_1 u_1^2 + b_{12} u_1 u_2 + \dots + \frac{1}{2} a_2 u_2^2 + \&c.$$

The kinetic energy, $\sum m u^2$, if expressed in terms of $r_1 \dots r_n$, would be a quadratic function of $r_1 \dots r_n$, with coefficients functions of α, β of (20).

31. I assume also that E is either constant, or varies very slowly with the time, while $u_1 \dots u_n$ in general vary very rapidly, so that they may go through cycles of changes while E is sensibly constant. E belongs to the class of variables which Max Planck calls "langsam veränderlich," while $s_1 \dots s_n$ belong to the class "schnell veränderlich."

Also I assume that the b , or correlation coefficients, are
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functions of *inter alia* a quantity v , which, like E , is “*langsam veränderlich*.” And that they possess the property that as v diminishes every b^2 increases *cæteris paribus*. That is, the b coefficients may, as mentioned in art. 8, be functions of the instantaneous distance between the variables to which they relate, *e. g.*, b_{pq} may be a function of ρ , the distance which at the instant separates u_p and u_q , or the things to which u_p and u_q relate, from each other. If then the distances ρ diminish by the diminution of v , which will generally be the case if v denote the volume of the system, the condition that every $\frac{db^2}{dv}$ is negative will be satisfied, if every $\frac{db^2}{d\rho}$ is negative.

32. Since the two functions $\phi(r_1 \dots r_n)$ and Ce^{-Q} , or let us say ϕ and F , are equal to one another throughout a certain range of values of v , for all of which D is positive, it follows that at every point or value of v within that range $\frac{d\phi}{dv} = \frac{dF}{dv}$, and all the derived coefficients of ϕ , as $\frac{d^2\phi}{dv^2}$, &c., are respectively equal to the corresponding derived coefficients of F . Therefore, by Taylor's theorem, $\phi = F$ for all values of v for which that theorem can be legitimately applied, with initial v within the given range. If, however, when a certain value of v , say $v = V$, is reached, a discontinuous change takes place in v , Taylor's theorem will in general at that point fail, and the equation $\phi = F$ will cease to be true.

33. Since e^{-Q} is maximum,

$$\frac{dQ}{du_1} \partial u_1 + \frac{dQ}{du_2} \partial u_2 + \&c. = 0,$$

and since E is constant,

$$m_1 u_1 \partial u_1 + m_2 u_2 \partial u_2 + \&c. = 0.$$

Whence, if λ be the indeterminate multiplier,

$$\frac{dQ}{du_1} = \lambda m_1 u_1, \quad \frac{dQ}{du_2} = \lambda m_2 u_2 \&c.,$$

or in the notation of (24)

$$\left. \begin{aligned} a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c. &= \lambda m_1 u_1 \\ b_{12} u_1 + a_2 u_2 + b_{23} u_3 + \&c. &= \lambda m_2 u_2 \\ \&c. \end{aligned} \right\} \dots \dots (A)$$

and λ is given by the determinantal equation of the n th degree,

$$\left. \begin{array}{cccc} \left(\frac{a_1}{m_1} - \lambda\right) & \frac{b_{12}}{m_1} & \frac{b_{13}}{m_1} & \dots \\ \frac{b_{21}}{m_1} & \left(\frac{a_2}{m_2} - \lambda\right) & \frac{b_{23}}{m_2} & \dots \\ \frac{b_{31}}{m_3} & \frac{b_{32}}{m_3} & \left(\frac{a_3}{m_3} - \lambda\right) & \dots \\ \dots & \dots & \dots & \dots \end{array} \right\} = 0 \quad \dots \quad (B)$$

If λ be any real root of equation (B), its substitution in (A) determines the ratios $\frac{u_2}{u_1}$, $\frac{u_3}{u_1}$, &c. in a system of values of $u_1 \dots u_n$ for which Q is minimum, given E.

Also

$$Q = \frac{1}{2} \sum u \frac{dQ}{du} = \frac{1}{2} \lambda \sum m u^2 = \lambda n E,$$

and this, with the ratios $\frac{u_2}{u_1}$, $\frac{u_3}{u_1}$, &c., determines an actual set of values of $u_1 \dots u_n$ which make Q minimum, given E.

34. Now expanding equation (B), we have

$$\lambda^n - \Delta_1 \lambda^{n-1} + \Delta_2 \lambda^{n-2} - \dots \pm \Delta_n = 0; \quad \dots \quad (C)$$

in which $\Delta_1 = \sum \frac{a}{m}$, and is the sum of the roots, Δ_n is the determinant B when $\lambda = 0$, and differs from D only by the factor $\frac{1}{m_1 m_2 \dots m_n}$, and is also the product of the n roots, and has the + or - sign prefixed, according as n is even or odd. Δ_{n-1} is the sum of the coaxial minors of D, having $(n-1)^2$ constituents, each divided by the product of $(n-1)$ m 's, and is equal to the sum of the products of the roots taken $n-1$ together, and has the opposite sign to that of Δ_n prefixed, and so on down to Δ_2 , which is the sum of the coaxial minors of D having 2^2 constituents, each divided by the product of two m 's, as $\frac{a_1 a_2 - b_{12}^2}{m_1 m_2}$, and is equal to the sum of the products of all pairs of the roots, and always has the positive sign prefixed.

It thus appears that the determinant D divided by

$m_1 m_2 \dots m_n$ is the product of n factors, which are the n roots of equation (B); also that the sum of the coaxial minors of D each of t^2 constituents, and each divided by the proper factors m , t in number, is equal to the sum of the products of the n factors of Δ_n taken t together, whatever number t may be less than n . For Δ_{n-1} is by the theory of equations the sum of the products of the n roots taken $n-1$ together. And it is also the sum of the coaxial minors of D each of $(n-1)^2$ constituents divided as aforesaid, &c. It is possible to construct a determinant which shall possess this property, but its demonstration would be too long for the present paper.

Since $Q = \lambda n E$, and E is positive, Q has the same sign as λ ; and since we are limited to positive values of Q , we are limited to positive values of λ . That is, equation (B) can have for our purpose no negative roots. But D is proportional to the product of the n roots of (B). Therefore we are limited to positive values of D .

35. Every maximum value of e^{-Q} determines a state of stable equilibrium for the system, that is, if $u_1 \dots u_n$ are changing rapidly, a state of stationary motion. There may be many such states corresponding to different real roots of (B). We may call the state corresponding to any particular real root λ , the state λ . If λ_1 and λ_2 be two real roots, and $\lambda_2 > \lambda_1$, then the state λ_1 is more probable than the state λ_2 in the ratio $e^{(\lambda_2 - \lambda_1)nE}$. If $\lambda_2 - \lambda_1$ is not nearly zero, and n very great, then the state λ_1 is more probable than the state λ_2 in a very high ratio, and the more so as nE increases. It follows that very great values of λ are in a very high degree improbable, and may be neglected. The state corresponding to the *least root* of B is the most probable of all the states of stable equilibrium—I define it to be the *normal state*.

36. I think the method thus investigated is applicable to determine the normal state of any material system whose parts mutually influence each other, and therefore become correlated. And is not this the case with almost all material systems in nature? A rare gas is perhaps the only known system to which the assumption of no correlation has been or can be legitimately applied. Further, a system of mutually acting, and therefore correlated, parts is a living system. On the other hand, if Q be reduced to a sum of squares, as Boltzmann's H theorem professes to prove, it would be, if left to itself, a dead system, for which no further change is possible.

On the Equipartition of Energy.

37. In the first equation of (19)

$$u_1 = \frac{D_{11}}{D} r_1 + \frac{D_{12}}{D} r_2 + \&c.,$$

multiply both sides by r_1 , and take mean values of both sides, remembering that since $r_1 \dots r_n$ are possible values of $s_1 \dots s_n$, $\overline{r_1^2} = \overline{s_1^2} = a_1$. Similarly $\overline{r_2^2} = a_2$ &c. and $\overline{r_p r_q} = b_{pq}$ &c. That gives

$$\begin{aligned} \overline{r_1 u_1} &= \frac{a_1 D_{11} + b_{12} D_{12} + b_{13} D_{13} + \&c.}{D} \\ &= \frac{D}{D} = 1. \end{aligned}$$

Similarly $\overline{r_2 u_2} = 1$ and $\overline{r_1 u_1} = \overline{r_2 u_2}$ &c. = $\overline{r_n u_n}$. Note that this result would fail if $\overline{s_1}, \overline{s_2}$ &c. were not zero, for then $\overline{r_1^2} \neq a_1$ &c.

Evidently $\overline{r_1 u_1} = u_1 \overline{\frac{dQ}{du_1}}$ &c. and $\overline{r_1 u_1} = r_1 \overline{\frac{dQ}{dr_1}}$ &c. Whence the law of equipartition of energy takes the general form

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} = \&c.$$

Let us now apply the results of making Q minimum above investigated. We have, since

$$Q = \lambda n E, \quad \frac{dQ}{du} = \lambda n \frac{dE}{du}, \quad \text{or} \quad u \frac{dQ}{du} = \lambda n u \frac{dE}{du},$$

for each u . Whence, if we may assume that

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} \&c.$$

for the normal or most probable state, we have

$$\overline{u_1 \frac{dE}{du_1}} = \overline{u_2 \frac{dE}{du_2}} \&c.$$

And if $2nE = \sum m_i \overline{u_i^2}$, $m_1 \overline{u_1^2} = m_2 \overline{u_2^2}$ &c. It may perhaps be objected that the law

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} \&c.$$

is proved only when the means are taken over *all* values of the variables, consistent with the constancy of E , and may

fail if confined to those values which make Q minimum. But E is constant for all possible values of $u_1 \dots u_n$, and therefore for the values that make Q minimum.

It appears then that the law of equipartition of energy is not necessarily dependent on Maxwell's law of the distribution of velocities, $e^{-\frac{1}{2}\sum mu^2}$, but depends on the conditions (1) that every variable s has zero for its mean, and (2) that $2nE$ has the form $\sum mu^2$.

Application of the Method to Gases.

38. I assume now that the system represents a gas, or two or more gases uniformly mixed, and that $u_1 \dots u_n$ are the vector velocities, $m_1 \dots m_n$ the masses, of the molecules. For the dimensions of $u_1 \dots u_n$ and the other variables in this case see art. 25.

I assume, further, that each of the gases forming the mixture, or the single gas if there be only one, is homogeneous as regards the constitution of its molecules, and that the mixture is homogeneous as regards the proportions in which different gases are mixed. Also that E represents its temperature, and is uniform throughout. And that v represents the volume containing a given number of molecules, and has, up to a certain point hereafter to be defined, the same value at all points. The definitions of *temperature at a point* and *density at a point* present no difficulty in a system of this homogeneous character.

Since the number of molecules in volume v is proportional to v , $\sum mu^2$ for the molecules in volume v is proportional to vE ; and instead of $Q = \lambda nE$ it is convenient to write $Q = \lambda vE$, using v to express so much of the space occupied by the gas as contains n molecules.

I proceed to prove that the velocities of any two molecules if sufficiently near to each other, say at distance less than ρ from each other, will be appreciably correlated, if at sufficiently great distance inappreciably. And I assume $-b_{pq}$ to be, for any distance ρ , proportional to the force R which at that distance the two molecules denoted by p and q exert on each other, and that for very small values of ρ such force is repulsive, and $-b_{pq}$ negative. This is consistent with art. 8.

39. In Clausius' Virial equation, let P denote pressure and v volume, u the vector velocity, m the mass of a molecule, R the force which acts between two molecules at distance ρ from each other, R being positive when the force is repulsive. Then Clausius' equation is

$$\frac{3}{2}Pv = \sum \frac{1}{2}mu^2 + \frac{1}{2}\sum\sum R\rho.$$

If the gas be rare enough, the last or Virial term is negligible, because R is negligible for all but an infinitely small proportion of the pairs of molecules at every instant. For such a gas then

$$\frac{3}{2} P v = \frac{1}{2} \sum m u^2 = n E$$

very approximately. That is, Boyle's law is fulfilled very approximately. This is the case for air and other gases at ordinary pressures. For such gases, since R is, for nearly all pairs, negligible, the correlation coefficients b are, for nearly all pairs, negligible, and Maxwell's law holds. But as the density of the gas increases, the resistance to compression increases beyond—and ultimately very far beyond—what it would be under Boyle's law (see Lord Kelvin's paper "On the Problem of a Spherical Gaseous Nebula," p. 260 note). For a sufficiently dense gas therefore the Virial term can no longer be neglected, as a considerable part of the pressure P , in $\frac{3}{2} P v$, is due to it. For the same reason the correlation coefficients b , which are proportional to the R 's, can no longer be neglected. Q becomes a complete quadratic function containing products as well as squares of the velocities.

40. Experimentally, by reason of the homogeneity, any portion of the gas, if containing a sufficiently great number of molecules, has the same properties as any other portion, or as the whole. Analytically the same result appears thus: About any point O in the gas as centre, suppose a sphere of radius c containing N molecules, and another concentric sphere of radius $c + c'$, containing $N + N'$ molecules, and both c and c' much greater than the radius of correlation. Let Q_N be the value of Q for the N molecules in the c sphere, $Q_{N+N'}$ its value for the $N + N'$ molecules within the sphere $c + c'$. Then, if N be very great,

$$\frac{Q_N}{Q_{N+N'}} = \frac{N}{N + N'}.$$

Therefore $\frac{Q}{NE}$, or λ , is independent of N . It is true the number of roots of equation B , as applied to the N molecules, is N , and as applied to the $N + N'$ molecules is $N + N'$, or N' new roots are introduced by taking in the additional N' molecules. The statement that λ is independent of N means therefore that all the roots of (B) , as applied to the N molecules, are also roots of (B) as applied to the $N + N'$ molecules. And in particular the least root of (B) is the same in the two cases. The correlation coefficients between the N' new variables and the original N are, except in an infinitely small proportion of the cases, evanescent. Therefore by art. 35 the normal state

is the same in the two cases ; and therefore the system of N molecules in the sphere c has the same physical properties as the system of $N+N'$ molecules in the sphere $c+c'$.

41. I have not up to this point assumed either sign for the correlation coefficients. That must depend on the physical relations of the system. If the forces between two molecules be repulsive at sufficiently small distances, as must be the case if discrete molecules are to exist permanently, let χ denote the potential at any point of all such forces. Then, according

to Boltzmann's law, $e^{-2\lambda\chi}$, where $\frac{3}{2h}$ is mean kinetic energy, must be a maximum, or $\bar{\chi}$ must be a minimum, given the total energy, when the motion is stationary. That will be the case if molecules very near each other move on average in the same direction, so that very near approaches involving high potentials are rare. That is, if, m_p and m_q are neighbouring molecules, the scalar product $u_p u_q$ is on average positive.

The molecules of the dense gas tend to move, not as in the ordinary rare gas each independently of all its neighbours, but *in streams*. And this tendency increases as the density increases.

Of two motions of a system with the same total energy, $\chi+T$, that one is the more probable for which $h\chi$ is the less. Let the potential of mutual action of two molecules m and m' , when distant r from each other, be $f(r)$. We might assume $f(r) = \frac{1}{r^q}$, where q is positive and not less than unity, or $f(r) = \frac{1}{r}e^{-\kappa r}$, where κ is positive. In either case if $x < r$

$$f(r+x) + f(r-x) > 2f(r) \quad \text{and} \quad \frac{d}{dx} \{f(r+x) + f(r-x)\}$$

is positive. It follows that if r is constant on average of time, the mean value of $f(r)$ for the same time is least when m and m' have no relative velocity, or $x=0$, and increases as the kinetic energy of their relative velocity increases. If therefore they have any common velocity, *i. e.* stream motion, the mean potential χ or $\Sigma f(r)$ is less than it would be if with the same total kinetic energy they had no stream motion.

I think the existence of the streams is thus proved. It is a plausible theory that, of all possible stationary motions which a material system may have, that one is the most probable, and therefore will be the actual motion, in which the mean value of $e^{-2\lambda\chi}$ is maximum, or $\bar{\chi}$ is minimum. That is analogous to the theorem, that of all

positions which a material system may have in statical equilibrium, that one will be chosen for which χ is minimum. In fact, by making h infinite, the first theorem is reduced to the statical theorem. If this theorem be not accepted in all its generality, still it must be accepted for a gas, for which in fact Boltzmann proved it. It amounts merely to saying that the system chooses, for given total energy, the motion of *least resistance*.

42. It is evident that, as v diminishes, and therefore the coefficients of products in Q increase in absolute magnitude, Q generally diminishes. For the only terms in Q that can be negative are the terms containing products. And such terms can always be made negative by making the correlation coefficient, e. g. b_{pq} , negative if u_p and u_q have the same sign, or by making it positive if they have opposite signs. As we have already seen, there are, if the system represents a gas, physical reasons why when b_{pq} is not inappreciable, u_p and u_q should in general have the same sign, and therefore why, if Q is to be as small as possible, b_{pq} should be negative.

But, at all events when v is very great, and therefore the correlation coefficients very small, $\frac{D_{pq}}{D}$, which is equal to $\frac{u_p u_q}{D}$, has the opposite sign to b_{pq} . For $D_{pq} = -b_{pq}$ multiplied by a coaxial minor of D which is positive, plus terms containing products of more than one b , which, v being great enough, are negligible compared with the term containing only b_{pq} . Therefore D_{pq} is generally of the opposite sign to $r_p r_q$, that is to b_{pq} . On the other hand, β_{pq} of (20) is equal to $\frac{u_p u_q}{D}$, and is positive.

Since, further, Q diminishes as v diminishes, and $Q = \lambda n E$, it follows that λ diminishes as v diminishes, or if λ be the least root of B , $\frac{d\lambda}{dv}$ is positive. Also, since as v diminishes the correlation terms in Q assume relatively more importance, $\frac{d\lambda}{dv}$ generally increases as v diminishes, and $\frac{d^2\lambda}{dv^2}$ is negative.

43. Having thus explained what I consider to be the general form of motion of the dense gas, I now assume that it is being compressed with constant temperature, that is v diminishes, E remaining constant.

I make an hypothesis, namely this:—That “the critical volume” V at which a gas under compression undergoes liquefaction, is the volume at which $D=0$, and that in such liquefaction v , the volume, changes discontinuously. And

that this is the discontinuous change in v at which the equation $\phi(r_1 \dots r_n) = Ce^{-Q}$ ceases to be true, expressed above analytically as the failure of Taylor's Theorem. It will appear that this hypothesis leads to results agreeing with experiment in the liquefaction of gases under pressure.

44. *On possible variations of density between different parts of the system, as a consequence of the diminution of v .*

If for the whole gas, or for any separate portion of it, v varies continuously, λ the least real root of equation (B), as applied to that portion, will in general vary continuously as a function of v .

Let us suppose the volume v of the gas to be divided into

N parts, each equal to $\frac{v}{N}$. Let v' be a volume less than v divided into N parts each equal to $\frac{v'}{N}$, and let λ' and $\frac{d\lambda'}{dv}$

denote the values of those functions for v' . Suppose that by the compression v is reduced to $v - \partial v$. This may

happen in either of two ways. Suppose $\partial v = \frac{v - v'}{N}$. Then,

event A, as v diminishes by ∂v , each of the N parts into which v is divided diminishes in the same ratio, so that the density, however varying with the time, remains constant in space throughout the whole volume v . Or, event B, as v is diminished by ∂v , some one of the N parts into which v is divided, is diminished by ∂v , and therefore, since

$\partial v = \frac{v - v'}{N}$, is reduced to $\frac{v'}{N}$, the other parts remaining unchanged in volume.

In event A, λ varies continuously throughout, and becomes for each diminution ∂v of v , $\lambda - \frac{d\lambda}{dv} \partial v$. The chance of this happening is therefore, for each ∂v , represented by

$$e^{-(\lambda - \frac{d\lambda}{dv} \partial v) v E}.$$

In event B, the chance is the product of two chances,

(1) that for $\frac{v}{N}$, λ shall become λ' , and (2) that for $v(1 - \frac{1}{N})$, λ shall be unchanged. The chance is therefore

$$e^{-\lambda' \frac{vE}{N}} e^{-\lambda(1 - \frac{1}{N}) v E}.$$

45. If P_A denote the probability of event A, P_B that of event B, we have

$$P_A = e^{-\left(\lambda - \frac{d\lambda}{dv}\right) \frac{vE}{N}},$$

$$P_B = e^{-\lambda' \frac{vE}{N}} e^{-\lambda \left(1 - \frac{1}{N}\right) vE},$$

and $P_B/P_A = e^{\left(\frac{\lambda - \lambda'}{v - v'} - \frac{d\lambda}{dv}\right) \frac{vE}{N}}.$

Event B is the more probable when $\lambda - \lambda'$ is greater than

$$(v - v') \frac{d\lambda}{dv}.$$

Now $\lambda' = \lambda - (v - v') \frac{d\lambda}{dv} + \frac{1}{2}(v - v')^2 \frac{d^2\lambda}{dv^2} + \&c.,$

and by making $v - v'$ very small we may reject all higher powers of it. And since $\frac{d^2\lambda}{dv^2}$ is negative (art. 42) $\frac{\lambda - \lambda'}{v - v'} \geq \frac{d\lambda}{dv}$

—that is $P_B \geq P_A$. Further, when $v' = V$, $\lambda' = 0$, because λ' is the least root of (B), and must therefore vanish when D, the product of the roots, becomes zero. Therefore ultimately

$$\frac{\lambda}{v - v'} > \frac{d\lambda}{dv}.$$

46. It thus appears that, consistently with the theory of probabilities, inequalities of density *may arise* at any time. But so long as the system remains gas, they will be dispersed by diffusion as fast as they arise. The uniformity of density is therefore stable. But as v approaches V , the portion of gas which assumes the greater density becomes liquid, and is not dispersed by diffusion. Liquefaction will therefore take place of separate portions of gas successively, other portions remaining as gas unaffected, until, with continuing compression, they undergo liquefaction in their turn. And the proportion of gas liquefied for a diminution, ∂v , of the volume of gas for the time being is $\frac{\partial v}{v - V}$.

47. It is true that we cannot define *a priori* what particular part of v will as $\frac{v}{N}$ bear the whole loss of volume ∂v . In the same way, when a gas liquefies under pressure, we know that as the diminishing volume approaches the critical volume, some portion, though we cannot define what specific portion, will be liquefied, the rest remaining as gas unaffected. In

fact the result above obtained agrees in this respect with the results obtained experimentally in the liquefaction of gases under compression.

Change of E .

48. As E increases with v constant, the axial constituents of the determinant D increase, while the anaxial constituents which depend on v remain generally constant. The effect of that is to increase D and all its coaxial minors. If therefore E be great enough, no amount of compression will reduce D to zero, and liquefaction cannot take place.

II. *Electrical Oscillations in Coupled Circuits.* By E. TAYLOR JONES, D.Sc., Professor of Physics in the University College of North Wales, Bangor*.

[Plate I.]

IT is well known that in a system consisting of two circuits containing capacity, self-inductance, and sufficiently great mutual inductance, each circuit has two natural periods of electrical oscillation. The problem of determining the constants of the two oscillations has been considered by Oberbeck †, M. Wien ‡, and Drude §. Drude showed that in addition to having different periods the two oscillations also have in general different damping coefficients, and calculated their values; he further showed how to calculate the potential at the terminals of the secondary circuit at any time after the application of a given potential-difference to those of the primary.

The present paper deals mainly with the case in which the oscillations are started by breaking a current in the primary. The expression for the secondary potential is deduced for this case by Drude's method, and compared with measurements of photographs obtained by means of the short-period electrometer (or "electrostatic oscillograph," as it has been called) described by the author ||. The instrument shows that the course of the variation of the secondary potential is much simpler when the oscillations are started

* Communicated by the Author.

† Oberbeck, *Wied. Ann.* lv. p. 623 (1895).

‡ M. Wien, *Wied. Ann.* lxi. p. 151 (1897).

§ Drude, *Ann. der Physik*, xiii. p. 512 (1904).

|| E. T. Jones, *Phil. Mag.* August, 1907.

in this way, owing to the absence of disturbance by the auxiliary sparking apparatus.

Some photographs and their measurements are also given for electrical oscillations in coupled circuits produced by the "musical arc" method.

(1) *Calculation of the Potential at the Terminals of the Secondary Condenser.*

The general equations as employed by Drude for two oscillatory circuits, when the coefficient of coupling M^2/L_1L_2 is not very small, are

$$\left. \begin{aligned} (T_1^2 + \theta_1^2) \frac{d^2 V_1}{dt^2} + 2\theta_1 \frac{dV_1}{dt} + V_1 &= p_{12} (T_1^2 + \theta_1^2) \frac{d^2 V_2}{dt^2} \\ (T_2^2 + \theta_2^2) \frac{d^2 V_2}{dt^2} + 2\theta_2 \frac{dV_2}{dt} + V_2 &= p_{21} (T_2^2 + \theta_2^2) \frac{d^2 V_1}{dt^2} \end{aligned} \right\} \quad (1)$$

where

$$\begin{aligned} \theta_1 &= \frac{1}{2} R_1 C_1, & T_1^2 + \theta_1^2 &= L_1 C_1, & p_{12} &= 2 \frac{L_{12} C_2}{L_1 C_1}, \\ \theta_2 &= \frac{1}{2} R_2 C_2, & T_2^2 + \theta_2^2 &= L_2 C_2, & p_{21} &= \frac{1}{2} \frac{L_{21} C_1}{L_2 C_2}. \end{aligned}$$

R_1, L_1, C_1 , and R_2, L_2, C_2 , are the resistances, self-inductances, and capacities in the primary and secondary circuits.

Drude's results are applicable to the case of a Tesla transformer, in the secondary of which the current varies from point to point along the wire. In the experiments described in the present paper the two coils were connected to condensers of considerable capacity, so that we may neglect the variation of current along the wires, and in consequence put $L_{12} = L_{21} = M$, the mutual inductance of the two coils.

V_1 and $2V_2$ are the potential-differences of the plates of the primary and secondary condensers.

Drude shows that the solution of (1) is

$$\begin{aligned} V_1 &= A_1 e^{t/y_1} + A_2 e^{t/y_2} + A_3 e^{t/y_3} + A_4 e^{t/y_4}, \\ V_2 &= B_1 e^{t/y_1} + B_2 e^{t/y_2} + B_3 e^{t/y_3} + B_4 e^{t/y_4}, \end{aligned}$$

where

$$\begin{aligned} y_1 &= \beta - \frac{\theta_1 + \theta_2}{2} + iT, & y_2 &= \beta - \frac{\theta_1 + \theta_2}{2} - iT, \\ y_3 &= -\beta - \frac{\theta_1 + \theta_2}{2} + iT', & y_4 &= -\beta - \frac{\theta_1 + \theta_2}{2} - iT'; \end{aligned}$$

$2\pi T$ and $2\pi T'$ are the longer and shorter periods of oscillation of the circuits, $i = \sqrt{-1}$, and β is given by

$$2\beta(T^2 - T'^2) = -(\theta_1 - \theta_2)(T_1^2 - T_2^2). \quad (2)$$

When the oscillations are started by breaking a current i_0 in the primary circuit, the initial conditions are

$$-C_1 dV_1/dt = i_0, \quad V_1 = R_1 i_0, \quad dV_2/dt = 0, \quad V_2 = 0, \quad \text{when } t = 0.$$

Hence

$$\Sigma A = R_1 i_0,$$

$$\Sigma \frac{A}{y} = -i_0/C_1,$$

$$\Sigma B = 0,$$

$$\Sigma \frac{B}{y} = 0.$$

It can also be shown* that

$$\begin{aligned} \Sigma B y &= p_{21}(T_2^2 + \theta_2^2) \Sigma \frac{A}{y} \\ &= -p_{21}(T_2^2 + \theta_2^2) i_0/C_1 \\ &= -G_1, \text{ say;} \end{aligned}$$

and that

$$\begin{aligned} \Sigma B y^2 &= p_{21}(T_2^2 + \theta_2^2) \Sigma A - 2\theta_2 \Sigma B y \\ &= p_{21}(T_2^2 + \theta_2^2) \cdot R_1 i_0 + p_{21}(T_2^2 + \theta_2^2) i_0/C_1 \\ &= 2p_{21}(T_2^2 + \theta_2^2) (\theta_1 + \theta_2) i_0/C_1 = G_2, \text{ say.} \end{aligned}$$

Hence, in order to determine the coefficients B we have the four equations:

$$B_1 + B_2 + B_3 + B_4 = 0,$$

$$\frac{B_1}{y_1} + \frac{B_2}{y_2} + \frac{B_3}{y_3} + \frac{B_4}{y_4} = 0,$$

$$B_1 y_1 + B_2 y_2 + B_3 y_3 + B_4 y_4 = -G_1,$$

$$B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_4^2 = G_2.$$

Solving these for B_1/y_1 , we find

$$\frac{B_1}{y_1} = \frac{G_1(y_2 + y_3 + y_4) + G_2}{(y_1 - y_2)(y_1 - y_3)(y_1 - y_4)}.$$

* Cf. Drude, *l.c.* p. 537.

Similarly,

$$\frac{B_2}{y_2} = \frac{G_1(y_1 + y_3 + y_4) + G_2}{(y_2 - y_3)(y_2 - y_4)(y_2 - y_1)},$$

$$\frac{B_3}{y_3} = \frac{G_1(y_1 + y_2 + y_4) + G_2}{(y_3 - y_4)(y_3 - y_2)(y_3 - y_1)},$$

$$\frac{B_4}{y_4} = \frac{G_1(y_1 + y_2 + y_3) + G_2}{(y_4 - y_1)(y_4 - y_2)(y_4 - y_3)}.$$

Inserting the values of y_1, y_2, y_3, y_4 , and remembering that $G_2/G_1 = 2(\theta_1 + \theta_2)$, we find after reduction,

$$B_1 = - \frac{G_1}{2(T+T')\{4\beta^2 + (T-T')^2\}} [2\beta T + (T-T')\lambda_2 - iT(T-T')]$$

where
$$\lambda_2 = \theta_1 + \theta_2 - \frac{2\beta T'}{T+T'}.$$

In this reduction quantities of the order β^2/T^2 and $\beta(\theta_1 + \theta_2)/T^2$ are neglected, these being, in all the cases considered below, very small.

In a similar manner we find

$$B_2 = - \frac{G_1}{2(T+T')\{4\beta^2 + (T-T')^2\}} [2\beta T + (T-T')\lambda_2 + iT(T-T')],$$

$$B_3 = + \frac{G_1}{2(T+T')\{4\beta^2 + (T-T')^2\}} [2\beta T' + (T-T')\lambda_1 - iT'(T-T')],$$

$$B_4 = + \frac{G_1}{2(T+T')\{4\beta^2 + (T-T')^2\}} [2\beta T' + (T-T')\lambda_1 + iT'(T-T')],$$

where
$$\lambda_1 = \theta_1 + \theta_2 + \frac{2\beta T}{T+T'}.$$

Also, again neglecting β^2/T^2 and $(\theta_1 + \theta_2)^2/T^2$ we have

$$\frac{1}{y_1} = - \frac{1}{T^2} \left(-\beta + \frac{\theta_1 + \theta_2}{2} \right) - \frac{i}{T},$$

$$\frac{1}{y_2} = - \frac{1}{T^2} \left(-\beta + \frac{\theta_1 + \theta_2}{2} \right) + \frac{i}{T},$$

$$\frac{1}{y_3} = - \frac{1}{T'^2} \left(\beta + \frac{\theta_1 + \theta_2}{2} \right) - \frac{i}{T'},$$

$$\frac{1}{y_4} = - \frac{1}{T'^2} \left(\beta + \frac{\theta_1 + \theta_2}{2} \right) + \frac{i}{T'}.$$

Hence

$$B_1 e^{t/y_1} = B_2 e^{t/y_2}$$

$$= \frac{-G_1 e^{-k_1 t}}{2(T+T')\{4\beta^2 + (T-T')^2\}} \left[\{2\beta T + (T-T')\lambda_2\} \cos \frac{t}{T} - T(T-T') \sin \frac{t}{T} \right]$$

and

$$B_3 e^{t/y_3} = B_4 e^{t/y_4}$$

$$= \frac{+G_1 e^{-k_2 t}}{2(T+T')\{4\beta^2 + (T-T')^2\}} \left[\{2\beta T' + (T-T')\lambda_1\} \cos \frac{t}{T'} - T'(T-T') \sin \frac{t}{T'} \right].$$

In the particular cases dealt with below $4\beta^2/(T-T')^2$ is never greater than .0004, and may be neglected.

Noticing that

$$2\beta T + (T-T')\lambda_2 = 2\beta T' + (T-T')\lambda_1,$$

and writing h for each of these quantities, also inserting the value of G_1 , we find

$$V_2 = -\frac{1}{2} \cdot \frac{M i_0 e^{-k_1 t}}{(T+T')(T-T')^2} \left[h \cos \frac{t}{T} - T(T-T') \sin \frac{t}{T} \right] \\ + \frac{1}{2} \frac{M i_0 e^{-k_2 t}}{(T+T')(T-T')^2} \left[h \cos \frac{t}{T'} - T'(T-T') \sin \frac{t}{T'} \right];$$

where k_1 and k_2 are written for the damping coefficients of the two oscillations, viz. :—

$$\frac{1}{T^2} \left(\frac{\theta_1 + \theta_2}{2} - \beta \right) \quad \text{and} \quad \frac{1}{T'^2} \left(\frac{\theta_1 + \theta_2}{2} + \beta \right).$$

Since h is small in comparison with $T(T-T')$ and $T'(T-T')$, we may express V_2 in the more convenient form

$$V_2 = + \frac{1}{2} \cdot \frac{M i_0 e^{-k_1 t} T}{(T^2 - T'^2) \cos \delta} \sin \left(\frac{t}{T} - \delta \right) \\ - \frac{1}{2} \frac{M i_0 e^{-k_2 t} T'}{(T^2 - T'^2) \cos \delta'} \sin \left(\frac{t}{T'} - \delta' \right).$$

where

$$\sin \delta = \frac{h}{T(T-T')}$$

and

$$\sin \delta' = \frac{h}{T'(T-T')}.$$

The difference of potential between the plates of the secondary condenser is therefore,

$$2V_2 = + \frac{Mi_0T}{(T^2 - T'^2) \cos \delta} e^{-\frac{t}{T^2} \left(\frac{\theta_1 + \theta_2}{2} - \beta \right)} \cdot \sin \left(\frac{t}{T} - \delta \right) \\ - \frac{Mi_0T'}{(T^2 - T'^2) \cos \delta'} e^{-\frac{t}{T'^2} \left(\frac{\theta_1 + \theta_2}{2} + \beta \right)} \cdot \sin \left(\frac{t}{T'} - \delta' \right). \quad (3)$$

(2) Experimental Arrangements.

A detailed description of a short-period electrometer designed by the author, suitable for the measurement of rapidly-varying potentials, was given in a previous paper *, and need not be repeated here. One or two improvements in the instrument may, however, be mentioned. The oscillating strip has been made shorter, the distance between the supporting glass rods being reduced to about 1.7 cm. This gives the strip and mirror a shorter natural period of oscillation. Instead of the oil-mixture described in the previous paper, paraffin-oil is now used. This has suitable viscosity, gives very high insulation, and is colourless. In its present form the instrument may be used for oscillations up to a frequency of 1500, and the present suspension has lasted for nearly a year with the strip under very considerable tension.

The present experiments were made with two circuits for which the values of the product self-inductance \times capacity were not very different. The secondary coil was the secondary of an induction-coil, of self-inductance, as measured by Rayleigh's method, 70.15×10^9 cm., and resistance at the time when the photographs were taken 14022 ohms. The primary coil consisted of about 1200 turns of No. 14 copper wire wound on a glass tube. Its resistance was 1.0378 ohms, and self-inductance (by the same method) $.004619 \times 10^9$ cm. This construction for the primary coil was adopted in order to make it suitable for the "musical arc" method.

Two leyden-jars were used as capacities in the secondary circuit, and two paraffin-paper condensers in the primary. The values of the secondary capacities were determined in the following manner. The terminals of the leyden-jar were connected to the secondary coil and to the electrometer. Oscillations were set up in this circuit by sparking to the terminals of the jar with an induction-coil whose primary circuit was broken in each revolution of the rotating mirror

* E. T. Jones, *l. c.* p. 238.

as the spots were crossing the photographic plate. The method of comparing the frequency, n , of the electrical oscillations with that of the tuning-fork, making allowance for difference of velocity of the electrometer and fork spots across the plate, was fully described in the previous paper. The self-inductance L_2 of the secondary coil being known, the capacity was calculated from the formula $C_2 = 1/4\pi^2 L_2 n^2$, which was shown in the previous paper to agree very closely with the results obtained from the photographs. This method has the advantage not only of giving the value of the secondary capacity under approximately the required experimental conditions, but also of giving at once the total capacity in the secondary circuit, including leyden-jar, electrometer, and coil.

This method was also used for measuring the coupling coefficient $M^2/L_1 L_2$ of the two coils. For this purpose one of the leyden-jars was connected to the secondary coil, and the oscillation curves were photographed with the primary coil open and closed, the primary condenser being removed. The ratio of the squares of the periods with the primary closed and open is then $1 - M^2/L_1 L_2$. The mutual inductance of the two coils was also deduced from these measurements, since L_1 and L_2 were known. The values so obtained were $M^2/L_1 L_2 = 0.1483$, $M = 0.2192 \times 10^9$ cm.

The method employed for the determination of the capacities of the paraffin-paper condensers was as follows. The condenser was connected directly to the terminals of the primary coil, whose circuit also included a battery and an interrupter consisting of a pointed zinc rod dipping under paraffin-oil into mercury. The terminals of the secondary coil were connected (without any condenser) to the electrometer. The mercury contact was broken, by means of a plate attached to the axle of the rotating mirror, as the spots were crossing the plate. Under these circumstances the secondary circuit had only the small capacity due to the electrometer and the coil, which was determined by methods described in the previous paper*. The frequency of the electrical oscillations in these circumstances is given by

$$n = \frac{1}{2\pi} \sqrt{\left(1 - \frac{M^2}{L_1 L_2} \cdot \frac{L_2 C_2}{L_1 C_1}\right) / L_1 C_1},$$

* *L. c.* p. 248. C_2 was here taken as the capacity of the electrometer $+ \frac{n-2}{n^2} C' + \frac{1}{12} C''$, where n is the number of sections in the coil, C' the capacity of one section on the next, C'' the capacity of the whole of the secondary coil with the primary coil inside it. The value of C_2 was .0000236 mfd.

a result which is deduced from the more general formula (4) below, by neglecting the square and higher powers of the small fraction L_2C_2/L_1C_1 . Now C_1 was known approximately, and this approximate value being used in the small term in the brackets, the inductances being also known, and n determined from the photographic plates, a more accurate value of C_1 was calculated from the above formula. The capacities of the paraffin-paper condensers were thus determined under conditions not differing greatly from those existing when the double-period oscillation curves were photographed. With either of the paraffin-paper condensers in the primary circuit, the frequency of the oscillations when the secondary capacity was very small was, in fact, intermediate between the two frequencies when one of the leyden-jars was connected to the secondary coil.

The values of the capacities are given in the accompanying tables.

(3) The Double-period Curves.

Four cases of coupled oscillating circuits were worked out, particulars of which are given in the following table.

TABLE I.

$L_1 = .004619 \times 10^9$ cm.	$L_2 = 70.15 \times 10^9$ cm.
$M = .2192 \times 10^9$ cm.	$M^2/L_1L_2 = .1483$.
$R_1 = 1.0378$ ohm.	$R_2 = 14022$ ohms.

	C_1 , microfarads.	C_2 , microfarads.	$L_1C_1 \times 10^8$.	$L_2C_2 \times 10^8$.	$\frac{\theta_1 + \theta_2}{2} \times 10^6$.
I.....	9.55	.001063	4.411	7.455	6.203
II.....	9.55	.000875	4.411	6.138	5.545
III.....	11.87	.001063	5.482	7.455	6.805
IV.....	11.87	.000875	5.482	6.138	6.147

The frequencies n_1 and n_2 of the two oscillations were calculated from Oberbeck's approximate formula

$$8\pi^2n^2 = \frac{1}{1 - M^2/L_1L_2} \left[\frac{1}{L_1C} + \frac{1}{L_2C_2} \pm \sqrt{\left\{ \left(\frac{1}{L_1C_1} - \frac{1}{L_2C_2} \right)^2 + \frac{4M^2}{L_1^2L_2^2C_1C_2} \right\}} \right]. \quad (4)$$

Hence the values of T and T' , *i. e.* $1/2\pi n_1$ and $1/2\pi n_2$, were obtained, and β was calculated from (2).

In Table II. are given the values of the frequencies n_1 , n_2 , of T and T', the damping coefficients k_1 and k_2 , β , and the phase angles δ , δ' in degrees, for the four cases.

TABLE II.

	n_1 .	n_2 .	$T \times 10^4$.	$T' \times 10^4$.	k_1 .	k_2 .	$\beta \times 10^9$.	δ .	δ' .
I....	542.2	882.7	2.9351	1.8030	80.21	169.1	-707	1.81	2.95
II....	582.8	905.0	2.7307	1.7585	77.50	171.8	-233	2.08	3.24
III....	527.0	814.7	3.0201	1.9535	77.24	172.0	-240	2.36	3.651
IV....	560.3	844.4	2.8405	1.8847	76.16	173.1	+1.79	2.475	3.743

The primary current, i_0 , was taken as 5.5 amperes, and values of the secondary potential, $2V_2$, were calculated from (3) for various values of t up to about .008 second. For the purpose of comparison with the photographs the values of $2V_2$ were squared and the results plotted in curves. Figs. 1, 2, 3, 4 show the curves so obtained for the four cases, the abscissæ representing the time in thousandths of a second measured from the moment when the primary circuit was broken, and the ordinates one millionth of the square of the potential-difference in volts of the plates of the secondary condenser.

In obtaining the photographs, the primary condenser was connected directly across the terminals of the primary coil whose circuit contained also a battery and the mercury break described above. The terminals of the secondary coil were connected to the leyden-jar and to the electrometer. The curve was photographed three or four times on each plate, this being moved vertically in its own plane between the exposures. In addition the two spots were photographed in a number of positions in order to obtain the ratio of their average velocities in any portion of the curve. The primary current was not measured before each discharge, but before and after a group of several discharges; in the four cases here considered the value so obtained was 5.5 amperes, but there might possibly have been small variations during each group of discharges which were not recorded.

Pl. I. figs. 5, 6, 7, 8 show the photographs obtained with the coils and condensers specified in Table I., the lower wave-curve in each representing the oscillations of a

768 tuning-fork photographed simultaneously. In general appearance the photographs closely resemble the calculated curves, figs. 1, 2, 3, 4, but in order to obtain a more accurate comparison the maximum ordinates and the time-intervals corresponding to the minima of the wave-curves were measured. These measurements were made on the negatives with a travelling microscope.

In determining the minima, a fine line ruled on the plate at right angles to the zero line of the curve was used as line of reference, and the time-interval between the first minimum and the eighth or ninth after the starting-point, determined by the method described in the previous paper. The intervals between the first minimum and the second, third, fourth, &c., were then taken as proportional to the corresponding lengths on the plate. The starting-point was not used in these measurements as, owing to the initial slope being zero, it is difficult to determine exactly the point at which the curve first begins to ascend.

In Table III. are given the time-intervals in thousandths of a second of the first eight or nine zeros (measured from the first) of the secondary potential as determined from the photographs and as calculated approximately from (3) and the curves, figs. 1, 2, 3, 4.

TABLE III.

No. of Zero.	CASE I.		CASE II.		CASE III.		CASE IV.	
	t 10 ³ obs.	t 10 ³ calc.	t 10 ³ obs.	t 10 ³ calc.	t 10 ³ obs.	t 10 ³ calc.	t 10 ³ obs.	t 10 ³ calc.
Origin	1.04	0.98	1.10	1.03
2	0.7553	0.76	0.7283	0.72	0.7773	0.78	0.7713	0.77
3	1.579	1.59	1.452	1.45	1.586	1.60	1.530	1.54
4	2.762	2.76	2.619	2.60	2.855	2.85	2.596	2.57
5	3.549	3.53	3.395	3.37	3.714	3.70	3.558	3.55
6	4.382	4.36	4.132	4.12	4.542	4.55	4.342	4.34
7	5.537	5.53	4.962	4.99	5.409	5.45	5.141	5.15
8	6.357	6.36	6.019	5.99	6.585	6.60	6.166	6.15
9	6.814	6.75	7.462	7.47	7.129	7.07

The zeros of $2V_2$ calculated from (3) therefore agree closely with those determined from the photographs.

Fig. 1.

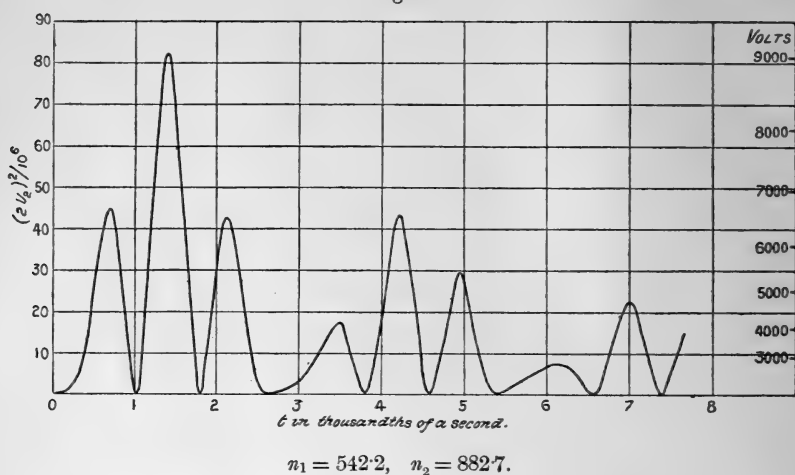
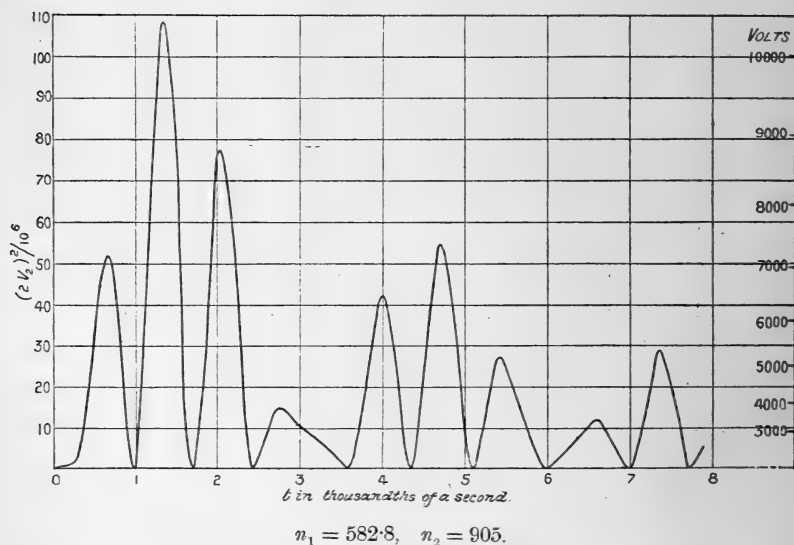


Fig. 2.



The maximum ordinates of the photographic curves were determined by turning the plate round, so that the zero-line of the curve was at right angles to the cross-wire, and measuring the height of each maximum above each of the two neighbouring minima. If the two latter were not quite

Fig. 3.

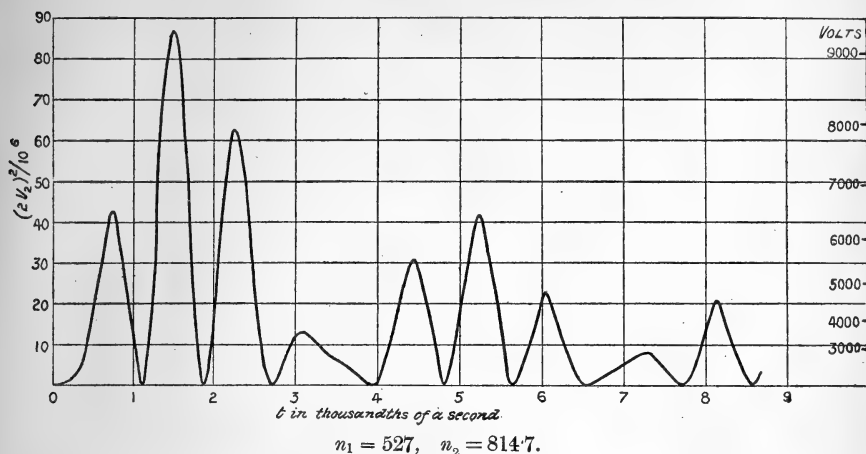
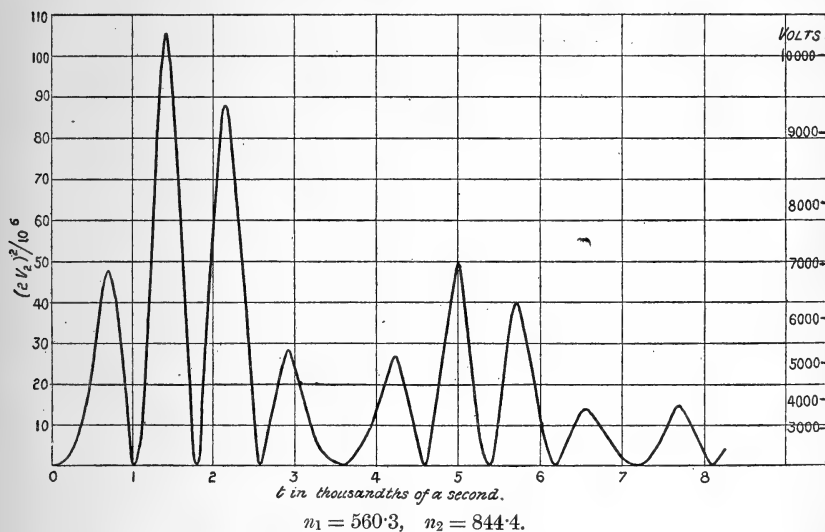


Fig. 4.



in the zero-line the mean of the two values was taken. The observed and calculated values of the maxima are given in Table IV., in which the first maximum of Case I. (4970 cm. in the photograph) is equalised to the corresponding calculated value 45, and all the other maxima of this and the other photographs are altered in the same proportion.

The numbers are given in millionths of the square of the volts at the plates of the secondary condenser.

TABLE IV.

No. of Maximum.	CASE I. Maxima.		CASE II. Maxima.		CASE III. Maxima.		CASE IV. Maxima.	
	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.
1.....	45	45	51.33	52.0	41.1	43.0	48.2	48.0
2.....	86.68	82.2	115.2	108.6	88.31	87.0	112.7	106.0
3.....	44.00	43.0	78.36	78.0	60.52	62.8	88.9	88.0
4.....	11.20	17.5	15.17	14.9	12.49	13.0	25.84	28.5
5.....	30.02	43.0	26.89	42.0	18.62	31.0	14.54	27.0
6.....	21.06	29.5	36.14	54.5	25.53	42.0	30.11	49.5
7.....	4.59	7.5	18.88	27.0	13.90	23.0	24.22	40.0
8.....	11.04	22.3	5.414	12.0	4.137	8.0	8.42	14.0
9.....	12.56	29.0	7.875	21.0	5.622	15.0

It will be seen that in the first group of waves, preceding the interval (about .003 to .0035 sec.), when the secondary potential is small and varies slowly, the maxima of the photographs are fairly closely proportional to the calculated values. In the second group the ratio of the observed to the calculated values of the maxima is considerably smaller, and there is a further drop in this ratio between the second and third groups. It seems, therefore, that the system loses energy rapidly in the intervals between the groups of waves of secondary potential. Now it is in these intervals that the energy exists mainly in the primary circuit, and that the potential at the plates of the primary condenser attains its highest values. Loss of energy, due to insufficient insulation or absorption in the paraffin-paper condensers, would therefore occur most rapidly during these intervals, and this appears to be the probable cause of the abnormal drop in the secondary potential from group to group of waves. Apart from this the wave of secondary potential represented by (3) agrees fairly well with the indications of the electrometer.

(4) *The Musical-arc Curves.*

If the primary coil, in series with the 11.87 mfd. condenser, is connected to the terminals of an arc lamp with

solid carbons (as in the Duddell musical-arc arrangement), and if the smaller leyden-jar is connected to the terminals of the secondary coil, either of two notes, differing in quality and separated in pitch by about a major sixth, may be heard. The pitch of the notes is not constant, but fluctuates to some extent, and can be varied by altering the length of the arc or the resistance in the supply circuit.

With these two condensers, which are the same as in Case IV. above, the two notes are equally stable, and the sound may change several times from one note to the other without any "feeding" of the arc, the change being always a sudden rise or fall of about a sixth.

If the primary capacity is increased to about 14.5 microfarads, two notes may again be heard but the higher note is less stable than before, and only comes out with a very short arc, and then with difficulty.

With the primary capacity 9.55 mfd. and the same secondary capacity (as in Case II. above) the lower note is rather less stable than the higher, while with $C_1=9.55$ and $C_2=.001063$ (Case I.) the lower note is rarely heard.

It appears therefore that with the coils used in these experiments, when L_1C_1 is nearly equal to L_2C_2 the two notes are equally stable; if the ratio of the primary to the secondary capacity is increased, the higher note becomes less stable; if this ratio is diminished, the lower note sounds less easily. If L_1C_1 is made either much smaller or much greater than L_2C_2 , only one note is heard.

Other pairs of condensers were tried, with similar results. In all cases in which the two notes were heard the interval between them was about a fifth or a sixth*.

In Pl. I., fig. 9, are shown specimens of the photographs obtained when the electrometer was connected to the secondary condenser. The upper curve represents the lower note and the lower curve the higher note, the capacities being $C_2=.000875$ and C_1 about 13.8 microfarads. In fig. 10 is shown the lower note curve for the case $C_1=11.87$, $C_2=.000875$ mfd.

It will be seen that while the higher note corresponds to an approximately simple oscillation, the curve for the lower note in fig. 9 consists of a succession of well-separated pairs of waves with sometimes two small waves appearing between the pairs. In fig. 10 the pairs of waves are not so well separated, and there are no small waves between them.

* The arc was supplied by the 200-volt continuous current mains through sufficient resistance. The current in the supply circuit was 3 or 4 amperes. The current in the primary circuit was generally greater with the low note than with the high note.

These curves appear to be due to the presence of the octave *, with varying intensity, along with the fundamental tone. Thus, in the lower note of fig. 9, the amplitude of the octave is approximately equal to that of the fundamental, while in fig. 10 the amplitude of the fundamental is about six times that of the octave. With given condensers in the circuits the relative intensity of the octave in the lower note can be varied by altering the length of the arc.

In Table V. are given the frequencies of the arc notes, along with the frequencies n_1 and n_2 calculated from the inductances and capacities, for the four combinations described in section (3). The arc frequencies were determined from one photograph only in each case, and must not be regarded as constant. The lower note in Case I. was rarely heard, and the corresponding curve was not photographed.

TABLE V.

Capacities in microfarads.		Singing arc.		Calculated frequencies.	
Primary.	Secondary.	Lower note.	Higher note.	n_1 .	n_2 .
9.55	.001063	691	542.2	882.7
9.55	.000875	542	780	582.8	905.0
11.87	.001063	470	723	527.0	814.7
11.87	.000875	445	745	560.3	844.4

Under the circumstances of these experiments, therefore, the musical arc gives, if the product of self-inductance and capacity is not very different in the primary and secondary circuits, two notes whose frequencies are generally considerably lower than the values calculated for the system from the inductances and capacities by (4): in addition the octave of the lower note is prominent.

It should be remembered, however, that in the calculated frequencies no account is taken of the resistance of the arc.

If the secondary condenser is removed (so that C_2 is very small) the arc gives one note only, the wave-curve being of the low-note type. With $C_1 = 9.55$ mfd. the arc frequency determined from one photograph was 659; breaking a current in the primary of this system (the arc being removed) gives a simple damped oscillation of frequency 755.5.

* The presence of harmonics in the high-frequency graphite arc has been detected by L. W. Austin (Electrician, Aug. 2, 1907).

With $C_1=11.87$ mfd. the frequency of the arc note was 578, and of the damped oscillation 678. In these cases also the arc frequencies are considerably less than those of the same circuits without the arc.

With $C_1=9.55$ and C_2 about .000275 mfd. one note was heard, and the secondary potential reached very high values during the oscillations, as was shown by copious "brushing" at the plates of the secondary condenser. The curve is shown in Pl. I. fig. 11. The amplitude was at times even greater than is there shown, and the zero-line had to be brought below the tuning-fork wave in order to keep the vibrating ray on the rotating mirror. The frequency is 613, and the octave appears to have an amplitude several times as great as that of the fundamental. The value of L_1C_1 is 4.411×10^{-8} , of L_2C_2 about 1.929×10^{-8} ; the frequencies calculated by (4) from the inductances and capacities are approximately 723 and 1300. The high potential at the plates of the secondary condenser in this case appears to be due to approximate agreement in frequency between the octave of the arc note and the higher natural vibration of the secondary coil. Similar evidences of "resonance" of a transitory kind were frequently observed with the other condensers when the arc note (either the high note or the low note) during its fluctuations fell rather low in pitch. All the effects here described may also be conveniently obtained by using a variable condenser with oil dielectric as the secondary capacity.

Bangor, September 1908.

III. *The Impulsive Motion of an Electrified Sphere.* By G. F. C. SEARLE, M.A., F.R.S., *University Lecturer in Experimental Physics, Cambridge* *.

§ 1. **A**T the end of a paper on "The Impulsive Motion of Electrified Systems,"† published in the *Philosophical Magazine* for January 1907, I have calculated the energy and momentum radiated when the velocity of a

* Communicated by the Author.

† I regret that when I wrote this paper I was unaware that Dr. Paul Hertz had obtained some of the results in his *Untersuchungen über unstetige Bewegungen eines Electrons*, Göttingen, 1904. His paper was in my possession at the time, but I was under the impression that it referred to the force required to set a charged sphere in motion and not to the problems which I was considering. Where my work overlaps that of Dr. Hertz, it gives an independent confirmation of his results, as it was done without any reference to his paper.

charged sphere is impulsively changed by an infinitesimal amount, the change of velocity making any angle with the initial velocity. In the present paper I complete the investigation by calculating the energy and momentum which are radiated when the velocity of the sphere is impulsively changed in *any* manner, the only restrictions being that both the initial and the final velocities are less than that of light, and that the sphere has no angular velocity about any axis through its centre. The results in the restricted case, in which the centre of the sphere moves along a single straight line throughout, have been given by Heaviside and by Paul Hertz.

§ 2. The investigation turns mainly upon a certain integral. In the evaluation of this integral I have been greatly aided by Mr. G. T. Bennett, Fellow of Emmanuel College. I had carried out the integration in the manner given in § 10 below, but I was unable to put the result into a symmetrical form. Mr. Bennett then devised the very elegant semi-geometrical process which is given in §§ 6, 7, 8, and, guided by the result, reduced the complicated expression arising in my method of integration to a symmetrical form. I am also indebted to Mr. F. J. W. Whipple, of the Merchant Taylors' School, for verifying the integration by an alternative method.

§ 3. Now let us consider a sphere of radius a , carrying a charge Q on its surface, and let us suppose that its velocity is impulsively changed from \mathbf{u}_1 to \mathbf{u}_2 , and that the angle between \mathbf{u}_1 and \mathbf{u}_2 is α , **heavy type** denoting vectors. Then it will be seen from § 31 of my earlier paper that, when the pulse generated by the change of velocity has travelled out through a distance r , which is very great compared with the diameter of the sphere, the electric force in the pulse is at right angles to the radius and is given by

$$\mathbf{E} = \frac{Q}{2Kra} \left\{ \frac{\mathbf{R}(\mathbf{u}_2\mathbf{R}) - \mathbf{u}_2}{v - \mathbf{u}_2\mathbf{R}} - \frac{\mathbf{R}(\mathbf{u}_1\mathbf{R}) - \mathbf{u}_1}{v - \mathbf{u}_1\mathbf{R}} \right\} \dots \quad (1)$$

Here \mathbf{R} denotes, for the moment, a unit vector along the radius, and $\mathbf{u}_1\mathbf{R}$ denotes the scalar product of \mathbf{u}_1 and \mathbf{R} , while K is the specific inductive capacity and v the velocity of light.

The magnetic force in the pulse is at right angles to both \mathbf{E} and \mathbf{R} , and is given by

$$\mathbf{H} = vK\mathbf{VRE}, \quad \dots \dots \dots (2)$$

where \mathbf{VRE} is a vector product.

If θ_1 and θ_2 be the angles between the radius and the directions of \mathbf{u}_1 and \mathbf{u}_2 , we have $\mathbf{u}_1\mathbf{R}=u_1 \cos \theta_1$ and $\mathbf{u}_2\mathbf{R}=u_2 \cos \theta_2$, and $\mathbf{u}_1\mathbf{u}_2=u_1u_2 \cos \alpha$, where α is the angle between \mathbf{u}_1 and \mathbf{u}_2 . Thus we find

$$E^2 = \frac{Q^2}{4K^2r^2a^2} \left\{ \frac{u_1^2 \sin^2 \theta_1}{(v-u_1 \cos \theta_1)^2} + \frac{u_2^2 \sin^2 \theta_2}{(v-u_2 \cos \theta_2)^2} - 2 \frac{u_1u_2 \cos \alpha - u_1u_2 \cos \theta_1 \cos \theta_2}{(v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2)} \right\}.$$

The numerator of the last term in this expression can be written in the form

$$2\{v(v-u_1 \cos \theta_1) + v(v-u_2 \cos \theta_2) - (v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2) - v^2 + u_1u_2 \cos \alpha\},$$

and thus we obtain, after an easy reduction,

$$E^2 = \frac{Q^2}{4K^2r^2a^2} \left\{ \frac{2(v^2 - u_1u_2 \cos \alpha)}{(v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2)} - \frac{v^2 - u_1^2}{(v-u_1 \cos \theta_1)^2} - \frac{v^2 - u_2^2}{(v-u_2 \cos \theta_2)^2} \right\}. \quad (3)$$

In the pulse, the magnetic energy per unit volume is equal to the electric energy per unit volume, and hence, remembering that the thickness of the pulse is $2a$, we find that W , the radiated energy, is given by

$$W = \frac{Kav^2}{2\pi} \int E^2 d\omega, \quad (4)$$

where $d\omega$ is an element of solid angle.

The last two terms in the expression for E^2 can be integrated at once. Thus we have

$$\begin{aligned} \int \frac{d\omega}{(v-u_1 \cos \theta_1)^2} &= 2\pi \int_0^\pi \frac{\sin \theta_1 d\theta_1}{(v-u_1 \cos \theta_1)^2} \\ &= \frac{4\pi}{v^2 - u_1^2}. \end{aligned}$$

The expression for W now becomes

$$W = \frac{Q^2}{8\pi aK} \left\{ \int \frac{2(v^2 - u_1u_2 \cos \alpha) d\omega}{(v-u_1 \cos \theta_1)(v-u_2 \cos \theta_2)} - 8\pi \right\}. \quad (5)$$

Writing n_1 for u_1/v and n_2 for u_2/v , and noting that

$Q^2/2\alpha K = U_0$, the electric energy of the sphere at rest, we have

$$W = U_0 \left\{ \frac{1 - n_1 n_2 \cos \alpha}{2\pi} \int \frac{d\omega}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_2)} - 2 \right\}. \quad (6)$$

The only difficulty lies in the evaluation of the integral in this expression. Since the integral is a function of n_1 , n_2 , and α , we shall write

$$F(n_1, n_2, \alpha) = \int \frac{d\omega}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_2)}, \quad (7)$$

where the integration extends over the surface of the unit sphere of which $d\omega$ is an element. Thus

$$W = U_0 \{ (1 - n_1 n_2 \cos \alpha) (2\pi)^{-1} F(n_1, n_2, \alpha) - 2 \}. \quad (8)$$

§ 4. When the directions of \mathbf{u}_1 and \mathbf{u}_2 are parallel and in the same sense, so that $\cos \alpha = 1$, the value of F is easily found, for we now have $\theta_2 = \theta_1$ and can take $2\pi \sin \theta_1 d\theta_1$ as the element of surface of the unit sphere. Thus

$$\begin{aligned} F(n_1, n_2, 0) &= 2\pi \int_0^\pi \frac{\sin \theta_1 d\theta_1}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_1)} \\ &= \frac{2\pi}{n_1 - n_2} \left\{ n_1 \int_0^\pi \frac{\sin \theta_1 d\theta_1}{1 - n_1 \cos \theta_1} - n_2 \int_0^\pi \frac{\sin \theta_1 d\theta_1}{1 - n_2 \cos \theta_1} \right\} \\ &= \frac{2\pi}{n_1 - n_2} \log \frac{(1 + n_1)(1 - n_2)}{(1 - n_1)(1 + n_2)}. \quad (9) \end{aligned}$$

When \mathbf{u}_1 and \mathbf{u}_2 are parallel but directed in *opposite* directions, so that $\cos \alpha = -1$, we have $\cos \theta_2 = -\cos \theta_1$, and hence the value of F can be found from (9) by changing the sign of n_2 . Then

$$F(n_1, n_2, \pi) = \frac{2\pi}{n_1 + n_2} \log \frac{(1 + n_1)(1 + n_2)}{(1 - n_1)(1 - n_2)}. \quad (10)$$

Substituting these values of F in (8) we obtain the following expressions for the radiated energy:—

When \mathbf{u}_1 and \mathbf{u}_2 are parallel and in the *same* direction, so that $\cos \alpha = 1$,

$$W = U_0 \left\{ \frac{1 - n_1 n_2}{n_1 - n_2} \log \frac{(1 + n_1)(1 - n_2)}{(1 - n_1)(1 + n_2)} - 2 \right\}. \quad (11)$$

When u_1 and u_2 are parallel but in opposite directions, so that $\cos \alpha = -1$,

$$W = U_0 \left\{ \frac{1+n_1n_2}{n_1+n_2} \log \frac{(1+n_1)(1+n_2)}{(1-n_1)(1-n_2)} - 2 \right\}. \quad (12)$$

When the motion is exactly reversed, so that $\cos \alpha = -1$ and $n_2 = n_1$

$$W = U_0 \left\{ \frac{1+n_1^2}{n_1} \log \frac{1+n_1}{1-n_1} - 2 \right\}. \quad (13)$$

The first result (11) was given by Dr. Oliver Heaviside*. These results have also been deduced by Paul Hertz† on dynamical principles from the values of the energy and momentum of a charged sphere in steady rectilinear motion.

§ 5. We now pass on to the general case in which the initial and final velocities are inclined at any angle. The integration of $F(n_1, n_2, \alpha)$ is now somewhat complicated, but on account of the fundamental importance of the integral two independent methods of evaluation are given below.

§ 6. The first method of evaluating $F(n_1, n_2, \alpha)$ is due to Mr. G. T. Bennett. Since this method is partly geometrical, it will be convenient to modify the integral so as to exhibit the geometrical quantities involved. We see, at once, from (7) that

$$F\left(\frac{R}{h_1}, \frac{R}{h_2}, \alpha\right) = \frac{h_1 h_2}{R^2} \cdot J, \quad (14)$$

where

$$J = \int \frac{R^2 d\omega}{(h_1 - R \cos \theta_1)(h_2 - R \cos \theta_2)}. \quad (15)$$

Here θ_1 and θ_2 are the angles between the radius defined by $d\omega$ and two fixed radii drawn from the centre of a sphere of radius R , and thus we can write

$$J = \int \frac{R^2 d\omega}{z_1 z_2}, \quad (16)$$

where z_1, z_2 are the perpendiculars drawn from any point on the surface of the sphere upon two fixed planes which cut the two fixed radii at right angles at distances h_1 and h_2 from the centre of the sphere. Since $n_1 < 1$ and $n_2 < 1$ we suppose

* 'Nature,' Nov. 6, 1902.

† *Untersuchungen über unetetige Bewegungen eines Electrons.* Göttingen, 1904. See also M. Abraham, *Theorie der Electricität*, vol. ii. p. 233.

that $h_1 > R$ and $h_2 > R$, and thus the planes do not cut the sphere.

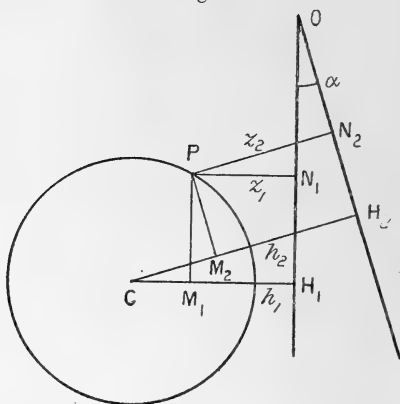
If we denote the mean value of $1/z_1 z_2$ for the surface of the sphere by $[1/z_1 z_2]_S$, we have

$$J = 4\pi R^2 \left[\frac{1}{z_1 z_2} \right]_S \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The first step is to cut the sphere in a circle by a plane perpendicular to both the fixed planes and to find the mean value of $1/z_1 z_2$ for this circle. The case in which the two fixed planes are parallel has been discussed in § 4. We may therefore now suppose that their line of intersection is at a finite distance from the centre of the sphere.

§ 7. Let the plane of the paper be normal to the line of intersection of the two fixed planes and let it cut the sphere in a circle of radius r , the centre of the circle being at a distance L from the line of intersection. Let OH_1 , OH_2 (fig. 1) be sections of the two fixed planes by the plane of

Fig. 1.



the paper, and let $CH_1 = h_1$ and $CH_2 = h_2$ be the perpendiculars from C , the centre of the circle, upon OH_1 and OH_2 . Then the angles H_1CH_2 and H_1OH_2 are each equal to α . If P be any point on the circle and PN_1 , PN_2 be perpendiculars from P upon the lines OH_1 , OH_2 , then $PN_1 = z_1$ and $PN_2 = z_2$.

Now draw PM_1 and PM_2 perpendicular to CH_1 and CH_2 . Then $CM_1 = h_1 - z_1$ and $CM_2 = h_2 - z_2$. Since M_1 and M_2 lie on a circle of which CP is the diameter, and since M_1M_2 subtends the angle α at C , it follows that M_1M_2 subtends an angle 2α at the middle point of CP . Hence

$$M_1M_2 = CP \sin \alpha = r \sin \alpha.$$

Since $CO=L$, we find in a similar manner

$$H_1H_2=L \sin \alpha.$$

From the triangles M_1CM_2 and H_1CH_2 we have

$$r^2 \sin^2 \alpha = (h_1 - z_1)^2 + (h_2 - z_2)^2 - 2(h_1 - z_1)(h_2 - z_2) \cos \alpha,$$

$$L^2 \sin^2 \alpha = h_1^2 + h_2^2 - 2h_1h_2 \cos \alpha;$$

and hence

$$(L^2 - r^2) \sin^2 \alpha = 2z_1z_2 \cos \alpha + 2z_1(h_1 - h_2 \cos \alpha) + 2z_2(h_2 - h_1 \cos \alpha) - z_1^2 - z_2^2.$$

Dividing by z_1z_2 , we obtain

$$\frac{(L^2 - r^2) \sin^2 \alpha}{z_1z_2} = 2 \cos \alpha + \frac{2(h_1 - h_2 \cos \alpha)}{z_2} + \frac{2(h_2 - h_1 \cos \alpha)}{z_1} - \frac{z_1}{z_2} - \frac{z_2}{z_1}. \quad (18)$$

The problem of finding the mean value of $1/z_1z_2$ for the circle is thus reduced to finding the mean values of $1/z_1$, $1/z_2$, z_1/z_2 and z_2/z_1 for the same circle.

If $[1/z_1]_C$ denote the mean value of $1/z_1$ for the circle of radius r , and if ϕ_1 denote the angle PCH_1 , we have

$$\left[\frac{1}{z_1}\right]_C = \frac{1}{\pi r} \int_0^\pi \frac{rd\phi_1}{h_1 - r \cos \phi_1},$$

since the integral from π to 2π is equal to that from 0 to π . Putting $\tan \frac{1}{2}\phi_1 = x$ and therefore $\cos \phi_1 = (1 - x^2)/(1 + x^2)$, we find

$$\begin{aligned} \left[\frac{1}{z_1}\right]_C &= \int_0^\infty \frac{2dx/\pi}{h_1 - r + (h_1 + r)x^2} = \frac{2/\pi}{(h_1^2 - r^2)^{\frac{1}{2}}} \left[\tan^{-1} x \sqrt{\frac{h_1 + r}{h_1 - r}} \right]_0^\infty \\ &= \frac{1}{(h_1^2 - r^2)^{\frac{1}{2}}} = \frac{1}{t_1}, \quad \dots \dots \dots (19) \end{aligned}$$

where t_1 is the length of the tangent from H_1 to the circle,

$$\text{Similarly} \quad [1/z_2]_C = 1/t_2. \quad \dots \dots \dots (20)$$

$$\text{Now} \quad \frac{z_2}{z_1} = \frac{PN_2}{PN_1} = \frac{PN_1 \cos \alpha + ON_1 \sin \alpha}{PN_1}.$$

But there are two positions of P for which PN_1 or z_1 has the same value, and the average value of ON_1 for these two positions is OH_1 . Thus we find for the whole circle

$$\begin{aligned} \left[\frac{z_2}{z_1}\right]_C &= \cos \alpha + OH_1 \sin \alpha \left[\frac{1}{z_1}\right]_C \\ &= \cos \alpha + \sin \alpha \cdot OH_1/t_1. \end{aligned}$$

Similarly, since

$$\frac{z_1}{z_2} = \frac{PN_1}{PN_2} = \frac{PN_2 \cos \alpha - ON_2 \sin \alpha}{PN_2},$$

we find

$$\left[\frac{z_1}{z_2} \right]_C = \cos \alpha - \sin \alpha \cdot OH_2/t_2.$$

$$\text{But } OH_1 \sin \alpha = CH_2 - CH_1 \cos \alpha = h_2 - h_1 \cos \alpha$$

$$OH_2 \sin \alpha = CH_2 \cos \alpha - CH_1 = h_2 \cos \alpha - h_1,$$

and thus

$$[z_2/z_1]_C = \cos \alpha + (h_2 - h_1 \cos \alpha)/t_1 \quad . \quad . \quad (21)$$

$$[z_1/z_2]_C = \cos \alpha + (h_1 - h_2 \cos \alpha)/t_2. \quad . \quad . \quad (22)$$

When we substitute in (18) the mean values shown in (19), (20), (21), and (22), we obtain

$$(L^2 - r^2) \sin^2 \alpha \left[\frac{1}{z_1 z_2} \right]_C = \frac{h_1 - h_2 \cos \alpha}{t_2} + \frac{h_2 - h_1 \cos \alpha}{t_1}.$$

$$\text{But } h_2 - h_1 \cos \alpha = OH_1 \sin \alpha = \sin \alpha \cdot (L^2 - h_1^2)^{\frac{1}{2}}. \quad . \quad (23)$$

$$h_2 \cos \alpha - h_1 = OH_2 \sin \alpha = \sin \alpha (L^2 - h_2^2)^{\frac{1}{2}}, \quad . \quad . \quad (24)$$

and hence

$$\left[\frac{1}{z_1 z_2} \right]_C = \frac{1}{(L^2 - r^2) \sin \alpha} \left\{ \frac{1}{t_1} (L^2 - h_1^2)^{\frac{1}{2}} - \frac{1}{t_2} (L^2 - h_2^2)^{\frac{1}{2}} \right\}. \quad (25)$$

§ 8. We can now find the mean value of $1/z_1 z_2$ for the surface of the sphere. If we measure x from the centre of the sphere parallel to the line of intersection of the two fixed planes, the area of the zone on the sphere defined by x and $x + dx$ is $2\pi R dx$, and thus

$$\left[\frac{1}{z_1 z_2} \right]_S = \frac{1}{2\pi R^2} \int_0^R \left[\frac{1}{z_1 z_2} \right]_C 2\pi R dx.$$

$$\text{Now } r^2 = R^2 - x^2,$$

$$\text{and hence } t_1^2 = h_1^2 - r^2 = h_1^2 - R^2 + x^2,$$

$$t_2^2 = h_2^2 - r^2 = h_2^2 - R^2 + x^2.$$

Thus we have

$$\left[\frac{1}{z_1 z_2} \right]_S = \frac{1}{R \sin \alpha} \int_0^R \frac{dx}{L^2 - R^2 + x^2} \left\{ \frac{(L^2 - h_1^2)^{\frac{1}{2}}}{(h_1^2 - R^2 + x^2)^{\frac{1}{2}}} - \frac{(L^2 - h_2^2)^{\frac{1}{2}}}{(h_2^2 - R^2 + x^2)^{\frac{1}{2}}} \right\}.$$

The integration can be effected by aid of the formula

$$\int \frac{(p^2 - q^2)^{\frac{1}{2}} dx}{(p^2 + x^2)(q^2 + x^2)^{\frac{1}{2}}} = \frac{1}{2p} \log \frac{p(q^2 + x^2)^{\frac{1}{2}} + x(p^2 - q^2)^{\frac{1}{2}}}{p(q^2 + x^2)^{\frac{1}{2}} - x(p^2 - q^2)^{\frac{1}{2}}}$$

if we put $p^2 = L^2 - R^2$, $q^2 = h^2 - R^2$,

and $p^2 - q^2 = L^2 - h^2$,

where h stands for either h_1 or h_2 . We thus obtain

$$\left[\frac{1}{z_1 z_2} \right]_s = \frac{1}{2R \sin \alpha (L^2 - R^2)^{\frac{1}{2}}} \left[\log A - \log B \right]_0^R,$$

where

$$A = \frac{(L^2 - R^2)^{\frac{1}{2}}(h_1^2 - R^2 + x^2)^{\frac{1}{2}} + x(L^2 - h_1^2)^{\frac{1}{2}}}{(L^2 - R^2)^{\frac{1}{2}}(h_1^2 - R^2 + x^2)^{\frac{1}{2}} - x(L^2 - h_1^2)^{\frac{1}{2}}}.$$

Thus

$$\left[\log A \right]_0^R = \log \frac{h_1(L^2 - R^2)^{\frac{1}{2}} + R(L^2 - h_1^2)^{\frac{1}{2}}}{h_1(L^2 - R^2)^{\frac{1}{2}} - R(L^2 - h_1^2)^{\frac{1}{2}}}.$$

For B we have only to substitute h_2 for h_1 . Hence

$$\left[\log \frac{A}{B} \right]_0^R = \log \frac{\{h_1(L^2 - R^2)^{\frac{1}{2}} + R(L^2 - h_1^2)^{\frac{1}{2}}\} \{h_2(L^2 - R^2)^{\frac{1}{2}} - R(L^2 - h_2^2)^{\frac{1}{2}}\}}{\{h_1(L^2 - R^2)^{\frac{1}{2}} - R(L^2 - h_1^2)^{\frac{1}{2}}\} \{h_2(L^2 - R^2)^{\frac{1}{2}} + R(L^2 - h_2^2)^{\frac{1}{2}}\}}.$$

We now substitute for $(L^2 - h_1^2)^{\frac{1}{2}}$ and $(L^2 - h_2^2)^{\frac{1}{2}}$ the values given in (23) and (24), and then carry out the multiplications in the numerator and denominator of the quantity under the logarithm. Replacing $h_1^2 + h_2^2 - 2h_1 h_2 \cos \alpha$ by $L^2 \sin^2 \alpha$, we then easily find

$$\begin{aligned} \left[\frac{1}{z_1 z_2} \right]_s &= \frac{1}{2R \sin \alpha (L^2 - R^2)^{\frac{1}{2}}} \left[\log \frac{A}{B} \right]_0^R \\ &= \frac{1}{2R \sin \alpha (L^2 - R^2)^{\frac{1}{2}}} \log \frac{h_1 h_2 - R^2 \cos \alpha + R(L^2 - R^2)^{\frac{1}{2}} \sin \alpha}{h_1 h_2 - R^2 \cos \alpha - R(L^2 - R^2)^{\frac{1}{2}} \sin \alpha} \end{aligned} \quad \dots (26)$$

where $L^2 \sin^2 \alpha = h_1^2 + h_2^2 - 2h_1 h_2 \cos \alpha$.

We have thus found the mean value of $1/z_1 z_2$ for points on the surface of the sphere.

Mr. Bennett has supplied the following remarks:—"The quantity under the logarithm in (26) is the cross-ratio of the pencil formed by the two planes together with the two tangent planes to the sphere through their line of intersection; and the result takes a simpler form if reciprocated with respect to the sphere itself. The perpendiculars from a variable point of the sphere to fixed planes are then replaced by (constant multiples of) perpendiculars drawn from fixed points within the sphere to a variable tangent plane. The result of the

mean-value problem may then be expressed concisely as follows :—

A sphere being given, AB being a fixed chord, and CD being fixed points on the chord, the mean value of the reciprocal of the product of the distances of C and D from a variable tangent plane is $\log (ABCD) \div AB \cdot CD$."

§ 9. To apply (26) to the electrical problem, we put

$$R=1, \quad h_1=1/n_1=v/u_1, \quad h_2=1/n_2=v/u_2,$$

It will be convenient to write

$$n_1^2 + n_2^2 - 2n_1n_2 \cos \alpha = m^2 = w^2/v^2, \quad \dots (27)$$

where w is the change of the velocity of the sphere, so that u_2 is the resultant of u_1 and w . Then

$$L^2 \sin^2 \alpha = \frac{1}{n_1^2} + \frac{1}{n_2^2} - \frac{2 \cos \alpha}{n_1 n_2} = \frac{m^2}{n_1^2 n_2^2}$$

and

$$(L^2 - R^2) \sin^2 \alpha = \frac{m^2 - n_1^2 n_2^2 \sin^2 \alpha}{n_1^2 n_2^2}.$$

By § 6 we have

$$F(n_1, n_2, \alpha) = \frac{4\pi}{n_1 n_2} \left[\frac{1}{z_1 z_2} \right]_B$$

provided h_1, h_2 and R have the values just written. Inserting those values in (26) we have

$$F(n_1, n_2, \alpha) = \frac{2\pi}{(m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{1 - n_1 n_2 \cos \alpha + (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{1 - n_1 n_2 \cos \alpha - (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \dots (28)$$

§ 10. We now proceed to give a second investigation of the value of

$$F(n_1, n_2, \alpha) = \int \frac{d\omega}{(1 - n_1 \cos \theta_1)(1 - n_2 \cos \theta_2)}$$

where the integration is taken over a sphere of unit radius.

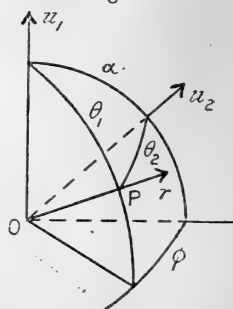
Let the plane containing the radius OP and the radius Ou_1 make an angle ϕ with the plane containing Ou_1 and Ou_2 , as shown in fig. 2. Then we have

$$\cos \theta_2 = \cos \theta_1 \cos \alpha + \sin \theta_1 \sin \alpha \cos \phi,$$

while

$$d\omega = \sin \theta_1 d\theta_1 d\phi.$$

Fig. 2.



Hence

$$F = \iint \frac{\sin \theta_1 d\theta_1 d\phi}{(1-n_1 \cos \theta_1)(1-n_2 \cos \alpha \cos \theta_1 - n_2 \sin \alpha \sin \theta_1 \cos \phi)}$$

where ϕ goes from 0 to 2π and θ_1 goes from 0 to π .

By § 7 we see that

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi}{1-n_2 \cos \alpha \cos \theta_1 - n_2 \sin \alpha \sin \theta_1 \cos \phi} \\ = \frac{2\pi}{\{(1-n_2 \cos \alpha \cos \theta_1)^2 - n_2^2 \sin^2 \alpha \sin^2 \theta_1\}^{\frac{1}{2}}} \end{aligned}$$

provided that the result is *real*. But

$$\begin{aligned} (1-n_2 \cos \alpha \cos \theta_1)^2 - n_2^2 \sin^2 \alpha \sin^2 \theta_1 \\ = (\cos \alpha - n_2 \cos \theta_1)^2 + \sin^2 \alpha (1-n_2^2), \end{aligned}$$

and hence this quantity is always positive, since $n_2^2 < 1$. The integration is therefore valid.

Replacing $\sin^2 \theta_1$ by $1 - \cos^2 \theta_1$ in the ϕ -integral, we find

$$F = \int_0^\pi \frac{2\pi \sin \theta_1 d\theta_1}{(1-n_1 \cos \theta_1)(1-n_2^2 \sin^2 \alpha - 2n_2 \cos \alpha \cos \theta_1 + n_2^2 \cos^2 \theta_1)^{\frac{1}{2}}}.$$

To reduce the integral to a simpler form, put

$$1-n_1 \cos \theta_1 = 1/x, \quad \cos \theta_1 = (x-1)/n_1 x.$$

$$\text{Then} \quad \sin \theta_1 d\theta_1 = -dx/n_1 x^2,$$

and $x = (1-n_1)^{-1}$ when $\theta=0$ and $x = (1+n_1)^{-1}$ when $\theta=\pi$. Making these substitutions, we easily find

$$F = 2\pi \int_A^B X^{-\frac{1}{2}} dx,$$

where

$$\begin{aligned} X &= p^2 x^2 - 2xn_2(n_2 - n_1\lambda) + n_2^2 \\ p^2 &= n_1^2 + n_2^2 - 2n_1n_2\lambda - n_1^2n_2^2 + n_1^2n_2^2\lambda^2 \quad . \quad . \quad (29) \\ \lambda &= \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30) \end{aligned}$$

and

$$A = (1+n_1)^{-1}, \quad B = (1-n_1)^{-1}.$$

Hence

$$F = \frac{2\pi}{p} \log Y,$$

where

$$\log Y = \left[\log \{p^2 x - n_2(n_2 - n_1\lambda) + pX^{\frac{1}{2}}\} \right]_A^B.$$

Using (29), we easily find that when $x=B=(1-n_1)^{-1}$

$$X^{\frac{1}{2}}=n_1(1-n_2\lambda)(1-n_1)^{-1},$$

and that when $x=A=(1+n_1)^{-1}$

$$X^{\frac{1}{2}}=n_1(1+n_2\lambda)(1+n_1)^{-1}.$$

Thus

$$Y = \frac{(1+n_1)\{p^2-n_2^2+n_1n_2\lambda+n_1n_2^2-n_1^2n_2\lambda+pn_1(1-n_2\lambda)\}}{(1-n_1)\{p^2-n_2^2+n_1n_2\lambda-n_1n_2^2+n_1^2n_2\lambda+pn_1(1+n_2\lambda)\}}.$$

When we substitute for p^2 from (29), we find that both numerator and denominator contain n_1 as a factor. When this factor is removed, we have

$$Y = \frac{(1+n_1)\{n_1+n_2^2-n_1n_2^2-n_2\lambda-n_1n_2\lambda+n_1n_2^2\lambda+p(1-n_2\lambda)\}}{(1-n_1)\{n_1-n_2^2-n_1n_2^2-n_2\lambda+n_1n_2\lambda+n_1n_2^2\lambda+p(1-n_2\lambda)\}}.$$

On carrying out the multiplications, we find that the five terms on the right side of (29) occur in both numerator and denominator. Replacing these terms by p^2 , we obtain

$$\begin{aligned} Y &= \frac{(1-n_1n_2\lambda)(n_1-n_2\lambda)+p(1-n_1n_2\lambda+n_1-n_2\lambda)+p^2}{(1-n_1n_2\lambda)(n_1-n_2\lambda)+p(1-n_1n_2\lambda-n_1+n_2\lambda)-p^2} \\ &= \frac{(1-n_1n_2\lambda+p)(n_1-n_2\lambda+p)}{(1-n_1n_2\lambda-p)(n_1-n_2\lambda+p)} \\ &= \frac{1-n_1n_2\lambda+p}{1-n_1n_2\lambda-p}. \end{aligned}$$

Since, by (29) and (27), $p^2 = m^2 - n_1^2 n_2^2 \sin^2 \alpha$ and since $\lambda = \cos \alpha$, we obtain, finally,

$$F(n_1, n_2, \alpha) = \frac{2\pi}{(m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{1 - n_1 n_2 \cos \alpha + (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{1 - n_1 n_2 \cos \alpha - (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}.$$

This result agrees with that found in § 9.

§ 11. We can now return to the electrical problem and obtain an expression for the energy radiated when the velocity of the sphere is impulsively changed.

Since, by (8)

$$W = U_0 \{ (2\pi)^{-1} (1 - n_1 n_2 \cos \alpha) F(n_1, n_2, \alpha) - 2 \}$$

we have, by (28)

$$\begin{aligned} W/U_0 &= \\ &= \frac{1 - n_1 n_2 \cos \alpha}{(m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{1 - n_1 n_2 \cos \alpha + (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{1 - n_1 n_2 \cos \alpha - (m^2 - n_1^2 n_2^2 \sin^2 \alpha)^{\frac{1}{2}}} - 2. \end{aligned}$$

In terms of the velocities the expression becomes

$$\frac{W}{U_0} = \frac{v^2 - u_1 u_2 \cos \alpha}{(v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}} \log \frac{v^2 - u_1 u_2 \cos \alpha + (v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{v^2 - u_1 u_2 \cos \alpha - (v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}} - 2.$$

It is easily seen that the values of W found in § 4 for $\alpha=0$ and for $\alpha=\pi$ may be deduced from the general expression for W .

It is convenient to write W/U_0 in the form

$$\frac{W}{U_0} = \frac{1}{Z} \log \frac{1+Z}{1-Z} - 2,$$

where

$$Z = \frac{(v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}}}{v^2 - u_1 u_2 \cos \alpha}.$$

The expression for W/U_0 has no meaning when Z , which is *positive*, is greater than unity. When Z is less than unity, we have

$$\frac{W}{U_0} = 2 \left\{ \frac{Z^2}{3} + \frac{Z^4}{5} + \frac{Z^6}{7} + \dots \right\}.$$

Since all the terms of this series are positive, W/U_0 increases as Z increases.

If we write $u_1^2 + u_2^2 - 2u_1 u_2 \cos \alpha$ for w^2 , we find

$$Z^2 = 1 - \frac{(v^2 - u_1^2)(v^2 - u_2^2)}{(v^2 - u_1 u_2 \cos \alpha)^2}.$$

Hence Z increases as α increases from 0 to π . Thus for given values of u_1 and u_2 , the radiated energy, W , increases as the angle between the directions of u_1 and u_2 increases, since both u_1 and u_2 are less than v .

For given values of u_1 and u_2 , the maximum value of Z occurs when $\alpha=\pi$. Then we have

$$Z_\pi = \frac{v(u_1 + u_2)}{v^2 + u_1 u_2} = 1 - \frac{(v - u_1)(v - u_2)}{v^2 + u_1 u_2},$$

and thus Z_π is positive and less than unity. Collecting these results we see that for all values of α and for all permissible values of u_1 and u_2 , Z is positive and less than unity.

§ 12. Since the loss of energy arises from the change in the velocity of the sphere, it will be convenient to express W as directly as possible in terms of w , the change of velocity. As we require two other quantities to fix the circumstances of the problem, we take u and ψ where $2u$ is the resultant of u_1 and u_2 and ψ is the angle between the positive

directions of \mathbf{u} and \mathbf{w} . For convenience, we write

$$u = nv. \quad \dots \dots \dots (31)$$

The relations which u_1 , u_2 , and α bear to u , w , and ψ are shown in fig. 3.

From the three equations

$$w^2 = u_1^2 + u_2^2 - 2u_1u_2 \cos \alpha,$$

$$u_1^2 = u^2 + \frac{1}{4}w^2 - uw \cos \psi,$$

$$u_2^2 = u^2 + \frac{1}{4}w^2 + uw \cos \psi,$$

we obtain

$$u_1u_2 \cos \alpha = u^2 - \frac{1}{4}w^2,$$

$$\text{or} \quad n_1n_2 \cos \alpha = n^2 - \frac{1}{4}m^2.$$

Writing down two equivalent expressions for the area of the parallelogram, we have

$$u_1u_2 \sin \alpha = uw \sin \psi,$$

or

$$n_1n_2 \sin \alpha = nm \sin \psi.$$

Hence

$$v^2w^2 - u_1^2u_2^2 \sin^2 \alpha = w^2(v^2 - u^2 \sin^2 \psi),$$

or

$$m^2 - n_1^2n_2^2 \sin^2 \alpha = m^2(1 - n^2 \sin^2 \psi).$$

When these values are inserted in the results of § 11, we obtain

$$\frac{W}{U_0} = \frac{1 - n^2 + \frac{1}{4}m^2}{m(1 - n^2 \sin^2 \psi)^{\frac{1}{2}}} \log \frac{1 - n^2 + \frac{1}{4}m^2 + m(1 - n^2 \sin^2 \psi)^{\frac{1}{2}}}{1 - n^2 + \frac{1}{4}m^2 - m(1 - n^2 \sin^2 \psi)^{\frac{1}{2}}} - 2,$$

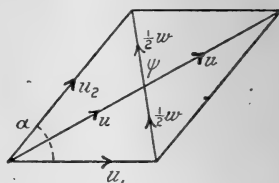
or

$$\frac{W}{U_0} = \frac{v^2 - u^2 + \frac{1}{4}w^2}{w(v^2 - u^2 \sin^2 \psi)^{\frac{1}{2}}} \log \frac{v^2 - u^2 + \frac{1}{4}w^2 + w(v^2 - u^2 \sin^2 \psi)^{\frac{1}{2}}}{v^2 - u^2 + \frac{1}{4}w^2 - w(v^2 - u^2 \sin^2 \psi)^{\frac{1}{2}}} - 2.$$

§ 13. We now pass on to calculate the momentum radiated in the pulse generated by an impulsive change of velocity of the sphere. It will not be necessary to employ the integral calculus, since the momentum can be deduced on dynamical principles from quantities which are already known.

To change the velocity of the sphere impulsively from \mathbf{u}_1 to \mathbf{u}_2 it is necessary to apply a force \mathbf{F}_{12} to the sphere as long as the pulse is passing over the sphere. When the pulse is clear of the sphere the force is no longer required. While the pulse is passing over the sphere, \mathbf{F}_{12} probably changes both in direction and in magnitude, but we are not concerned

Fig. 3.



with the value of \mathbf{F}_{12} at any instant, as we shall eliminate \mathbf{F}_{12} from the equations.

Let U_1 , T_1 , and U_2 , T_2 be the electric and magnetic energies of the sphere when moving with the velocities u_1 and u_2 , and let W_1 be the energy radiated when the velocity u_1 is suddenly destroyed, and W_2 the energy radiated when u_2 is destroyed. Since the force which stops the sphere does no work, the total energy in the electromagnetic field is unchanged. Before the sphere is stopped the energy is $U_1 + T_1$, and after it is stopped the energy is $U_0 + W_1$, where U_0 is the electrostatic energy. Hence

$$\begin{aligned} U_1 + T_1 &= U_0 + W_1 \\ \text{and} \quad U_2 + T_2 &= U_0 + W_2. \end{aligned}$$

The values of W_1 and W_2 , which may be deduced from the results of § 4, are given by

$$\begin{aligned} W_1 &= U_0 \left(\frac{v}{u_1} \log \frac{v+u_1}{v-u_1} - 2 \right) \\ &= U_0 \left(\frac{2n_1^2}{3} + \frac{2n_1^4}{5} + \frac{2n_1^6}{7} + \dots \right). \end{aligned} \quad (32)$$

$$W_2 = U_0 \left(\frac{v}{u_2} \log \frac{v+u_2}{v-u_2} - 2 \right). \quad (33)$$

Since $u_1 = n_1 v$, we can write

$$W_1 = U_0 \left\{ \frac{1}{n_1} \log \frac{1+n_1}{1-n_1} - 2 \right\}. \quad (34)$$

If we prefer to do so, we may express W_1 in terms of $\frac{1}{2}m_0v^2$, by the formula $U_0 = \frac{3}{2} \cdot \frac{1}{2}m_0v^2$, where m_0 is the electromagnetic mass for infinitesimal speeds.

TABLE I.—Values of $\frac{W_1}{U_0}$.

n_1 .	W_1/U_0 .	n_1 .	W_1/U_0 .
0.1	0.00671	0.6	0.31049
0.2	0.02733	0.7	0.47800
0.3	0.06346	0.8	0.74653
0.4	0.11824	0.85	0.95565
0.5	0.19722	0.9	1.27160

§ 14. Now consider the impulsive change of velocity from \mathbf{u}_1 to \mathbf{u}_2 . Before the force \mathbf{F}_{12} acted the energy in the electromagnetic field was $U_1 + T_1$, and after the force has acted the energy is $U_2 + T_2 + W$. Hence, by § 13, the gain of energy is $W_2 - W_1 + W$, and this must be equal to the work done by the force. Thus, in vector notation,

$$\begin{aligned} W_2 - W_1 + W &= \int \mathbf{F}_{12} \mathbf{u}_2 dt \\ &= \mathbf{u}_2 \int \mathbf{F}_{12} dt, \quad . \quad . \quad . \quad . \quad . \quad (35) \end{aligned}$$

since \mathbf{u}_2 is constant, the last expression being the scalar product of \mathbf{u}_2 and $\int \mathbf{F}_{12} dt$.

Similarly if \mathbf{F}_{21} be the force required to change the velocity from \mathbf{u}_2 to \mathbf{u}_1 , we have

$$W_1 - W_2 + W = \mathbf{u}_1 \int \mathbf{F}_{21} dt, \quad . \quad . \quad . \quad . \quad (36)$$

since the energy radiated is the same for the change from \mathbf{u}_2 to \mathbf{u}_1 as for the change from \mathbf{u}_1 to \mathbf{u}_2 .

From these equations of energy we pass to the equations of momentum. When the sphere is in steady motion, the resultant momentum of the electromagnetic field is in the same direction as the velocity of the sphere. If \mathbf{M}_1 be the momentum when the velocity is \mathbf{u}_1 and \mathbf{M}_2 when the velocity is \mathbf{u}_2 , it is known that,*

$$\mathbf{M}_1 = \frac{U_0}{2u_1} \left\{ \frac{v^2 + u_1^2}{vu_1} \log \frac{v + u_1}{v - u_1} - 2 \right\} \quad . \quad . \quad (37)$$

$$\mathbf{M}_2 = \frac{U_0}{2u_2} \left\{ \frac{v^2 + u_2^2}{vu_2} \log \frac{v + u_2}{v - u_2} - 2 \right\} \quad . \quad . \quad (38)$$

Since $u_1 = n_1 v$, we can write

$$\begin{aligned} \mathbf{M}_1 &= \frac{U_0}{v} \left[\frac{1}{2} \left(1 + \frac{1}{n_1^2} \right) \log \frac{1 + n_1}{1 - n_1} - \frac{1}{n_1} \right] \\ &= \frac{U_0}{v} \left\{ \frac{4n_1}{1.3} + \frac{8n_1^3}{3.5} + \frac{12n_1^5}{5.7} + \dots \right\}. \quad (39) \end{aligned}$$

If we prefer to do so, we may express \mathbf{M}_1 in terms of $m_0 v$ by means of the formula $U_0/v = 3m_0 v/4$, where m_0 is the electromagnetic mass for infinitely slow motion.

* These results may be deduced from § 19 of my previous paper, Phil. Mag. Jan. 1907.

TABLE II.—Values of $\frac{M_1 v}{U_0}$.

n_1 .	$M_1 v/U_0$.	n_1 .	$M_1 v/U_0$.
0.1	0.1339	0.6	0.9519
0.2	0.2710	0.7	1.2087
0.3	0.4153	0.8	1.5652
0.4	0.5715	0.85	1.8183
0.5	0.7465	0.9	2.1787

Let \mathbf{P} be the momentum in the pulse, so that

$$\mathbf{P} = \frac{\mu K}{4\pi} \iiint \mathbf{V} \mathbf{E} \mathbf{H} \, dx \, dy \, dz, \quad . \quad . \quad . \quad (40)$$

where $\mathbf{V} \mathbf{E} \mathbf{H}$ denotes the vector product of the electric and magnetic forces and the integration extends throughout the volume of the pulse. Then, since the momentum of the electromagnetic field is changed from \mathbf{M}_1 to $\mathbf{M}_2 + \mathbf{P}$ by the action of the force \mathbf{F}_{12} ,

$$\mathbf{M}_2 - \mathbf{M}_1 + \mathbf{P} = \int \mathbf{F}_{12} dt. \quad . \quad . \quad . \quad (41)$$

When the change of velocity of the sphere is effected in the reverse direction, both \mathbf{E} and \mathbf{H} in the pulse are simply reversed in direction without change of magnitude, as appears from (1) and (2), and hence \mathbf{P} remains unchanged. Hence we have

$$\mathbf{M}_1 - \mathbf{M}_2 + \mathbf{P} = \int \mathbf{F}_{21} dt. \quad . \quad . \quad . \quad (42)$$

Combining these equations with (35) and (36), and writing

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2,$$

where \mathbf{P}_1 is in the same direction as \mathbf{u}_1 and \mathbf{P}_2 in the same direction as \mathbf{u}_2 , we have

$$\mathbf{u}_2(\mathbf{M}_2 - \mathbf{M}_1 + \mathbf{P}_1 + \mathbf{P}_2) = W_2 - W_1 + W$$

$$\mathbf{u}_1(\mathbf{M}_1 - \mathbf{M}_2 + \mathbf{P}_1 + \mathbf{P}_2) = W_1 - W_2 + W.$$

Working out the scalar products, we obtain

$$P_1 \cos \alpha + P_2 = M_1 \cos \alpha - M_2 + (W_2 - W_1 + W)u_2^{-1} = X_1$$

$$P_1 + P_2 \cos \alpha = M_2 \cos \alpha - M_1 + (W_1 - W_2 + W)u_1^{-1} = X_2.$$

Solving these equations, we have

$$P_1 = \frac{X_2 - X_1 \cos \alpha}{\sin^2 \alpha}, \quad P_2 = \frac{X_1 - X_2 \cos \alpha}{\sin^2 \alpha}.$$

The magnitude of P is given by

$$\begin{aligned} P^2 &= P_1^2 + P_2^2 + 2P_1P_2 \cos \alpha \\ &= P_2X_1 + P_1X_2 \\ &= \frac{X_1^2 + X_2^2 - 2X_1X_2 \cos \alpha}{\sin^2 \alpha}. \end{aligned}$$

If we substitute for M_1 , M_2 , W_1 , and W_2 the values given in (37), (38), (32), and (33), and write

$$\log \frac{v+u_1}{v-u_1} = L_1, \quad \log \frac{v+u_2}{v-u_2} = L_2,$$

and make a slight re-arrangement of the factor of L_1 , we find that

$$\begin{aligned} P_2 + P_1 \cos \alpha &= \frac{W}{u_2} + \frac{U_0}{v} \left[\left\{ \frac{(v^2 - u_1^2) \cos \alpha}{2u_1^2} - \frac{v^2 - u_1u_2 \cos \alpha}{u_1u_2} \right\} L_1 \right. \\ &\quad \left. + \frac{v^2 - u_2^2}{2u_2^2} L_2 + \frac{v(u_1 - u_2 \cos \alpha)}{u_1u_2} \right]. \quad (43) \end{aligned}$$

Similarly, we find

$$\begin{aligned} P_1 + P_2 \cos \alpha &= \frac{W}{u_1} + \frac{U_0}{v} \left[\left\{ \frac{(v^2 - u_2^2) \cos \alpha}{2u_2^2} - \frac{v^2 - u_1u_2 \cos \alpha}{u_1u_2} \right\} L_2 \right. \\ &\quad \left. + \frac{v^2 - u_1^2}{2u_1^2} L_1 + \frac{v(u_2 - u_1 \cos \alpha)}{u_1u_2} \right]. \quad (43a) \end{aligned}$$

We can, without difficulty, write out the value of P^2 , but the expression becomes very complicated in the general case. We shall therefore consider its value in special cases only.

The component parallel to u_1 of the momentum in the pulse is $P_1 + P_2 \cos \alpha$. If we make $u_2 = 0$, we obtain the momentum in the pulse which is formed when u_1 , the initial velocity of the sphere, is suddenly destroyed. The expression for $P_1 + P_2 \cos \alpha$ (not $P_2 + P_1 \cos \alpha$) becomes indeterminate when we put $u_2 = 0$, but this difficulty can be overcome by first expanding L_2 in the form $2n_2 + 2n_2^3/3 + \dots$, and then putting $n_2 = 0$ in the terms which do not cancel. Using the value of W_1 given by (34) and using $u_1 = n_1v$, we find for

the resultant momentum in the pulse formed on stopping the sphere

$$P_1 = \frac{U_0}{v} \left\{ \left(\frac{3}{2n_1^2} - \frac{1}{2} \right) \log \frac{1+n_1}{1-n_1} - \frac{3}{n_1} \right\} \quad . \quad . \quad (44)$$

$$= \frac{U_0}{v} \left\{ \frac{4n_1^3}{3 \cdot 5} + \frac{8n_1^5}{5 \cdot 7} + \frac{12n_1^7}{7 \cdot 9} + \dots \right\}.$$

This result agrees with that given in § 19 of my earlier paper. In Table III. the numerical values of $P_1 v/U_0$ are given as well as the ratio of P_1 to M_1 , *i. e.* the ratio of the momentum carried off in the pulse to the momentum of the field before the velocity is destroyed. It will be easily seen from (39) and (44) that, when n_1 is very small, $P_1/M_1 = n_1^2 \cdot 5$. On the other hand, when n_1 approaches unity, only the logarithmic parts of the expressions for P_1 and M_1 need be considered, and we find that although both P_1 and M_1 tend to infinity, their ratio P_1/M_1 tends to unity.

TABLE III.—Values of $\frac{P_1 v}{U_0}$ and of $\frac{P_1}{M_1}$.

n_1 .	$P_1 v/U_0$.	P_1/M_1 .	n_1 .	$P_1 v/U_0$.	P_1/M_1 .
0	0	0	0.6	0.08308	0.08728
0.1	0.00027	0.00201	0.7	0.15699	0.12988
0.2	0.00221	0.00816	0.8	0.30113	0.19239
0.3	0.00780	0.01878	0.85	0.43029	0.23664
0.4	0.01977	0.03459	0.9	0.64711	0.29702
0.5	0.04237	0.05676	1.0	∞	1.00000

If the "impulse" of the force which stops the motion of the sphere be I_1 , then

$$M_1 - I_1 = P_1,$$

and hence, by (39) and (44),

$$I_1 = M_1 - P_1$$

$$= \frac{U_0}{v} \left\{ \left(1 - \frac{1}{n_1^2} \right) \log \frac{1+n_1}{1-n_1} + \frac{2}{n_1} \right\}.$$

Since $(1-n_1) \log(1-n_1)$ tends to zero as n_1 tends to unity, it follows that the limiting value of I is $2U_0/v$.

Thus it appears that if a charged sphere moving at the

speed of light be arrested by an obstacle, it only communicates a finite part of its infinite momentum to the obstacle. From Table II. we see that the momentum communicated to the obstacle is equal to that possessed by the sphere when moving at a speed of about 0.87 v .

The "impulse" of the force required to suddenly give the sphere a velocity equal to that of light is $P_1 + M_1$, and thus is infinite.

When the sphere is stopped, the pulse ceases to cut the sphere after a time $2a/v$, and then no force is required to hold the sphere at rest.

But when the sphere is started into motion at the speed of light, the sphere is always just enclosed in its own pulse, and consequently the force required to maintain its motion does not vanish until the motion has continued for an infinite time.

The component in the direction opposite to the sphere's motion of the electric force in the pulse is *

$$E = \frac{Qu \sin \theta}{2Kra(v-u \cos \theta)} \sin \theta$$

provided r be very great compared with a . When $u=v$ and θ is very small,

$$E = \frac{Q}{Kra}.$$

If x be measured along the direction of motion from the centre of the sphere, the retarding mechanical force experienced by the sphere in consequence of the action of the pulse tends to the value

$$\int_{-a}^{+a} \frac{Q}{Kra} \cdot \frac{Qdx}{2a} = \frac{Q^2}{Kar}$$

or to the value

$$\frac{Q^2}{Kavt},$$

when t , the time since the sphere was started, tends to infinity. I stated this result in a review of Dr. Heaviside's 'Electromagnetic Theory' in the 'Physical Review,' July 1900, where, however, by some error the result is printed $Q^2/2avt$ †.

When a charged sphere is suddenly set into motion with velocity u at the time $t=0$, the force required to maintain that velocity does not cease until $t=2a/(v-u)$, *i. e.* until the

* Phil. Mag. January 1907, p. 124.

† The printers did not furnish me with a "proof" of the review.

pulse generated by the sudden change of velocity no longer cuts the sphere. Dr. Paul Hertz* has obtained a complete solution for the force required in starting the sphere at any time from $t=0$ to $t=2a/(v-u)$, and, by extending his method, I have found the force required to stop the sphere †.

§ 15. The momentum in the pulse can also be obtained by direct integration. Since in the pulse $\mathbf{H} = v\mathbf{K}\mathbf{V}\mathbf{R}\mathbf{E}$, it follows that $\mathbf{V}\mathbf{E}\mathbf{H} = v\mathbf{K}\mathbf{V}\mathbf{E}\mathbf{V}\mathbf{R}\mathbf{E}$, where \mathbf{R} is a unit-vector along the radius. But \mathbf{E} is perpendicular to \mathbf{R} , and thus $\mathbf{V}\mathbf{E}\mathbf{H} = v\mathbf{K}\mathbf{E}^2\mathbf{R}$. Hence $\mathbf{V}\mathbf{E}\mathbf{H}$ is in the same direction as \mathbf{R} . Thus, by (40) the momentum in the pulse is

$$\mathbf{P} = \frac{K}{4\pi v} \iiint \mathbf{E}^2 \mathbf{R} \, dx \, dy \, dz,$$

the integration extending throughout the volume of the pulse.

Since the thickness of the pulse is $2a$, it follows that, if r be the radius of the pulse and $d\omega$ an element of solid angle,

$$\mathbf{P} = \frac{Kr^2a}{2\pi v} \int \mathbf{E}^2 \mathbf{R} \, d\omega.$$

The value of \mathbf{E}^2 is given in (3). For brevity we write

$$\begin{aligned} v - u_1 \cos \theta_1 &= h_1, & v - u_2 \cos \theta_2 &= h_2, \\ v^2 - u_1 u_2 \cos \alpha &= S. \end{aligned}$$

Noting that $Q^2/2Ka = U_0$, we obtain

$$\mathbf{P} = \frac{U_0}{4\pi v} \int \left(\frac{2S}{h_1 h_2} - \frac{v^2 - u_1^2}{h_1^2} - \frac{v^2 - u_2^2}{h_2^2} \right) \mathbf{R} \, d\omega.$$

The component of \mathbf{P} parallel to \mathbf{u}_2 is $P_2 + P_1 \cos \alpha$, with the notation of § 14, and thus, since the angle between \mathbf{R} and \mathbf{u}_2 is θ_2 , we have

$$P_2 + P_1 \cos \alpha = \frac{U_0}{4\pi v} \int \left(\frac{2S}{h_1 h_2} - \frac{v^2 - u_1^2}{h_1^2} - \frac{v^2 - u_2^2}{h_2^2} \right) \cos \theta_2 \, d\omega.$$

* *Untersuchungen über unetetige Bewegungen eines Electrons.*

† "On the Force required to stop a Moving Electrified Sphere," Proc. Royal Society, A. vol. lxxix. p. 550. In this paper I have given a sketch of Dr. Hertz's method and have stated his results.

Now $\cos \theta_2 = (v - h_2)/u_2$, and hence

$$\int \frac{\cos \theta_2 d\omega}{h_1 h_2} = \frac{1}{u_2} \left\{ v \int \frac{d\omega}{h_1 h_2} - \int \frac{d\omega}{h_1} \right\}.$$

But, by (7),
$$v \int \frac{d\omega}{h_1 h_2} = \frac{1}{v} F(n_1, n_2, \alpha)$$

and
$$\int \frac{d\omega}{h_1} = 2\pi \int_0^\pi \frac{\sin \theta_1 d\theta_1}{v - u_1 \cos \theta_1} = \frac{2\pi}{u_1} L_1,$$

where
$$L_1 = \log \frac{v + u_1}{v - u_1} \quad \text{and} \quad L_2 = \log \frac{v + u_2}{v - u_2}.$$

With the notation of § 10,

$$\cos \theta_2 = \cos \theta_1 \cos \alpha + \sin \theta_1 \sin \alpha \cos \phi,$$

and thus

$$\int \frac{\cos \theta_2 d\omega}{h_1^2} = \iint \frac{(\cos \theta_1 \cos \alpha + \sin \theta_1 \sin \alpha \cos \phi) \sin \theta_1 d\theta_1 d\phi}{h_1^2},$$

where ϕ goes from 0 to 2π and θ_1 from 0 to π . The term involving $\cos \phi$ vanishes on integration with respect to ϕ . Writing $(v - h_1)/u_1$ for $\cos \theta_1$, we obtain

$$\begin{aligned} \int \frac{\cos \theta_2 d\omega}{h_1^2} &= \frac{2\pi \cos \alpha}{u_1} \left\{ v \int_0^\pi \frac{\sin \theta_1 d\theta_1}{h_1^2} - \int_0^\pi \frac{\sin \theta_1 d\theta_1}{h_1} \right\} \\ &= \frac{2\pi \cos \alpha}{u_1} \left\{ \frac{2v}{v^2 - u_1^2} - \frac{L_1}{u_1} \right\}. \end{aligned}$$

Writing $(v - h_2)/u_2$ for $\cos \theta_2$, we obtain

$$\begin{aligned} \int \frac{\cos \theta_2 d\omega}{h_2^2} &= \frac{2\pi}{u_2} \left\{ v \int_0^\pi \frac{\sin \theta_2 d\theta_2}{h_2^2} - \int_0^\pi \frac{\sin \theta_2 d\theta_2}{h_2} \right\} \\ &= \frac{2\pi}{u_2} \left\{ \frac{2v}{v^2 - u_2^2} - \frac{L_2}{u_2} \right\}. \end{aligned}$$

Collecting these results, and substituting for $F(n_1, n_2, \alpha)$ the value derived from (8), we have

$$\begin{aligned} P_2 + P_1 \cos \alpha &= \frac{W}{u_2} + \frac{U_0}{v} \left[\left\{ \frac{(v^2 - u_1^2) \cos \alpha}{2u_1^2} - \frac{v^2 - u_1 u_2 \cos \alpha}{u_1 u_2} \right\} L_1 \right. \\ &\quad \left. + \frac{v^2 - u_2^2}{2u_2^2} L_2 + \frac{v(u_1 - u_2 \cos \alpha)}{u_1 u_2} \right]. \end{aligned}$$

This result is identical with (43), which was obtained in § 14 by dynamical principles.

§ 16. When w , the change of velocity, is small compared with $(v^2 - u^2)^{\frac{1}{2}}$, where $2\mathbf{u}$ is the resultant of \mathbf{u}_1 and \mathbf{u}_2 , we can obtain approximate expressions for W and for \mathbf{P} which may be useful for many purposes.

The energy radiated when w is small is easily deduced from the expression given in § 12. Since, when y/x is less than 1,

$$\frac{x}{y} \log \frac{x+y}{x-y} - 2 = 2 \left(\frac{y^2}{3x^2} + \frac{y^4}{5x^4} + \dots \right),$$

it follows that, when m or w/v is small,

$$W = U_0 \frac{2m^2(1 - n^2 \sin^2 \psi)}{3(1 - n^2 + \frac{1}{4}m^2)^2}.$$

But $U_0 = Q^2/2Ka = \mu v^2 Q^2/2a$, and thus as far as terms in w^2 ,

$$W = \frac{2U_0 w^2(1 - n^2 \sin^2 \psi)}{3v^2(1 - n^2)^2} = \frac{\mu Q^2 w^2(1 - n^2 \sin^2 \psi)}{3a(1 - n^2)^2},$$

a value identical with that obtained when w is treated as small throughout*. Thus the effect of a given change of velocity depends upon the initial velocity. If W_0 be the energy radiated when the velocity is changed from 0 to w , we have†,

$$W_0 = \frac{\mu Q^2 w^2}{3a};$$

and hence

$$\frac{W}{W_0} = \frac{1 - n^2 \sin^2 \psi}{(1 - n^2)^2}.$$

The following table shows the value of W/W_0 for a few values of n and of ψ .

It will be seen from the table that the angle between the initial velocity and the change of velocity has only a small effect when the initial velocity is small. For initial velocities less than $\frac{1}{5}v$ the energy radiated does not differ by as much as 10 per cent. from the energy radiated when the initial

* Phil. Mag. Jan. 1907, p. 146.

† Phil. Mag. Jan. 1907, p. 131 (or § 13 above).

velocity is zero, the small change of velocity being the same in either case.

TABLE IV.—Values of $\frac{W}{W_0}$.

n .	$\psi=0$. $\psi=180^\circ$.	$\psi=30^\circ$. $\psi=150^\circ$.	$\psi=60^\circ$. $\psi=120^\circ$.	$\psi=90^\circ$.
0.1	1.020	1.018	1.013	1.010
0.2	1.085	1.074	1.053	1.042
0.3	1.208	1.180	1.126	1.099
0.4	1.417	1.361	1.247	1.190
0.5	1.778	1.667	1.444	1.333
0.6	2.441	2.222	1.782	1.562
0.7	3.845	3.374	2.432	1.961
0.8	7.716	6.481	4.012	2.778
0.85	12.986	10.640	5.949	3.604
0.9	27.701	22.091	10.873	5.263

§ 17. In § 32 of my previous paper I obtained approximate expressions for the components of the radiated momentum, in directions parallel and perpendicular to \mathbf{u} , by treating \mathbf{w} as small from the beginning. It will be convenient to express these results in terms of $m_0 = 2\mu Q^2/3a$, the electromagnetic mass of the sphere for infinitesimal speeds.

If P_u^* be the component in the direction of \mathbf{u} , then

$$\begin{aligned}
 P_u &= \frac{m_0 w^2}{v} \cdot \frac{3}{8n^4} \left[\log \frac{1+n}{1-n} - \frac{2n(3-5n^2)}{3(1-n^2)^2} \right. \\
 &\quad \left. - \sin^2 \psi \left\{ \frac{3+n^2}{2} \log \frac{1+n}{1-n} - \frac{n(9-12n^2-n^4)}{3(1-n^2)^2} \right\} \right] \\
 &= \frac{m_0 w^2}{v} (A - B \sin^2 \psi),
 \end{aligned}$$

where A and B are functions of n .

For numerical calculations it is convenient to express P_u in the form of a series, which we may use when n is small. Thus

$$\begin{aligned}
 P_u &= \frac{m_0 w^2}{v} \left[\left\{ \frac{1.2}{5} n + \frac{2.3}{7} n^3 + \frac{3.4}{9} n^5 + \dots \right\} \right. \\
 &\quad \left. - \frac{1}{2} \sin^2 \psi \left\{ \frac{3}{3.5} n + \frac{34}{5.7} n^3 + \frac{117}{7.9} n^5 + \frac{276}{9.11} n^7 + \dots \right\} \right].
 \end{aligned}$$

* In the previous paper (Phil. Mag. Jan. 1907) the momentum in the direction of \mathbf{u} was denoted by P_1 and the momentum perpendicular to \mathbf{u} by P_2 .

If P_t be the transverse component of the momentum, *i. e.* the component perpendicular to \mathbf{u} in the direction making an acute angle with \mathbf{w} ,

$$P_t = \frac{m_0 w^3}{16v} \cdot \frac{3 \sin 2\psi}{16n^4} \left[\frac{2n(3-2n^2)}{3(1-n^2)} - \log \frac{1+n}{1-n} \right]$$

$$= \frac{m_0 w^2}{v} C \sin 2\psi,$$

where C is a function of n .

Expressing the result in a series, we have

$$P_t = \frac{m_0 w^2}{v} \cdot \frac{\sin 2\psi}{4} \left[\frac{n}{5} + \frac{2n^3}{7} + \frac{3n^5}{9} + \frac{4n^7}{11} + \dots \right].$$

I have verified that the values of P_1 and P_2 given in § 14 lead to the above formula for P_u by expanding all the quantities in (43) in powers of w as far as w^2 , the process involving a good deal of labour. The verification of the formula for P_t requires the expansions to be carried as far as w^3 , and this led to expressions which were so complicated that I abandoned the work.

Tables V. and VI., which have been prepared for me by my friend and former colleague R. S. Cole, show the values of $vP_u/m_0 w^2$ and of $vP_t/m_0 w^2$ for a few values of n and of ψ .

TABLE V.—Values of $\frac{vP_u}{m_0 w^2} = A - B \sin^2 \psi$.

n .	A.	B.	$vP_u/m_0 w^2$.				
			$\psi=0^\circ$. $\psi=180^\circ$.	$\psi=30^\circ$. $\psi=150^\circ$.	$\psi=45^\circ$. $\psi=135^\circ$.	$\psi=60^\circ$. $\psi=120^\circ$.	$\psi=90^\circ$.
0.1	0.0409	0.0105	0.0409	0.0382	0.0356	0.0330	0.0304
0.2	0.0873	0.0242	0.0873	0.0813	0.0752	0.0692	0.0631
0.3	0.1468	0.0457	0.1468	0.1354	0.1240	0.1125	0.1011
0.4	0.2322	0.0835	0.2322	0.2114	0.1905	0.1696	0.1487
0.5	0.3695	0.1559	0.3695	0.3305	0.2915	0.2525	0.2135
0.6	0.6204	0.3099	0.6204	0.5430	0.4655	0.3880	0.3105
0.7	1.1680	0.6924	1.1680	0.9948	0.8217	0.6486	0.4755
0.8	2.7651	1.9461	2.7651	2.2786	1.7921	1.3056	0.8190
0.85	5.0427	3.8667	5.0427	4.0760	3.1094	2.1427	1.1760
0.9	11.6575	9.7421	11.6575	9.2220	6.7865	4.3510	1.9154

TABLE VI.—Values of $\frac{vP_t}{m_0w^2} = C \sin 2\psi$.

n .	C.	vP_t/m_0w^2 .			
		$\psi = 0^\circ$. $\psi = 90^\circ$.	$\psi = 15^\circ$. $\psi = 75^\circ$.	$\psi = 30^\circ$. $\psi = 60^\circ$.	$\psi = 45^\circ$.
0.1	0.0051	0	0.0025	0.0044	0.0051
0.2	0.0106	0	0.0053	0.0092	0.0106
0.3	0.0172	0	0.0086	0.0149	0.0172
0.4	0.0256	0	0.0128	0.0222	0.0256
0.5	0.0375	0	0.0187	0.0325	0.0375
0.6	0.0560	0	0.0280	0.0485	0.0560
0.7	0.0888	0	0.0444	0.0769	0.0888
0.8	0.1606	0	0.0803	0.1391	0.1606
0.85	0.2382	0	0.1191	0.2063	0.2382
0.9	0.4039	0	0.2020	0.3498	0.4039

§ 18. I shall now consider the energy and the momentum which are radiated when the impulsive change of velocity affects the direction but not the magnitude of the velocity. If the direction of the velocity is changed by the angle α , we have

$$u_2 = u_1 = n_1 v,$$

$$w^2 = 2u_1^2(1 - \cos \alpha) = 4n_1^2 v^2 \sin^2 \frac{1}{2}\alpha,$$

$$(v^2 w^2 - u_1^2 u_2^2 \sin^2 \alpha)^{\frac{1}{2}} = 2n_1 v^2 \sin \frac{1}{2}\alpha (1 - n_1^2 \cos^2 \frac{1}{2}\alpha)^{\frac{1}{2}}.$$

Hence, by § 11, we have

$$\frac{W}{U_0} = \frac{1}{Z} \log \frac{1+Z}{1-Z} - 2,$$

where

$$Z = \frac{2n_1 \sin \frac{1}{2}\alpha (1 - n_1^2 \cos^2 \frac{1}{2}\alpha)^{\frac{1}{2}}}{1 - n_1^2 \cos \alpha}.$$

When n_1 is small, the numerical calculation of W/U_0 is best made by the series

$$\frac{W}{U_0} = 2 \left\{ \frac{Z^2}{3} + \frac{Z^4}{5} + \frac{Z^6}{7} + \dots \right\},$$

which, by § 11, is always valid when n_1 is less than unity.

If the denominator of each term of the last series be expanded in powers of n_1^2 , we obtain a *double* series in n_1^2 . This can be rearranged in the form of a *single* power series in n_1^2 .

provided that the double series remains convergent when the modulus of every term is substituted for the term itself. This will be the case if

$$\frac{4n_1^2 \sin^2 \frac{1}{2}\alpha (1 + n_1^2 \cos^2 \frac{1}{2}\alpha)}{(1 - n_1^2 |\cos \alpha|)^2} < 1,$$

where $|\cos \alpha|$ is the modulus of $\cos \alpha$.

The quantity on the left increases with n_1^2 , and hence the inequality will be satisfied if n_1 be less than the least real value obtained from the equation

$$4n_1^2 \sin^2 \frac{1}{2}\alpha (1 + n_1^2 \cos^2 \frac{1}{2}\alpha) = (1 - n_1^2 |\cos \alpha|)^2.$$

When α is not greater than $\frac{1}{2}\pi$ this equation leads to $(\sqrt{2}-1)^{\frac{1}{2}}$, or 0.643... as the value of n . When α is greater than $\frac{1}{2}\pi$, the value is $\sqrt{2}-1$, or 0.414.... Hence, if $n < \sqrt{2}-1$ the double series can certainly be rearranged as a single series for all values of α .

On rearranging this series, we obtain

$$\begin{aligned} \frac{W}{U_0} = & 8n_1^2 S^2 \left\{ \frac{1}{3} + (5 - 3S^2) \frac{n_1^2}{3 \cdot 5} + (35 - 42S^2 + 16S^4) \frac{n_1^4}{3 \cdot 5 \cdot 7} \right. \\ & \left. + (105 - 189S^2 + 144S^4 - 40S^6) \frac{n_1^6}{5 \cdot 7 \cdot 9} + \dots \right\}, \quad (45) \end{aligned}$$

where $S = \sin \frac{1}{2}\alpha$.

Mr. R. S. Cole has prepared Table VII., giving the values of W/U_0 for some values of n_1 and of α .

TABLE VII.—Values of $\frac{W}{U_0}$.

n_1 .	$\alpha=0^\circ$.	$\alpha=30^\circ$.	$\alpha=60^\circ$.	$\alpha=90^\circ$.	$\alpha=120^\circ$.	$\alpha=150^\circ$.	$\alpha=180^\circ$.
0.1	0	0.0018	0.0067	0.0134	0.0201	0.0250	0.0268
0.2	0	0.0074	0.0276	0.0547	0.0818	0.1013	0.1084
0.3	0	0.0177	0.0650	0.1281	0.1895	0.2333	0.2492
0.4	0	0.0338	0.1235	0.2404	0.3515	0.4293	0.4572
0.5	0	0.0588	0.2118	0.4050	0.5325	0.7038	0.7465
0.6	0	0.0983	0.3464	0.6454	0.9082	1.0820	1.1423
0.7	0	0.1653	0.5623	1.0087	1.3779	1.6125	1.6922
0.8	0	0.2968	0.9475	1.6030	2.1021	2.4040	2.5043
0.85	0	0.4221	1.2773	2.0704	2.6433	2.9804	3.0911
0.9	0	0.6540	1.8161	2.7765	3.4285	3.8008	3.9216

The radiated momentum is, by symmetry, along the direction bisecting the angle between the initial and final velocities. In this case the two components P_1 and P_2 , which are parallel to u_1 and u_2 , have equal values, and thus

$$P = 2P_1 \cos \frac{1}{2}\alpha.$$

Hence, since $M_2 = M_1$ and $W_2 = W_1$, we find by § 14 that,

$$\begin{aligned} P &= 2 \cos \frac{1}{2}\alpha \frac{M_1(\cos \alpha - 1) + W/u_1}{1 + \cos \alpha} \\ &= \left(\frac{W}{u_1} - 2M_1 \sin^2 \frac{1}{2}\alpha \right) \sec \frac{1}{2}\alpha. \end{aligned}$$

Hence

$$\frac{Pv}{U_0} = \left[\frac{W}{n_1 U_0} - \frac{2M_1 v}{U_0} \sin^2 \frac{1}{2}\alpha \right] \sec \frac{1}{2}\alpha. \quad \dots (46)$$

If we wish, we can express P in terms of $m_0 v$ by means of the formula $U_0/v = 3m_0/4$, where m_0 is the electromagnetic mass for infinitesimal speeds.

If we make use of the series (39) for M_1 and combine it with the series (45) for W/U_0 , we obtain the series

$$\begin{aligned} \frac{Pv}{U_0} &= 8n_1^2 S^2 \sqrt{1-S^2} \left\{ \frac{1}{5} + (26-16S^2) \frac{n_1^2}{3 \cdot 5 \cdot 7} \right. \\ &\quad \left. + (85-104S^2+40S^4) \frac{n_1^4}{5 \cdot 7 \cdot 9} + \dots \right\}, \end{aligned}$$

where, as before, $S = \sin \frac{1}{2}\alpha$. This series is certainly valid when $n_1 < 0.414 \dots$.

In Table VIII., which was prepared by Mr. A. J. Bamford of Emmanuel College, the values of Pv/U_0 for $n_1 = 0.1, n = 0.2$, and $n_1 = 0.3$ were calculated by aid of this series. For the remaining values of n_1 , Pv/U_0 was calculated by (46) from the values of W/U_0 given in Table VII. and those of $M_1 v/U_0$ given in Table II.

When $\alpha = \pi$, so that the motion is just reversed, the value of P given by (46) becomes indeterminate since then $\cos \frac{1}{2}\alpha = 0$ and $W = 2u_1 M_1$, as appears from (13) and (39). But in this case symmetry demands that P should vanish.

TABLE VIII.—Values of $\frac{Pv}{U_0}$.

n_1 .	$\alpha=0^\circ$.	$\alpha=30^\circ$.	$\alpha=60^\circ$.	$\alpha=90^\circ$.	$\alpha=120^\circ$.	$\alpha=150^\circ$.	$\alpha=180^\circ$.
0.1	0	0.00010	0.00035	0.00057	0.00060	0.00039	0
0.2	0	0.00087	0.00289	0.00468	0.00493	0.00316	0
0.3	0	0.00312	0.01031	0.01654	0.01723	0.01095	0
0.4	0	0.0082	0.0266	0.0417	0.0430	0.0264	0
0.5	0	0.0182	0.0581	0.0898	0.0905	0.0564	0
0.6	0	0.0376	0.1171	0.1751	0.1716	0.1046	0
0.7	0	0.0768	0.2297	0.3285	0.3108	0.1859	0
0.8	0	0.1670	0.4640	0.6202	0.5596	0.3257	0
0.85	0	0.2619	0.6854	0.8733	0.7646	0.4381	0
0.9	0	0.4501	1.0722	1.2817	1.0828	0.6091	0

In conclusion I desire to thank those friends who have assisted me in the preparation of this paper. But for the help of Mr. G. T. Bennett I should have given up the problem as unmanageable. To Mr. R. S. Cole and to my colleague Mr. T. G. Bedford, and to Mr. A. J. Bamford I am indebted for much help in the heavy arithmetical work involved in the construction and verification of the numerical tables. Mr. Bamford has also verified the mathematical work in the paper.

IV. *A Modified Theory of Gravitation.*

By C. V. BURTON, *D.Sc.**

1. SO many questions arise in connexion with any theory which aims at assigning a dynamical basis for gravitation, that it may conduce to clearness if some elements of this paper are first presented without mathematical treatment.

The theory now put forward is pulsatory, and may in fact be regarded as a development of that which has been suggested by Prof. Hicks †. The æther, even where modified

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† W. M. Hicks, "On the Problem of two Pulsating Spheres in a Fluid," *Proc. Camb. Phil. Soc.* iii. p. 277 (Oct. 1879).

by the presence of atomic matter, is assumed to be so nearly incompressible that the bulk-modulus of elasticity enormously transcends every other elastic modulus ; so that, in dealing with any compressional-rarefactional disturbances, we may treat the medium as a fluid.

In two particulars, however, there is an essential difference between the views now tentatively put forward and the more familiar form of pulsatory hypothesis. These differences are indicated in the following five paragraphs.

2. It is a well-known hydrodynamical result * that if two spheres, placed at some distance apart in a frictionless incompressible fluid, are by any means caused to execute periodic pulsations (increase and decrease of volume), the pulsation-period being the same for the two spheres, the average force which either sphere causes to be exerted upon the other will be an attraction, a repulsion, or zero, according as the phase-difference of the pulsations is less than, greater than, or equal to a quarter-period. Hence, if we are to construct on these lines a model illustrative of gravitation, we must assume agreement of phase amongst all the pulsating centres, or at least we must suppose that the phases of the various centres are more nearly in agreement than would be the case if the phase-distribution were purely random. This seems to suggest that the source of pulsatory motion is not to be sought for in the free vibrations of individual atoms or electrons, but rather in something external to these, and acting on them all in common.

Let us suppose, for example, that free æther is very nearly incompressible, and that every region where atomic matter exists is a region of somewhat enhanced compressibility. On this supposition, an increase of pressure applied by any ideal means to all the æther within a given volume would cause only a very minute contraction except where atomic matter was present ; the expansion resulting from a diminution of the pressure applied to the æther being similarly localised.

3. It appears that we might thus account for attractions of gravitational type by supposing the ætherial pressure to be undergoing a *secular* change. A gradually increasing (or decreasing) pressure would cause every region containing atomic matter to behave as a sink (or as a source) in the medium. For our purpose, however, *accelerated* inflow (or outflow) of æther at each sink (or source) would be necessary ; that is to say, the strength of each sink (or

* Hicks, *loc. cit.* Cf. also Basset's "Hydrodynamics," vol. i. chap. xi., where further references are given.

source) must be steadily increasing or steadily decreasing with the time—no matter which. The suggested possibility of accounting for gravitation by means of a slow secular change taking place throughout the universe appeared at first sight very alluring ; but calculation shows at once that, feeble as gravitational forces are, they are incomparably greater than could be accounted for in this way.

4. What, then, would be the result of periodic or quasi-periodic fluctuations of the ætherial pressure? Every portion of atomic matter would constitute a centre of pulsatory motion in the æther ; and throughout any region not too immense the pulsations would agree sensibly in phase. It is by assuming the existence of a fluctuating ætherial pressure that I have here sought to obtain a consistent illustration of gravitational attraction ; and, as furnishing most readily an assignable cause for such fluctuations of pressure, the propagation of compressional-rarefactional waves through the nearly incompressible æther is suggested. These waves are supposed to travel with so great a velocity that, even though all effective periods involved be very small, the effective wave-lengths, measured even by astronomical standards, are very great. Though the origin of the assumed wave-trains were unexplained, these might still be regarded as contributing to the explanation of gravity ; for when once the constitution of a dynamical system has been defined, the mere supposition that the system is in motion is hardly to be viewed as a piling-up of hypotheses. And if we regard the question broadly, extending our consideration to that wider range of space and motion wherein the whole of our explorable universe must needs be treated as an infinitesimal volume-element, sensibly homogeneous as regards its ætherial content, we may admit our complete ignorance of this greater universe, and of the forms of disturbance which might emanate from it. One might perhaps conceive of compressional waves as proceeding from a more primitive and chaotic condition of the “primal æther,” from which an æther such as we know, with electromagnetic qualities, may be in process of formation. As regards the sensible uniformity in time which must characterize the primary disturbance, if so unchanging a phenomenon as gravitation is thus to be accounted for, we may think of it as a quality naturally to be looked for in any activities whose scale is sufficiently vast ; just as the energy of the radiation passing into space from the sun varies but little in the course of an hour, or as the turbulence of an ocean presents much the same aspect from minute to minute.

In any case it is instructive to trace out some of the consequences which might be expected to follow from the propagation of compressional waves through a very slightly compressible æther; the more so as we are led to certain conclusions which, although essentially based on the ordinary postulates of dynamics, are at variance with some traditional or instinctive views.

5. The second innovation referred to concerns the nature and mode of progression of the centres of enhanced compressibility, which are assumed to be associated with atomic matter. If the nucleus of an electron is taken to be a vacuous cavity, the question of free mobility of this nucleus through the æther presents some difficulty, even when we suppose that the electron itself is essentially constituted by a self-equilibrating distribution of strain in the æther surrounding the nucleus. But now, discarding for the moment a too minute scrutiny of ætherial constitution, so that we may regard the æther as a continuous medium, let us suppose that the nucleus of an electron *, instead of being vacuous, is merely a region of somewhat diminished density. (Though increased compressibility may not be inevitably associated with diminished density, it is convenient to assume such a relation for preliminary descriptive purposes. In the sequel this question is treated more generally.) This implies that, in the complete strain-distribution which constitutes the electron, there is included some degree of expansion in the nuclear region. An electron thus constituted may more readily be conceived as freely mobile through the æther, no vacuous cavity being present to complicate the problem; so that any infinitesimal displacement of the electron is simply equivalent to the impressing of a differential strain upon the medium.

6. This assumption as to the nature of an electronic nucleus is admittedly gratuitous, but apart from the difficulty regarding mobility which it was designed to remove, it has the advantage of greatly simplifying the dynamics of the problem proposed. So far as we are concerned with the distribution of resultant ætherial motion from point to point, we may treat each moving electron as a doublet comprising a source and a neighbouring sink, while it is shown that the resultant of the hydrodynamical (gravitational) forces †

* The necessity for distinguishing, in this connexion, between positive and negative electrons is considered in Appendices A and B.

† That is to say of those forces which are concerned in palpable ætherial motions; our scrutiny being minute enough to take account of variations of ætherial qualities through the region occupied by an electron, but not so minute that the æther ceases to appear to us as a continuous medium.

tending to accelerate the motion of the electron simply follows the gradient of diminishing pressure and is proportional thereto. It further appears that the assumption referred to might enable the elementary requirements of the gravitation problem to be satisfied with the contribution of only a very minute non-electromagnetic term to the total effective inertia of an electron (Appendix D).^c

7. Let F be a physical magnitude characteristic of all electrically neutral (atomic) matter, and defined as follows: if m is the mass of atomic matter included in any given region, then $Fm\rho$ is the defect of mass of ætherial substance contained within that region, as compared with the mass included in an equal region filled with free æther; the density of the æther being denoted by ρ . (As we shall suppose the average density of ætherial substance to be but little different in a vacuum and in the densest atomic matter, no more precise definition of ρ is needed in this connexion.) Otherwise thus: if, within a region bounded by a fixed ideal surface and originally free from atomic matter, a mass m of matter could be created, a volume Fm of æther would flow outward across the bounding surface. F may be called the extrusion of æther per unit mass of matter.

8. Consider further the application of a small additional normal pressure δp to a surface bounding some definite portion of the ætherial medium, which is modified by the presence of a mass m of atomic matter. A reduction of volume will result, and we shall define another constant H by the statement that the reduction of volume is *greater* by $Hm\delta p$ than if the original volume had been wholly occupied by free æther. Thus evidently

$$H = -dF/dp. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

9. Let a particle of atomic matter be present where from any cause there is a pressure-gradient in the æther. When the particle is displaced from one point to a second neighbouring point, there is a flow of æther inward at the first point, and a flow outward at the second point. If the ætherial normal pressure is greater at the first point than at the second, work on the whole will be done by the pressure during the displacement.

Consider a particle m of matter at a point (x, y, z) where the pressure is p , and let the particle be displaced to $(x + \delta x, y, z)$ where the pressure is $p + \partial p / \partial x \cdot \delta x$. Also let the particle be acted upon by such ideal force as may be necessary to ensure that the displacement takes place very slowly. There is work done by the pressure p at (x, y, z)

corresponding to a reduction of volume Fm ; this work being pFm . There is also work done against the pressure $p + \delta p$ at $(x + \delta x, y, z)$ corresponding to an increment of volume $(F + \delta F)m$; this work is, to a first order,

$$(pF + p\delta F + F\delta p)m,$$

where $\delta p = \partial p / \partial x \cdot \delta x$,

and $\delta F = -H\delta p = -H\partial p / \partial x \cdot \delta x$.

On the whole, then, the work done by the normal pressure of the medium in the virtual displacement δx is

$$\left(H \frac{\partial p}{\partial x} \cdot p - F \frac{\partial p}{\partial x} \right) m \delta x.$$

Of this the first term represents the work done in compression, the potential energy of the system being increased to that extent, while the remaining term is the work done against the ideal force which was applied to the particle to prevent its moving with appreciable acceleration. Thus if a particle of mass m is at rest in the æther (or, as below, in motion) at a point where an ætherial pressure-gradient exists, there will be a force

$$-mF \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) \cdot \cdot \cdot \cdot \cdot \quad (2)$$

exerted on the particle. If the particle were allowed to move with a finite acceleration, the associated movements of expansion and contraction would be correspondingly accelerated, and work would thus be done in imparting this form of kinetic energy to the æther. This only implies, however, the existence of a corresponding term * in the total effective inertia of the particle, the remaining terms being probably of electromagnetic origin. Since here, as in the usual acceptance of the term, the mass m signifies the *total* inertia of the particle in question, we may remove the restriction as to the particle being at rest, (2) being in any case the force-components acting on m . It should be particularly noticed that the virtual displacement from which (2) was deduced was a displacement of m with respect to the æther, and that consequently

$$-mF \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) \text{ is the force tending to} \\ \text{accelerate the particle } m \text{ with respect to the æther} \quad \cdot \quad (3)$$

* The possible magnitude of this term is considered in Appendix D.

It will be necessary later to examine the forces which material bodies might be expected to exert on one another owing to their motion through the æther.

10. Properly speaking, there are two distinct ways in which matter may be conceived of as moving with respect to the æther. For simplicity, consider only a single electron, and let us agree to give the name "nucleus" to a certain definite central region of the electron. At any instant the nucleus comprises a certain identical portion of ætherial substance, and if this portion were bodily displaced, with respect to the surrounding æther (which thereby became strained), it might be said that the electron had suffered a displacement of like amount. But such displacement, even through distances far less than the diameter of the electronic nucleus, would presumably be met by an opposing stress of enormous magnitude, arising from the deformation of the circumnuclear æther; while the alternative type of electronic motion, involving only transference of strain from one part of the æther to another, can take place freely without evoking any opposing stress. (The forces mutually exerted by electrons and assemblages of electrons not coming here into consideration.) We shall suppose that when, from any cause, matter experiences a tendency to move, or to change its motion with respect to the æther at large, the resulting motion may, with sufficient accuracy, be treated as exclusively of the strain-transference type assumed in the last paragraph. It should be remarked that, whereas vortex motion would be involved in a bodily displacement of that identical portion of ætherial substance which is instantaneously co-extensive with the nucleus of an electron, the type of motion dealt with in § 9 above is irrotational. The latter motion may be supposed to be entirely distinct from that which, arising from the charge of the moving electron, constitutes a surrounding magnetic field. Such a supposition is justifiable if, as I imagine, a magnetic field involves no bodily motion of the æther.

11. Let us now conceive the æther to be traversed by trains of compressional-rarefactional waves, and consider first how the motion of matter will be *directly* affected thereby. As already stated, the æther is regarded as so slightly compressible that, where compressional waves are concerned, any stresses which may be involved, over and above the variations of normal pressure, are relatively insignificant: the æther, in regard to such waves, behaving like a compressible fluid, whose motion is sensibly irrotational. To fix ideas, let the wave-motion be limited to a single

harmonic train of plane-waves, propagated with velocity \mathcal{V} in the direction of x -increasing. It will be convenient to define the waves in terms of the variations of pressure taking place in the æther ; thus

$$p = \bar{p} + B \sin \frac{2\pi}{\lambda}(x - \mathcal{V}t + \epsilon); \quad . \quad . \quad . \quad (4)$$

so that B is the semi-amplitude of pressure-variation, \mathcal{V} the velocity of the disturbance, λ the wave-length, and λ/\mathcal{V} the period ; while ϵ is a linear magnitude serving to fix the otherwise undefined question of phase.

12. As a particular case, suppose that F , the extrusion of æther per unit-mass of atomic matter (§ 7), is zero, any given volume of space thus containing precisely the same amount of ætherial substance, whether atomic matter is present in that space or not. It is then evident from general considerations that the æther affected by the wave-motion, in surging to and fro, will merely carry with it whatever atomic matter may be present ; no tendency arising for those strain-distributions which constitute matter to lag behind or to outrun the bodily excursions of the ætherial plenum. The same conclusion is reached when we put $F = 0$ in (3) ; there are then no forces belonging to the class there concerned, or, in other words, the wave-motion has no tendency to accelerate the motion of atomic matter with respect to the æther ; so that every portion of atomic matter will continue in its state of rest, or of uniform or accelerated motion *with respect to the æther*, precisely as if no wave-motion were taking place. Under this particular assumption that $F = 0$, the wave-motion (at least in the case of sufficiently great wave-lengths) would appear to be without influence on any of the phenomena which come within the range of our observation, and to belong to that class of activities for which the name *apphenomenal* has been suggested *. Apart from any question as to the correspondence between our special assumptions and the conditions obtaining in the physical universe, it is instructive to realize that the picture here presented—of vibratory motions affecting all matter without influencing the phenomena of ordinary dynamics—is in no way incompatible with the fundamental concepts of dynamical science.

13. If, instead of $F=0$, as in § 12, we suppose F to be finite, so that the mean density of ætherial substance in a region containing atomic matter is different from the density in free æther, it is evident that a general bodily acceleration

* But see also Appendix B.

of the æther will tend to produce an acceleration of atomic matter with respect to the æther. From (3) and (4) we find for the acceleration of any free mass m of neutral matter with respect to the æther

$$a = \left\{ -FB \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (x - \mathfrak{V}t + \epsilon), 0, 0 \right\}; \quad (5)$$

while the absolute acceleration A of the æther itself is given by

$$A = -\frac{1}{\rho} \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) = \frac{1}{\rho} \left\{ -B \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (x - \mathfrak{V}t + \epsilon), 0, 0 \right\}. \quad (6)$$

In this case then, and (as we easily see) generally for small compressional disturbances of the æther, the ratio

$$\frac{\text{acceleration of free matter with respect to the æther}}{\text{acceleration of the æther}} = F\rho \quad (7)$$

14. There is another aspect of the hypothetical wave-motion which deserves some attention. At any point fixed with respect to the æther, changes of pressure are supposed to occur, accompanied by changes of density, and as we consider the compressibility of the medium in bulk to be very slight, the proportional changes of pressure and of constitutive (potential) energy will be great compared with the proportional changes of density. From this it may be surmised that corresponding changes in the velocity of radiation would be involved, and if the amplitude of these changes were not a quite insignificant fraction of the whole velocity, effects might result which would be palpable to observation. On the other hand, all effective wave-lengths are assumed to be so immense that, throughout even an astronomically considerable region, the ætherial pressure, and therewith the velocity of radiation, may be treated as uniform at each instant. It is perhaps conceivable that the effect of increased ætherial pressure would be to accelerate not only radiation but *all* phenomena in the same proportion; so that, for example, two universes identically constituted otherwise, but differing from one another as to the pressure prevailing throughout the æther and the rapidity of phenomena in general, might present identical aspects to their respective inhabitants. But even so, in the question whether the pressure of the æther throughout our universe could be gradually or suddenly changed without giving rise to

observable effects, some nice points of dynamics would seem to be involved*.

15. The primary effects of the hypothetical compressional waves having been for the moment disposed of, the secondary or gravitational effects may be considered. Every portion of atomic matter, in so far as its presence implies altered compressibility of the ætherial plenum ($H \neq 0$), behaves as a centre of pulsatory motion, or as an aggregation of such centres. Thus the motion of the æther, to a second approximation, is to be found by superposing on the wave-motion proper a pulsatory motion due to a distribution of alternating source-sink centres, to which should be added (except, as will appear, when $F=0$) a distribution of source-sink doublets arising from the *motion* of atomic matter through the æther. It will be assumed that the whole region of space under consideration is so circumscribed that its greatest dimension is very small compared with the shortest effective wavelength of the compressional disturbance, so that, at any given instant, the primary pressure-variation is sensibly in the same phase throughout, while, to the same order of approximation, the source-sink centres just referred to may be treated as if they existed in an incompressible medium. In accordance with the assumptions explained in §§ 1, 11 above, the variations of normal (hydrostatic) pressure are taken to outweigh in importance all other stresses evoked by varying compression of the medium, and the whole motion to be investigated is accordingly irrotational.

16. Let σ be the density of atomic matter at any point (x, y, z); then the total compressibility of matter-encumbered æther contained in the volume-element $dx dy dz$ exceeds the total compressibility of a like volume of free æther by

$$H\sigma dx dy dz, \dots \dots \dots (8)$$

in accordance with our definition of H (§ 8). Further, so far as the primary disturbance is concerned, let the pressure p be expressed by

$$p = \bar{p} + \mathbf{p}; \dots \dots \dots (9)$$

the presence of atomic matter in the volume $dx dy dz$ is thus equivalent to a *source* of strength

$$-H \frac{\partial \mathbf{p}}{\partial t} \sigma dx dy dz; \dots \dots \dots (10)$$

* Among the values which, in § 33 below, are tentatively assigned to the various physical magnitudes involved, the variations of potential energy of the æther are set down as so small in comparison with the whole constitutive energy that the special difficulty above referred to seems to be evaded.

which expression must therefore be equated to $\nabla^2 \phi \, dx \, dy \, dz$, ϕ being the velocity-potential of the secondary motion now considered, while $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

With due regard to boundary conditions, this leads to

$$\phi = \frac{H}{4\pi} \frac{\partial \mathbf{p}}{\partial t} \iiint \frac{\sigma'}{r} \, dx' \, dy' \, dz'; \quad . \quad . \quad . \quad (11)$$

σ' being the density of atomic matter at any point (x', y', z') , while $r^2 \equiv (x-x')^2 + (y-y')^2 + (z-z')^2$. The integration must be conducted through a sufficiently extended region to include all pulsatory centres which contribute appreciably to the value of ϕ at (x, y, z) . In accordance with the assumption mentioned in § 15 above, the factor $\partial \mathbf{p}/\partial t$ is placed outside the sign of integration; while sources and sinks other than those corresponding to the expression (10) are assumed to be absent from the secondary motion.

17. Treating the primary wave-motion more generally than hitherto, (4) may be taken as expressing a single typical constituent; no restriction being imposed on the direction of propagation of the waves, or on their periods or phases, provided that all effective wave-lengths are supposed sufficiently great, and all limited sources of the primary disturbance sufficiently remote. The form of our equations of course involves the implicit assumption that, in the primary wave-motion, all relations are linear. It is now similarly assumed that, in dealing with the secondary disturbance expressed by (11), non-linear terms may be omitted from the equations, the principle of superposition being accordingly applicable to the primary and secondary motions. Thus, to our order of approximation, we may write

$$p = \bar{p} + \mathbf{p} + \varpi \quad . \quad . \quad . \quad (12)$$

and

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial t} &= - \int \frac{d\rho}{\rho} + \text{func. } (t) \\ &= - \int \frac{d(\mathbf{p} + \varpi)}{\rho} + \text{func. } (t); \quad . \quad . \quad (13) \end{aligned}$$

where ϕ , ϖ are respectively the velocity-potential and pressure-increment corresponding to the secondary motion, ψ , \mathbf{p} being the velocity-potential and pressure-increment corresponding to the primary disturbance. Hence

$$\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial t} \right) = - \frac{1}{\rho} \frac{\partial \rho}{\partial x} = - \frac{1}{\rho} \left(\frac{\partial \mathbf{p}}{\partial x} + \frac{\partial \varpi}{\partial x} \right); \quad . \quad (14)$$

and subtracting from this the equation proper to the primary motion, there remains

$$\frac{\partial^2 \phi}{\partial t \cdot \partial x} = -\frac{1}{\rho} \frac{\partial \varpi}{\partial x}; \quad . \quad . \quad . \quad (15)$$

which with (11) gives

$$\begin{aligned} \frac{\partial \varpi}{\partial x} &= -\frac{H\rho}{4\pi} \cdot \frac{\partial^2 \mathbf{p}}{\partial t^2} \iiint \sigma' \frac{\partial}{\partial x} \left(\frac{1}{r} \right) dx' dy' dz' \\ &= \frac{H\rho}{4\pi} \frac{\partial^2 \mathbf{p}}{\partial t^2} I_x, \quad . \quad . \quad . \quad (16) \end{aligned}$$

where

$$I_x \equiv \iiint \frac{\sigma'}{r^2} \frac{x-x'}{r} dx' dy' dz'. \quad . \quad . \quad . \quad (17)$$

Hence the secondary motion gives rise to a force upon a mass m of atomic matter at (x, y, z) , the x -component of which is, by (2),

$$-mF \frac{\partial \varpi}{\partial x} = -m \frac{HF\rho}{4\pi} \frac{\partial^2 \mathbf{p}}{\partial t^2} I_x = mX \text{ (say),} \quad . \quad (18)$$

with corresponding expressions for the y - and z -components.

In (18) F may be written $\bar{F} + \delta F$, where \bar{F} is the mean value of F , and δF , to a sufficient approximation, represents a variable term, arising from and proportional to the variable pressure-term \mathbf{p} , which corresponds to the primary disturbance. Thus

$$F = \bar{F} + \delta F = \bar{F} - H\mathbf{p}, \quad . \quad . \quad . \quad (19)$$

by (1); so that (18) becomes

$$X = \frac{\rho}{4\pi} \left(-H\bar{F} \frac{\partial^2 \mathbf{p}}{\partial t^2} + H^2 \mathbf{p} \frac{\partial^2 \mathbf{p}}{\partial t^2} \right) I_x. \quad . \quad . \quad (20)$$

The mean value of X is therefore given by

$$\bar{X} = \text{av.} \left[\frac{\rho}{4\pi} H^2 \mathbf{p} \frac{\partial^2 \mathbf{p}}{\partial t^2} I_x \right]. \quad . \quad . \quad . \quad (21)$$

18. As the primary wave-motion is now supposed to involve only very great wave-lengths, so that the (primary) pressure-variation throughout the region of space considered is sensibly a function of the time only, and not of x, y, z , we may write

$$\mathbf{p} = \Sigma B_s \sin(st + \epsilon_s); \quad . \quad . \quad . \quad (22)$$

each B being of the nature of a pressure (in general infinitesimal), while the ϵ 's are phase-terms. Hence

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = -\Sigma B_s s^2 \sin(st + \epsilon_s); \quad . \quad . \quad . \quad (23)$$

and (20) becomes

$$\begin{aligned} X &= \frac{\rho}{4\pi} H \bar{F} \bar{\Sigma} B_s s^2 \sin(st + \epsilon_s) \cdot I_x \\ &- \frac{\rho}{4\pi} H^2 \bar{\Sigma} B_s \sin(st + \epsilon_s) \cdot \bar{\Sigma} B_s s^2 \sin(st + \epsilon_s) \cdot I_x. \end{aligned} \quad (24)$$

There is thus a gravitative field at (x, y, z) defined by the average values

$$\begin{aligned} (\bar{X}, \bar{Y}, \bar{Z}) &= -\frac{H^2 \rho}{8\pi} \bar{\Sigma} B_s^2 s^2 (I_x, I_y, I_z) \\ &= -\frac{H^2 \rho}{8\pi} \bar{\Sigma} B_s^2 s^2 \iiint \frac{\sigma' x - x', y - y', z - z'}{r^2} dx' dy' dz' \end{aligned} \quad (25)$$

Within the limits of our assumptions, there is seen to be universal mutual attraction of electrically neutral matter, the Newtonian constant being

$$G = \frac{H^2 \rho}{8\pi} \bar{\Sigma} B_s^2 s^2. \quad (26)$$

19. It has been pointed out above, as an assumption essential to the theory proposed, that every effective wave-length of the primary ætherial disturbance must be very great compared with the distance between any mutually attractive bodies for which the Newtonian law of inverse squares has been closely verified. It may be worth while to indicate very briefly what would result if this condition were not realized. By way of illustration, let the primary disturbance take the form of plane progressive waves of harmonic type and of definite wave-length λ . Two bodies whose line of centres was perpendicular to the direction of propagation of the waves, and which were at a distance r apart, would attract one another with a force proportional to

$$r^{-2} \cos 2\pi r/\lambda,$$

which, as r is increased, changes sign periodically at intervals of $\frac{1}{2}\lambda$. (If, in place of a single wave-length λ , there were a continuous distribution of wave-energy over a wide range of wave-lengths, there would simply be a falling off of attraction between the two bodies at a rate more rapid than that of the inverse square of the distance.)

With the primary disturbance in the form of progressive waves travelling in one direction (as above) the case of two bodies in a line not perpendicular to that direction would be more involved; but if the primary waves were travelling indifferently in all directions, there would merely be a gradual

extinction of gravitational attraction as the bodies concerned were removed to distances apart not wholly inconsiderable in comparison with the smallest effective wave-length of the primary disturbance.

It will be remarked that, when the distance between two bodies is small compared with the mean effective wave-length of the primary disturbance, the attraction between them exhibits only a *second order* deviation from the Newtonian law of gravitation.

Motional Forces.

20. The next question to be considered concerns the forces arising from the motion of neutral matter with respect to the æther. The expressions for these forces, as will appear, contain a factor F^2 , F being (as defined in § 7) the extrusion of æther per unit mass of matter: so that if the mean value \bar{F} were zero, these forces—which correspond to nothing so far observed—would disappear.

Let (x, y, z) be the co-ordinates of a particle of mass m at time t , and let (x', y', z') be the simultaneous co-ordinates of another particle of mass m' , all referred to axes fixed with respect to the general mass of æther. As in § 10, the type of ætherial motion which here concerns us, and which arises from the motion of m and m' , with respect to the æther, is sensibly irrotational.

Let us consider the forces exerted on m owing to the motion of m' , and let ϕ be the velocity-potential at (x, y, z) due to the motion of m' only, while $r^2 \equiv (x-x')^2 + (y-y')^2 + (z-z')^2$. Then, the effect of the motion of m' being representable by a source-sink doublet at (x', y', z') we readily find

$$\phi = -\frac{Fm'}{4\pi} \cdot \frac{d'}{dt} \left(\frac{1}{r} \right), \quad \dots \dots \dots (27)$$

where d'/dt operating on any quantity represents the time-flux of that quantity due to variations of x', y', z' only. Hence also,

$$\begin{aligned} \dot{\phi} &= -\frac{Fm'}{4\pi} \frac{d'^2}{dt^2} \left(\frac{1}{r} \right) \\ &= \frac{Fm'}{4\pi} \left\{ \left(\frac{1}{r^3} - \frac{3(x-x')^2}{r^5} \right) \left(\frac{dx'}{dt} \right)^2 + \dots + \dots \right. \\ &\quad - \frac{6(y-y')(z-z')}{r^5} \frac{dx'}{dt} \cdot \frac{dz'}{dt} - \dots - \dots \\ &\quad \left. - \frac{x-x'}{r^3} \frac{d^2x'}{dt^2} - \frac{y-y'}{r^3} \frac{d^2y'}{dt^2} - \frac{z-z'}{r^3} \frac{d^2z'}{dt^2} \right\}. \quad \dots \dots (28) \end{aligned}$$

21. In accordance with (3), the force components on the particle m at (x, y, z) are

$$-mF \left(\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \right) = m(X, Y, Z) \text{ say;}$$

or, to a first approximation, neglecting squares and products of ætherial velocities,

$$\begin{aligned} m(X, Y, Z) &= -mF\rho \left(\frac{\partial \dot{\phi}}{\partial x}, \frac{\partial \dot{\phi}}{\partial y}, \frac{\partial \dot{\phi}}{\partial z} \right) \\ &= -\frac{mm'F^2\rho}{4\pi} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{d'^2}{dt^2} \left(\frac{1}{r} \right); \end{aligned}$$

that is

$$\begin{aligned} mX &= -\frac{mm'F^2\rho}{4\pi} \left\{ \left(\frac{9(x-x')}{r^5} - \frac{15(x-x')^3}{r^7} \right) \left(\frac{dx'}{dt} \right)^2 \right. \\ &+ \left(\frac{3(x-x')}{r^5} - \frac{15(y-y')^2(x-x')}{r^7} \right) \left(\frac{dy'}{dt} \right)^2 + \left(\frac{3(x-x')}{r^5} - \frac{15(z-z')^2(x-x')}{r^7} \right) \left(\frac{dz'}{dt} \right)^2 \\ &+ \left(\frac{6(y-y')}{r^5} - \frac{30(x-x')^2(y-y')}{r^7} \right) \frac{dx'}{dt} \frac{dy'}{dt} + \left(\frac{6(z-z')}{r^5} - \frac{30(x-x')^2(z-z')}{r^7} \right) \frac{dx'}{dt} \frac{dz'}{dt} \\ &- \frac{30(x-x')(y-y')(z-z')}{r^7} \frac{dy'}{dt} \frac{dz'}{dt} + \left(\frac{1}{r^3} - \frac{3(x-x')^2}{r^5} \right) \frac{d^2x'}{dt^2} - \frac{3(y-y')(x-x')}{r^5} \frac{d^2y'}{dt^2} \\ &\left. - \frac{3(z-z')(x-x')}{r^5} \frac{d^2z'}{dt^2} \right\}, \end{aligned}$$

with similar expressions for mY, mZ (29)

22. From these equations it follows that, when two bodies are moving in the same direction through the æther, and with the same unaccelerated velocity, the motional forces experienced by the bodies will be equal and opposite. They will not, however, have the same line of action, except when the motion is either in the line of centres or perpendicular thereto. When the common velocity \mathbf{v} of the two bodies is *in the line of centres*, there is *repulsion* between the bodies amounting to

$$\frac{3}{2\pi} \frac{mm'F^2\rho}{r^4} \mathbf{v}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

When the common velocity \mathbf{v} is *perpendicular to the line of centres* there is *attraction* amounting to

$$\frac{3}{4\pi} \cdot \frac{mm'F^2\rho}{r^4} \mathbf{v}^2 = K \frac{mm'}{r^4} \mathbf{v}^2, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where

$$K \equiv \frac{3}{4\pi} F^2\rho. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

There is neither attraction nor repulsion when the common velocity of the two bodies is inclined at an angle $\tan^{-1}\sqrt{2}$, or approximately $54^\circ 44'$ to the line of centres. But in addition to the forces in the line of centres, there are in general forces perpendicular to this line, and if we imagine the two bodies connected by a rigid immaterial (or material) bar, the tendency will be to set the line of centres *perpendicular* to the direction of motion; the couple acting on the system only vanishing when the direction of motion is parallel or perpendicular to the line of centres. When the direction of motion is inclined 45° to the line of centres, the couple has its maximum value, namely,

$$\frac{3}{4\pi} \cdot \frac{mm'F^2\rho}{r^3} v^2 \quad \text{or} \quad K \frac{mm'}{r^3} v^2. \quad . \quad . \quad . \quad (33)$$

23. If we consider a bar made of given material and of definite proportions, and so suspended as to have a given period of oscillation about an axis through its centre of figure, then the angular deviation of the bar arising from a given velocity of translation through the æther will, at its maximum, be inversely proportional to the square of the linear dimensions of the bar. Though no experiments may have been specially made with the object of detecting motional forces of this kind, it seems certain that if more than a very slight residual effect existed, it must have become apparent in other observations. None the less it would be interesting to make some definite experiments to put the question to the test, the apparatus required being simple. Incidentally, if the couples proved to be of measurable magnitude, we could readily obtain the data necessary for a determination of our motion relatively to the æther. This would involve no contradiction of the principle of relativity in electromagnetism; the phenomena with which we are here concerned not being electromagnetic in character.

24. Though we do not know what may be the value of the constant K defined by (32), or even whether it be different from zero, we can make some attempt to assign an upper limit to its value. Imagine two compact masses of 1 gram each, connected by a bar of negligible mass, the distance between their centres being 2 centimetres. Let this system be suspended by a quartz fibre, so that its axis is horizontal, and the period of its oscillations 30 seconds. Let us further suppose that we are able to detect any deviation as great as one second of arc on either side of a mean position, when

the suspended system turns about a vertical axis. Now in virtue of the earth's orbital motion, the highest value attained by the velocity v must be at least as great as 32 kilometres per second ($= 3.2 \times 10^5$ cm. per second), so that by (33) our suspended system, when most favourably placed and oriented, could experience a couple as great as

$$\frac{1}{8}(3.2 \times 10^5)^2 K \doteq 1.28 \times 10^{12} K. \quad . \quad . \quad . \quad (34)$$

Also the restoring moment when the system is displaced through one second of arc about a vertical axis is 4.4×10^{-7} dyne-cm.; and for the experiment to give a null result, we should accordingly have

$$K \gtrsim 3.4 \times 10^{-19} \text{ in c.g.s. measure.} \quad . \quad . \quad (35)$$

Assuming, in advance, that a null result would be obtained in this case, the value which, in the sequel, will be provisionally assumed is

$$K = 2 \times 10^{-20}. \quad . \quad . \quad . \quad (36)$$

25. This tentative value of K may be compared with one suggested by another way of regarding the question. If the negative electron could be regarded as having a vacuous spherical nucleus of radius 10^{-13} cm., and if we assume the atom of hydrogen to contain only one negative electron, while the number of molecules in 1 c.c. of gas at 0° C. and 1 atmosphere pressure is taken to be 3.9×10^{19} ; then it appears that, on the score of negative electrons alone, F , the extrusion of æther per unit mass of matter (§ 7), would be 4×10^{-15} c.c. of æther per gram of matter; equivalent to .004 gram of æther per gram of matter, when an ætherial density of 10^{12} is assumed*. Though the existence of a vacuous nucleus in an electron does not appear reconcilable with some of the views adopted in this paper, it may be worth while to remark that the value of F just mentioned leads to $K = 4 \times 10^{-18}$, which is 200 times as great as the estimate (36), and about 12 times as great as might just suffice to give positive results in the hypothetical experiment of § 24.

Conjectural Estimates.

26. The values of a further number of physical magnitudes will have to be conjectured before a quantitative view of the theory can be attempted. Some of these magnitudes are

* Cf. § 35 below.

introduced in the following seven paragraphs, which are devoted to establishing certain simple relations amongst the quantities concerned; the immediate object being to discover whether there is any consistent set of values which does not bring our theory into conflict with experience.

27. From (32)

$$F = 2\sqrt{\left(\frac{\pi K}{3\rho}\right)}. \quad . \quad . \quad . \quad . \quad (37)$$

As regards the primary disturbance, the problem (as in § 11) may be simplified by limiting our consideration to a single train of plane waves of definite wave-length λ , the greatest linear dimension of the region with which we are concerned being but a small fraction of λ , so that, to a sufficient degree of approximation, the pressure variations at any point within that region are given by

$$p = B \sin(2\pi nt - \epsilon), \quad . \quad . \quad . \quad . \quad (38)$$

n denoting the frequency of the disturbance. Equation (26) now becomes

$$G = \frac{H^2\rho}{8\pi} B^2 s^2 = \frac{\pi}{2} \cdot H^2 \rho B^2 n^2, \quad . \quad . \quad . \quad (39)$$

since we are now limited to a single s , which is identical with $2\pi n$.

F , the extrusion of æther per unit mass of matter (§ 7) suffers variations given by

$$\delta F = -Hp = -HB \sin(2\pi nt - \epsilon). \quad . \quad . \quad (40)$$

Let the greatest numerical value of δF be ζF ; then by (40), (39),

$$\zeta^2 F^2 = H^2 B^2 = 2G/\pi\rho n^2; \quad . \quad . \quad . \quad (41)$$

or, remembering (37),

$$n = \frac{1}{\pi\zeta} \sqrt{\left(\frac{3G}{2K}\right)}. \quad . \quad . \quad . \quad (42)$$

28. Let κ denote the bulk-modulus of elasticity of the æther, or $1/\kappa$ its compressibility. By § 8 above, $H\sigma$ is the *additional* ætherial compressibility where there are σ grams of atomic matter per cubic centimetre; let this be equivalent to $\mu\sigma/\kappa$, so that $\mu\sigma$ is a pure number. Thus

$$H = \mu/\kappa. \quad . \quad . \quad . \quad (43)$$

In virtue of our assumption that κ is overwhelmingly greater than any other elastic modulus of the æther, we have for the square of the velocity of compressional waves

$$n^2\lambda^2 = \kappa/\rho,$$

whence

$$\kappa = n^2\lambda^2\rho. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

From (39), (43), (44),

$$B = \frac{n\lambda^2}{\mu} \sqrt{\left(\frac{2\rho G}{\pi}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\frac{B}{\kappa} = \frac{1}{\mu n} \sqrt{\left(\frac{2G}{\pi\rho}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and

$$\frac{B^2}{2\kappa} = \frac{\lambda^2}{\mu^2} \frac{G}{\pi E} \cdot E, \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where E is the constitutive energy per cubic centimetre of the æther. Thus the factor of E in the right-hand member of (47) expresses the maximum potential energy per unit volume due to the wave-motion as a proportion of the constitutive energy of the medium. Moreover, this maximum potential energy per unit volume corresponds to an increment δF , equal to ζF , above the mean value \bar{F} , and is thus, to our order of approximation, proportional to ζ^2 . We may write, then,

$$\frac{1}{2}\zeta^2 = \mathfrak{S} \frac{\lambda^2}{\mu^2} \cdot \frac{G}{\pi E},$$

$$\text{or} \quad \zeta = \frac{\lambda}{\mu} \sqrt{\left(\frac{2G\mathfrak{S}}{\pi E}\right)}; \quad . \quad . \quad . \quad . \quad (48)$$

which serves to define a numerical constant \mathfrak{S} now first introduced. This constant might, not improbably, be unity, or a number of that order of magnitude, and for illustrative purposes it is later assumed that

$$\mathfrak{S} = 1. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

29. From (42), (48),

$$n = \frac{\mu}{2\lambda} \sqrt{\left(\frac{3E}{\pi K\mathfrak{S}}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (50)$$

the velocity of compressional waves in free æther being

$$V = n\lambda = \frac{\mu}{2} \sqrt{\left(\frac{3E}{\pi K\mathfrak{S}}\right)}. \quad . \quad . \quad . \quad . \quad (51)$$

Using (50) in conjunction with (44) and (45) respectively,

$$\kappa = \frac{3\mu^2\rho E}{4\pi K S}, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$B = \frac{\lambda}{\pi} \sqrt{\left(\frac{3EG\rho}{2KS}\right)}; \quad . \quad . \quad . \quad (53)$$

while from (43)

$$H = \frac{4\pi KS}{3\mu\rho E}. \quad . \quad . \quad . \quad . \quad . \quad (54)$$

30. An expression is readily found for the maximum translational velocity attained by any element of æther in the primary wave-motion, half the square of this velocity multiplied by the density of the æther being equal to the maximum kinetic or maximum potential energy per unit volume; that is, equal to $B^2/2\kappa$. From (47) the desired expression is at once obtained, and only requires to be divided by $2\pi n$ to give the semi-amplitude of vibration (maximum displacement), and multiplied by $2\pi n$ to give the maximum acceleration of any æther-element. Remembering (50) the three expressions are:—

$$\text{Maximum displacement from mean position} = \frac{\lambda^2}{\pi\mu^2} \sqrt{\left(\frac{2KG S}{3\rho E}\right)}, \quad (55)$$

$$\text{Maximum velocity} = \frac{\lambda}{\mu} \sqrt{\left(\frac{2G}{\pi\rho}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$\text{Maximum acceleration} = \sqrt{\left(\frac{6GE}{S\rho K}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

31. Bearing in mind § 9, and in particular equation (3), we have also, *for any particle of neutral matter considered with respect to the æther,*

$$\text{Maximum displacement from mean position} = \frac{2\lambda^2 K}{3\mu^2} \sqrt{\left(\frac{2S G}{\pi E}\right)}, \quad (58)$$

$$\text{Maximum velocity} = \frac{2\lambda}{\mu} \sqrt{\left(\frac{2GK}{3}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$\text{Maximum acceleration} = 2\sqrt{\left(\frac{2\pi G E}{S}\right)}; \quad . \quad . \quad . \quad . \quad . \quad (60)$$

these three equations of course concern only that motion of matter with respect to the æther which is directly due to the primary wave-motion.

32. From (46), (50) we have for the range of proportional changes of volume of the æther, above and below mean value,

$$\frac{B}{\kappa} = \frac{2\lambda}{\mu^2} \sqrt{\left(\frac{2GK\mathfrak{S}}{3\rho E}\right)}; \quad \dots \dots (61)$$

while for the rate (in ergs per second) at which wave-energy is being propagated through any square centimetre of surface whose plane is parallel to the wave-fronts, we must take the product of the wave-energy per c.c. multiplied by the velocity of propagation. Now throughout our purely progressive wave-train the *total* energy per unit volume is $B^2/2\kappa$, the maximum value attained by the potential energy per unit volume. Thus multiplying (47) by (51) the rate at which wave-energy is being propagated, in ergs per second per square centimetre, is found to be

$$\mathfrak{V} B^2/2\kappa = \frac{\lambda^2 G}{2\mu} \sqrt{\left(\frac{3E}{\pi^3 K \mathfrak{S}}\right)}. \quad \dots \dots (62)$$

33. All the quantities with which we are now concerned are expressed in terms of the following seven quantities:—

- (i.) G the Newtonian gravitation constant $= 6.66 \times 10^{-8}$.
- (ii.) K the motional constant, defined by (31) $[2 \times 10^{-20}]$.
- (iii.) ρ the density of the æther $[10^{12}]$.
- (iv.) E the constitutive energy per c.c. of the æther $[10^{33}]$.
- (v.) λ the wave-length of the primary disturbance $[10^4 \text{ astronomical units} = 1.5 \times 10^{17} \text{ cm.}]$.
- (vi.) μ the proportional increase of ætherial compressibility arising from the presence of 1 gram of atomic matter per c.c., as defined at the beginning of § 28 [20].
- (vii.) \mathfrak{S} a numeric defined by (48) [1].

Of these seven, only the first, the gravitation constant, is known, the values indicated (in square brackets) for the remaining six being merely conjectural. If no set of values could be found which did not lead to demonstrably false results, we should have to conclude that the theory in the form here suggested was untenable. The considerations leading to the choice of the values in question will be best understood after the corresponding values of some related quantities have been computed. The table (p. 92) gives in a collected form the relations obtained in §§ 27–32.

TABLE.

Values of certain quantities connected with the hypothetical (primary) wave-motion, expressed in terms of G , K , ρ , E , λ , μ , and \mathfrak{S} , together with the numerical values for the same quantities derived from those already assigned to G , K , &c.

Nature of quantity.	Symbol.	Equivalent expression.	Value in C.G.S. units.	Value otherwise expressed.
For any element of æther	{	Maximum displacement. ... $\frac{\lambda^2}{\mu^2} \sqrt{\left(\frac{29KG}{3\rho E}\right)}$	6.9×10^{-5}	69 micron.
		Maximum velocity. ... $\frac{\lambda}{\mu} \sqrt{\left(\frac{2G}{\pi\rho}\right)}$	5.6×10^6	56 kilom. per sec.
		Maximum acceleration. ... $\sqrt{\left(\frac{6GE}{9\rho K}\right)}$	4.5×10^{17}	
For any element of atomic matter, with respect to the æther	{	Maximum displacement. ... $\frac{2\lambda^2 K}{3\mu^2} \sqrt{\left(\frac{29G}{\pi E}\right)}$	2×10^{-8}	1/5000 micron.
		Maximum velocity. ... $\frac{2\lambda}{\mu} \sqrt{\left(\frac{2GK}{3}\right)}$	1.6×10^3	16 metres per sec.
		Maximum acceleration. ... $2 \sqrt{\left(\frac{2\pi GE}{9}\right)}$	1.3×10^{14}	
Frequency of primary disturbance.	n	$\frac{\mu}{2\lambda} \sqrt{\left(\frac{3E}{\pi 9K}\right)}$	1.29×10^{16}	
Velocity of compressional waves in æther.	\mathfrak{V}	$\frac{\mu}{2} \sqrt{\left(\frac{3E}{\pi 5K}\right)}$	2.19×10^{27}	2.3×10^9 light-years per second.
Extrusion of æther per unit mass of matter (§ 7).	F	$2 \sqrt{\left(\frac{\pi K}{3\rho}\right)}$	2.9×10^{-16}	
Mass of æther extruded per unit mass of matter.	$F\rho$	$2 \sqrt{\left(\frac{\pi \rho K}{3}\right)}$	2.9×10^{-4}	
Maximum proportional deviation of F from its mean value.	ζ	$\frac{\lambda}{\mu} \sqrt{\left(\frac{29G}{\pi E}\right)}$	5.6×10^{-5}	
Bulk-modulus of elasticity of the æther.	κ	$\frac{3\mu^2 \rho E}{4\pi 9K}$	4.78×10^{66}	4.72×10^{60} atmospheres.
Additional ætherial compressibility per unit density of matter present.	H	$\frac{4\pi 9K}{3\mu \rho E}$	4.2×10^{-66}	
Maximum deviation of ætherial pressure from its mean value.	B	$\frac{\lambda}{\pi} \sqrt{\left(\frac{3EG\rho}{29K}\right)}$	3.9×10^{45}	3.8×10^{39} atmospheres.
Maximum proportional deviation of ætherial density from its mean value.	B/κ	$\frac{2\lambda}{\mu^2} \sqrt{\left(\frac{29GK}{3\rho E}\right)}$	8.1×10^{-22}	
Total energy of primary disturbance per cubic centimetre.	$B^2/2\kappa$	$\frac{\lambda^2 G}{\pi \mu^2}$	1.6×10^{24}	$1/(6 \times 10^8)$ of the constitutive energy of the medium.
Energy propagated through 1 sq. cm. of normal surface per second.	$\mathfrak{V} B^2/2\kappa$	$\frac{\lambda^2 G}{2\mu} \sqrt{\left(\frac{3E}{\pi 9K}\right)}$	3.4×10^{51} ergs.	3.4×10^{41} kilowatts per sq. cm.

34. We may now review briefly the tentative values attributed in the last paragraph to K , ρ , E , λ , μ , and \mathfrak{S} . The value 2×10^{-20} is assumed for the motional constant K , in

accordance with (35); the supposition being that K is small enough for a null result to be obtained in the test suggested in § 24, but not with any great margin to spare. It might be that the test referred to would give a positive result, and in any case a much more sensitive disposition could easily be devised. But since no such positive result has yet been recorded, we cannot on this score assign any lower limit to the value of K , nor is our choice of something near the highest of seemingly admissible values made necessary by the quantities given in the table. The experimental determination of the motional constant K , or of an upper limit to its value, would be interesting quite apart from its bearing on this theory; the question involved being, essentially, whether or not the total of ætherial substance comprised in any given volume is modified by the presence of atomic matter within that volume.

35. The values assumed for ρ and E (the density and the constitutive energy per unit volume of the æther) are those lately suggested by Sir O. Lodge*, who has given very convincingly his reasons for supposing that, in order of magnitude, the true values may not be very different from these estimates. Incidentally it may be remarked that Lodge's estimate of the constitutive energy per unit volume of the æther is about 6 times the electrostatic energy per unit volume, close to the surface of a negative electron, calculated on the usual assumptions that practically the whole inertia of the electron is electromagnetic, and that the ordinary linear relations of the electromagnetic field hold good right up to the surface of the electron.

36. For the wave-length of the primary disturbance, we must assume a value so great that, even at distances of at least several astronomical units, the inverse square law of gravitational attraction is sensibly accurate. By putting $\lambda = 10^4$ astronomical units, this condition seemed to be amply fulfilled, since for distances small compared with λ , the deviations from the inverse-square law would be only of the second order (§ 19). The chief consideration against assigning a very much greater value to λ is that ζ must be kept small to preserve linearity of relations, and also, I imagine, for general plausibility.

37. The same consideration (keeping ζ small) leads us to assign a fairly high value to μ (defined in § 28). It is difficult to see what sort of value μ should be expected to have: whether the proportional increase of ætherial compressibility due to a space being "filled" with water instead of

* Phil. Mag. vol. xiii., 1907; Nature, lxxv. p. 519, 1907.

being vacuous should be a large number or a small fraction. The value -20 —chosen for μ would make the phase-difference of pressure-fluctuation of about the same order between the surface and the centre of the sun as between the surface of the sun and that of the earth, under maximum conditions.

The somewhat vague grounds on which the value unity is assigned to \S were indicated at the end of § 28.

38. Some of the derived quantities in the table above may now be considered. In the first place, take ζ , which represents the maximum deviation from its mean value of F the “specific displacement” of atomic matter (§ 7). If the primary disturbance is slight enough to be adequately represented by linear equations, we should expect on general grounds that ζ would be small, and the value for ζ derived from our assumptions is 5.6×10^{-5} , while the energy of the primary disturbance per unit volume, under the same assumptions, appears as $1/(6 \times 10^8)$ part of the constitutive energy of the æther.

39. For n , the frequency of the primary disturbance, 1.29×10^{10} per second is found, leading to 2.19×10^{27} cm. per second, or over two thousand million light-years per second, for V , the velocity of propagation of a compressional-rarefactional disturbance through the æther. This would amply suffice for the sensibly instantaneous character of gravitational attraction.

40. The bulk-modulus of elasticity of the æther, denoted by κ is, with our assumptions, represented by 4.78×10^{66} dynes per sq. cm., or 4.72×10^{60} atmospheres. Comparing this with the assumed constitutive energy of the æther (10^{33} ergs per c.c.), which is taken to be the basis of dielectric elasticity (the reciprocal of dielectric capacity) in free æther, the comparison appears to be consistent with our assumption that the bulk-modulus is enormously greater than any other elastic modulus of the æther; an assumption in accordance with which the æther, in relation to compressional-rarefactional motions, has been treated as a fluid.

Direct Effects of the Primary Disturbance.

41. Considering next the motion of the æther constituting the primary disturbance, and the directly resulting motion of matter with respect to the æther, the question arises: what phenomena, if any, would present themselves to our observation if such motions were actually taking place? In the first place, bearing in mind § 12 above, it will be evident that a bodily vibratory motion of the æther, of

sufficiently great wave-length, *with atomic matter equally partaking in such motion*, could give rise to no observable phenomena; if any effects are to be made manifest, as a direct consequence of the primary disturbance, these must arise from the motion of matter with respect to the æther. So far as I have been able to see, after long and careful consideration, there would be nothing capable of affecting interference phenomena, no heating effect, and no production of electromagnetic waves, provided only that one condition were realized. That condition is that the positive and negative electrons should have identical accelerations impressed on them by the direct action of the primary disturbance. The discrimination between positive and negative electrons in relation to gravitational agency is a question presenting many aspects for consideration. Some of these are touched upon in Appendices A and B.

42. Perhaps the most surprising of the tentative numerical values tabulated above, is that suggested as the rate at which energy is being propagated through the æther, per square centimetre of surface normal to the direction of propagation; the energy so propagated in one second exceeding by many million times the Sun's entire store of available heat. The problem has of course been simplified by limiting the primary disturbance to a single progressive wave-train of definite wave-length; but the result would not have been greatly different if a series of wave-lengths had been included as in § 18, and it is of course quite immaterial whether we assume the primary waves to be travelling indiscriminately in all directions, or predominantly or exclusively in a single direction. It must indeed be admitted that the basis of the estimates put forward in § 33 is nothing better than guess-work; but, after making all allowances for the very great uncertainty attaching to such conjectures, it appears to be an inseparable feature of the theory proposed that energy should be travelling through the æther on a prodigious scale.

43. The theory of gravitation which forms the subject of this paper has a good deal in common with its predecessors. In the first place, it is made to appear that gravitational attraction is not an essential and inseparable attribute of matter, the Newtonian constant being theoretically susceptible of increase or diminution, or even of entire suppression through a change of external circumstances. In this respect the present theory is comparable with Le Sage's hypothesis of ultramundane corpuscles, as well as with a scheme put

forward by Challis *, which likewise assumed the existence of compressional waves in the æther, although the function attributed to these waves was very different. But the theory now advanced is much more akin to that of Prof. Hicks, which indeed suggested it, and of which it may be regarded as a development. The introduction of ætherial compressional waves of great wave-length to actuate the pulsatory movements of atomic matter, and thus secure the necessary agreement of phase amongst the pulsating centres, is one of the main modifications suggested; another being the manner in which the capacity for pulsatory motion, assumed to be associated with the electrons, is represented as mobile through the æther. In this way we avoid the necessity for supposing that any element of ætherial substance ever deviates by more than a minute amount from its mean position, the motion throughout being of "stationary" type, while at the same time the problem with which we have to deal is essentially one of hydrodynamics, capable of being worked out to a first order by means of a simple analysis. We are enabled to treat the æther as if it were a fluid, not because we assume it to yield freely to certain types of stress, but because the distortions involved in the motions considered are so excessively minute that no appreciable opposing stresses are called up, save only the changes of hydrostatic pressure, which owe their importance to the enormous value attributed to the bulk-modulus of elasticity of the æther. Apart from the simplicity thus attained, it appears to me most desirable that we should discard, if possible, the conception of matter, or of any parts or properties associated with matter, as grossly ploughing a course through a reluctant æther. All that we know of æther and of matter seems to indicate that the mobility of matter is absolute, and that the elastic properties of the æther remain perfect, notwithstanding the motions of material bodies, or even of detached negative electrons travelling with velocities approaching that of radiation. And the only way in which I have been able to conceive of matter as travelling through the æther, without doing unwarrantable violence to the structure of the medium, is by supposing the entire phenomenon of motion to be reducible to a transference of strain, so that no event in our universe involves a progressive yielding of the æther.

44. By Maxwell † it was felt to be an objection to such hypotheses as those of Le Sage and of Challis, that they

* See a review by Maxwell, 'Nature,' vol. viii.; Scientific Papers, vol. ii. p. 338.

† *Loc. cit.*

involved a continual expenditure of energy for the maintenance of gravitational attraction, the conservation of energy in such cases being "apparent only." The present theory is equally open to this criticism; but the objection is a metaphysical rather than a physical one, and in view of recent developments it has hardly the force which it might have appeared to have some forty years ago. We have become accustomed to the idea that our rapid motion through a medium of enormous density not only fails to provide us with a useful source of energy, but defies the most refined attempts to detect it by means of terrestrial observations.

And there is nothing more inherently improbable in the notion that our universe may be traversed by waves of enormous energy, perceptible to us only by means of a minute secondary effect—gravitation. That we should be unable, even in our dealings with this feeble residual phenomenon, to extract with continual profit the minutest portion of the energy so abundantly propagated, is a view which may appear to us somewhat ironical, but which is not out of harmony with the trend of modern physical conceptions.

45. Whatever may be the difficulties of the theory, this attempt to contribute something towards the explanation of gravitation has appeared to me sufficiently suggestive to be worth publication; the more so as incidentally certain questions are raised, some of which, it may be hoped, are capable of being decided by experiment.

APPENDIX A.

Electromagnetic Phenomena which might conceivably arise from Gravitation.

46. Since in this paper gravitation is regarded as a secondary or residual effect, due to the influence of atomic matter on the propagation of compressional ætherial waves, it might have seemed more logical to discuss the electromagnetic phenomena possibly produced by such waves, before considering effects of a like character which might be supposed to arise from gravitational attraction. In the first place, however, gravitation is known to exist, and in the second place, the effects to be considered, being statical, are comparatively simple. Accordingly gravitation may be dealt with first in this connexion.

47. Independently of any theory as to the cause of gravitation, if we suppose that ordinary matter is made up of electrons, then it seems reasonable to conclude that any influence exerted on matter can be analysed into influences

exerted on electrons. Again, experiment indicates unmistakably that, whatever the nature of positive and negative electrons may be, they are far from being symmetrical opposites. Thus it is natural to inquire what effects are to be expected when we discriminate between positive and negative electrons under gravitational influence.

48. To take the simplest case, consider a body moving without constraint in a uniform field of gravity, the acceleration of the body f thus agreeing in direction and magnitude with the strength of the field. Suppose that, electrically, the condition of the body is not changing; then on the whole both positive and negative electrons are moving with the acceleration f . Now if the forces exerted in a field of gravity are in the same sense on positive and negative electrons, and of magnitudes proportional respectively to the masses of such electrons, the identical acceleration of the two denominations will follow, without additional forces of electrostatic type being called into play. But if the gravitational forces on positive and negative electrons do not conform to the condition just referred to, they can always be resolved into two sets: one set of like forces proportional respectively to the masses of the positive and negative electrons on which they act, and another set of forces acting equally and oppositely on the positive and the negative. This latter set produces on the body under consideration the same effect as would result from a uniform electrostatic field, the body becoming electrically polarized if of dielectric material, or, if a conductor, acquiring a surface charge of electrification without internal polarization.

49. The state of things here suggested is somewhat different from anything ordinarily contemplated in electrostatics; the gravitational quasi-electromagnetic intensity which acts throughout the substance of a conducting body being balanced by the true electromotive intensity arising from the surface distribution of electricity, so that we have an electrostatic field of force exerted in a conducting medium in equilibrium. In these circumstances, it may not be superfluous to point out that, in estimating the electric field-intensity from the surface charges, no dielectric constant other than that of free æther comes into play. For in an electrostatic field, the intensity at any point is determined jointly by the signs and positions of all the electrons concerned, and by nothing else, each electron contributing to the field-intensity the same component as if the other electrons were non-existent. As is well understood, the electronic theory affords on these lines an account of the dielectric qualities of different

media; such qualities, however, being without any bearing on the problem, so long as *on the whole* there are no forces tending to displace relatively to one another the positive and the negative electrons which together make up the medium under consideration.

50. Let μ_1, μ_2 denote the *masses* of the positive and negative electrons respectively, contained in unit mass of matter; then

$$\mu_1 + \mu_2 = 1. \quad . \quad . \quad . \quad . \quad . \quad (63).$$

Also let the total *charge* of the positive electrons in unit mass of matter be $\epsilon\chi$, the total charge of the negative electrons being accordingly $-\epsilon\chi$: where ϵ is the quantity of electricity required to liberate one gram of hydrogen electrolytically, and χ is a numeric.

Further, in a field of gravity whose intensity is measured by the acceleration f , let

gravitational attraction on mass μ_1 of positive electrons be $\mu_1 \gamma_1 f$.
 „ „ „ *μ_2 of negative electrons be $\mu_2 \gamma_2 f$.*

We then readily find, for a mass moving freely in the field f , that the electrical condition is the same as if an electrostatic field were exerted, whose intensity, measured in the same direction as f , is

$$(-\mu_1 + \mu_1 \gamma_1) f / \epsilon \chi = (\mu_2 - \mu_2 \gamma_2) f / \epsilon \chi; \quad . \quad . \quad (64)$$

this equivalent field-intensity being expressed in electrostatic or electromagnetic measure, according as ϵ is in electrostatic or electromagnetic measure.

51. The (single) condition that this quasi-electromotive-intensity should vanish is $\gamma_1 = 1 = \gamma_2$, in which case the gravitational forces acting on masses μ_1, μ_2 of positive and negative electrons in a field of intensity f are respectively $\mu_1 f, \mu_2 f$; that is to say, in the proportion of the masses. As will be seen in Appendix B, the gravitation theory of this paper appears almost certainly to demand that the condition just referred to should be satisfied; but putting that theory for the time being on one side, it is interesting to trace out some consequences of a less restricted view as to the relative behaviour of positive and negative electrons under the action of gravity.

52. When the body with which we have to deal is not moving freely in a sensibly uniform gravitational field, the corresponding results cannot be written down without making some assumption as to the relative mobility of positive and negative electrons in a conducting or dielectric mass. For

example, let us trace the consequences of supposing that the negative electrons alone possess sufficient mobility to enable them to migrate through unrestricted distances within the mass of a solid conducting substance; the negative electrons thus serving to convey the whole of any current which may be flowing in the substance. In a cylindrical conductor whose axis is vertical, let a current be flowing upward; consider what happens when negative electrons whose aggregate charge is $-\epsilon\chi$ pass downward through the cylinder, and in particular through a certain stratum of the cylinder, bounded above and below by horizontal planes, whose distance apart is h . According to our notation, $-\epsilon\chi$ is the aggregate charge of the negative electrons which are comprised in one gram of neutral matter, and which have collectively the mass μ_2 , experiencing in a gravitational field of intensity g the force $\mu_2\gamma_2g$, measured downward, so that the work done by gravity while the quantity $-\epsilon\chi$ moves downward through the stratum h is $\mu_2\gamma_2gh$. There is thus virtually, corresponding to a height h of the conductor, an electromotive force

$$-\mu_2\gamma_2gh/\epsilon\chi \text{ (measured downward) ; } \dots (65)$$

the quasi-electromotive-intensity due to the action of gravity being therefore

$$-\mu_2\gamma_2g/\epsilon\chi \text{ (measured downward). } \dots (66)$$

53. If, in place of gravity acting, an acceleration f is impressed on an isolated conductor, then the negative electrons of total charge $-\epsilon\chi$, having an aggregate mass μ_2 , must experience, when there is electrical equilibrium, a resultant force μ_2f , and in virtue of the assumed mobility of the negative electrons, this resultant force μ_2f must be due to an electrical distribution on the conductor in question. But an electrical distribution which would exert the force μ_2f on certain electrons is precisely that distribution which would arise from such an external field as would by itself exert the force $-\mu_2f$ on the same electrons. In other words, the conducting body caused to move with acceleration f , becomes electrified precisely as if it were placed in an electrostatic field of intensity

$$\mu_2f/\epsilon\chi. \dots (67)$$

If we take f to be vertically downward, and identical in value with g , we find by combining (66) and (67) an expression for the virtual electrostatic field due to the action of

gravity on a freely falling body ; and this expression agrees with (64) which was obtained independently of any assumption regarding the respective mobilities of positive and negative electrons.

54. On the other hand, if we were to assume that, while currents of conduction were carried exclusively by the negative electrons, it was the positive electrons alone which were acted upon by gravitational attraction, we should be led to the conclusion that bodies at rest relatively to (say) the earth would acquire no electrical charges through the action of gravity.

55. Reverting now to the illustrative assumption of § 52, we may attempt to form some idea of the magnitude of the electromotive effects to be expected when the negative electrons are supposed to be not only the exclusive carriers of currents of conduction, but also the only objects influenced by gravity. A further assumption is needed as to the total of positive or of negative charges carried by the electrons in unit mass of matter. Let it be assumed, for example, that $\chi=1$ (see § 50), so that the negative electrons comprised in one gram of hydrogen (or of other substance) will have an aggregate charge equal to ϵ , the quantity of electricity required to liberate one gram of hydrogen by electrolysis. The expression (66) for the downwardly-directed quasi-electromotive intensity resulting from the direct action of gravity becomes $-\mu_2\gamma_2g/\epsilon$, where $\mu_2\gamma_2g$ is the downward force exerted in the earth's gravitational field upon the mass μ_2 of negative electrons comprised in one gram of matter. But under our present assumptions this latter force is simply the weight of one gram of matter, and is equivalent to g ; so that finally the downwardly-directed quasi-electromotive intensity is measured by $-g/\epsilon$. Now ϵ is roughly 10^4 e.m. units of quantity per gram, and g may be taken as about 980 cm./sec.² ; consequently, for the quasi-electromotive intensity affecting stationary bodies at the earth's surface the estimate obtained is 980×10^{-4} e. m. units of potential per cm., or 9.8×10^{-10} volt per cm.

56. On applying the axiom that gravity cannot be made to furnish an unlimited supply of energy, it is evident that the total gravitational electromotive force round any closed circuit of conducting bodies must vanish, however small may be the conductivity of those bodies ; so that any attempt to detect such electromotive effects galvanometrically must fail. The only methods conceivably available would be those depending upon the convection of electric charges by insulated

conductors moving between one level and another, and making contact intermittently with the extremities of an elongated vertical conductor. It would be easy to devise a machine for performing continually a cycle of operations, whereby any electromotive influence due to gravity would be rendered effective in gradually imparting contrary charges to a pair of insulated hollow conductors. But apart from the excessive minuteness of the effect to be looked for, and the thermoelectric and other disturbing influences almost necessarily encountered, another consideration steps in to render any such attempt nugatory. To guard against stray electric influences, our whole apparatus would require to be enclosed in a conducting envelope, the charge induced on which by gravitational influence would exactly neutralize, throughout the interior of the envelope, the quasi-electromotive intensity due to the direct action of gravity. Thus, even if we suppose positive and negative electrons to be oppositely acted upon by a gravitational field (in which case, for example, the aggregate attractive force on the negative electrons in a body might greatly exceed the weight of the body), it would still be impossible, by means of laboratory experiments, to detect any electromotive effects due to the earth's gravity.

57. The case of the earth moving in its orbit under the sun's attraction may be regarded as that of a body moving freely under a sensibly uniform field of gravity. Making any assumptions that suggest themselves as to the number of electrons in a gram of matter, the relative masses of positive and negative electrons, and the forces experienced by these under gravitational influence, (64) may be used to estimate the resulting electrical distribution on the earth's surface (or in the upper strata of the atmosphere) arising directly from the gravitational attraction of the sun. But with such assumptions as I have tried, it appears that the effect to be expected is very minute. For example, making the same assumptions as in § 52, and in addition assuming that nearly all the inertia of neutral matter is the inertia of positive electrons, the extreme difference of potential between opposite poles of the earth comes out as about 80 microvolts. Moreover, the shifting of this feeble electrical distribution, owing to the earth's (solar) diurnal rotation, could not have any sensible influence on the phenomena of terrestrial magnetism.

APPENDIX B.

*Electromagnetic and Thermal Effects produced by
Compressional Ætherial Waves.*

58. Returning more particularly to the view adopted in the body of this paper, the results obtained in §§ 16-18 above may be readily revised so as to discriminate between the possibly different gravitative behaviour of positive and negative electrons.

As in Appendix A, let unit mass of neutral matter contain mass μ_1 of positive and μ_2 of negative electrons (so that $\mu_1 + \mu_2 = 1$), the aggregate charge of mass μ_1 (or μ_2) of positive (or negative) electrons being as before denoted by $\epsilon\chi$ (or $-\epsilon\chi$).

Let

	volume of æther "extruded" by mass μ_1 of positive electrons	$= F_1\mu_1$,
	" " " " " μ_2 " negative "	$= F_2\mu_2$;
then		
mass	" " " " μ_1 " positive "	$= F_1\mu_1\rho$,
and		
mass	" " " " μ_2 " negative "	$= F_2\mu_2\rho$.

Also let $-dF_1/dp = H_1$, $-dF_2/dp = H_2$; . . (68)

then evidently

$$F_1\mu_1 + F_2\mu_2 = F(\mu_1 + \mu_2) = F \quad \text{and} \quad H_1\mu_1 + H_2\mu_2 = H. \quad (69)$$

Precisely as in (7), we can write down, in terms of the acceleration of the æther, the corresponding forces on positive and negative electrons. Writing A for the acceleration of the æther at any instant, arising from the propagation of the primary waves, the corresponding force tending to accelerate the mass μ_1 of positive electrons relatively to the æther is $F_1\mu_1\rho A$, while the mass μ_2 of negative electrons similarly experiences a force $F_2\mu_2\rho A$. Now

$$\begin{aligned} F_1\mu_1\rho A &= A\{F\mu_1\rho + (F_1 - F)\mu_1\rho\} \\ F_2\mu_2\rho A &= A\{F\mu_2\rho + (F_2 - F)\mu_2\rho\} \end{aligned} \quad . \quad . \quad (70),$$

the aggregate forces acting respectively on the positive and on the negative electrons in unit mass of matter being thus resolvable into forces $F\mu_1\rho A$, $F\mu_2\rho A$ proportional to the masses on which they act, together with the pair of forces $(F_1 - F)\mu_1\rho A$ and $(F_2 - F)\mu_2\rho A$, which by (68) are equal and opposite. The first two constituents named tend to accelerate each element of matter with respect to the æther as a whole,

the total force exerted per unit mass of matter being $F\rho A$; while the remaining equal and opposite forces, acting respectively on positive and negative electrons whose aggregate charges are $\epsilon\chi$, $-\epsilon\chi$, have the same effect on the motion of those electrons as would be produced by an electromotive intensity

$$T = \frac{(F_1 - F)\mu_1\rho A}{\epsilon\chi} = \frac{(F - F_2)\mu_2\rho A}{\epsilon\chi}, \quad . . . (71)$$

measured in the direction of the acceleration A .

59. As in Appendix A, let it be assumed (merely by way of example) that $\chi=1$, ϵ being as before the quantity of electricity required to liberate one gram of hydrogen electrolytically. Assume also that the extrusion of æther F per unit mass of neutral matter is due entirely to negative electrons; the æther extruded by one gram of matter being therefore that extruded by a mass μ_2 of negative electrons, whose total charge is $-\epsilon\chi$ ($=-\epsilon$). Assume, that is to say,

$$F_1=0 \quad \text{and} \quad F_2\mu_2=F\mu=F(\mu_1+\mu_2). \quad . . . (72)$$

If we further assume (as in Appendix A, § 57) that nearly all the inertia of neutral matter is the inertia of positive electrons, so that $\mu_1=1$ approximately, then (71) becomes

$$T = -\frac{F\rho A}{\epsilon}, \quad (73)$$

which, with our present special assumptions, is the quasi-electromotive intensity due to an acceleration A of the æther.

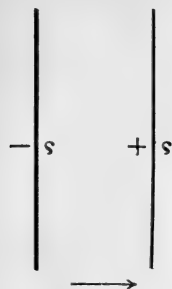
60. As regards the thermal effects arising from compressional waves in the æther, let us simplify the problem to be considered by reducing the disturbance (as in § 27) to a single plane wave-train of very great wave-length λ , and of frequency n . Putting s in place of $2\pi n$, we may write for any region of dimensions very small compared with λ ,

$$T = T_0 e^{ist}, \quad (74)$$

where it is to be understood that T and T_0 are expressed in absolute *electromagnetic* measure.

61. Consider in particular the case of a flat parallel-faced slab of conducting material, whose length and breadth are very great in comparison with its thickness, the main faces of the slab being perpendicular to the direction in which the compressional ætherial waves are being propagated, and

perpendicular therefore to the direction of the quasi-electromotive intensity T , which we shall suppose to be measured positively from left to right in the figure, as indicated by the arrow.



At any instant let s be (in electromagnetic measure) the surface-density of electrification on the right-hand face, $-s$ being that on the left-hand face; s will evidently be a periodic function of the time of the form

$$s = s_0 e^{ist}, \quad (75)$$

where s_0 is a constant, in general complex.

Between the faces of the slab there is, at any time t , a conduction current whose density, referred to unit of area, is ds/dt , and a polarization current whose density is $-ds/dt^*$, the total effective current being thus zero. The same result is otherwise evident; for at any point exterior to the slab the total electromotive intensity arising from the surface-densities s and $-s$ vanishes at each instant. Hence outside the slab there is no polarization and no polarization-current, and of course no conduction-current; so that the total effective current is zero outside the slab. The solenoidal or "stream" character of the total current accordingly necessitates a zero value for this vector within the substance of the slab also.

62. Thus everywhere, except near the edges of the slab (which we ignore) magnetic force is absent, and electromagnetically speaking our system is without kinetic energy †. There must of course be some kinetic energy involved in the reciprocating relative motions of positive and negative electrons, but this kinetic energy, even though a close enough analysis would reveal its electromagnetic character, has no place in the ordinary equations of electromagnetism, and a moment's consideration suffices to indicate its excessive minuteness.

* Cf. Appendix A, § 49.

† There will accordingly be no production of electromagnetic waves, except near the edges of the slab where the same simple conditions are not realized. But in general electromagnetic waves will be produced wherever there are bodies of dielectric or conducting material in the path of the primary compressional waves. If the primary disturbance, as in the foregoing table, is assumed to be periodic and of frequency 1.29×10^{10} , the electromagnetic waves, being of the same frequency, will in free æther have a wave-length of 2.33 cm. Without having considered these waves in any detail, I think it may be said that they will be quite insignificant, provided the electromotive effects of the primary waves are too slight to give rise to appreciable thermal phenomena.

63. Within the substance of the slab there is at each instant a quasi-electromotive intensity T and a true electromotive intensity $-4\pi V^2 s$, V being the velocity of radiation. For simplicity it will now be supposed that the only opposing forces are due to electrical resistance; thus in the absence of appreciable inertia effects, the relation

$$(T - 4\pi V^2 s)c = ds/dt \quad . \quad . \quad . \quad (76)$$

is always satisfied; c being the conductivity of the substance of the slab. Using (74) and (75),

$$s_0 = T_0 c / (4\pi V^2 c + is) \quad . \quad . \quad . \quad (77)$$

This gives for the current-density

$$I = is s_0 e^{ist} = \frac{Ac(s^2 + i \cdot 4\pi V^2 sc)}{16\pi^2 V^4 c^2 + s^2} e^{ist}; \quad . \quad . \quad (78)$$

or, separating the real from the imaginary problem, we have, corresponding to

$$T = T_0 \cos st, \quad . \quad . \quad . \quad (79)$$

$$I = \frac{T_0 cs}{16\pi^2 V^4 c^2 + s^2} (s \cos st - 4\pi V^2 c \sin st) \quad . \quad . \quad (80)$$

64. The rate, in ergs per second, at which energy is being dissipated as heat in each cubic centimetre of the conductor is accordingly

$$\frac{I^2}{c} = \frac{T_0^2 cs^2}{16\pi^2 V^4 c^2 + s^2} (s \cos st - 4\pi V^2 c \sin st)^2; \quad . \quad (81)$$

so that

$$\text{average of } \frac{I^2}{c} = \frac{1}{2} \frac{T_0^2 cs^2}{16\pi^2 V^4 c^2 + s^2} \quad . \quad . \quad (82)$$

The last-written expression is a maximum with respect to c when $c = s/4\pi V^2$, in which case the average of I^2/c becomes

$$T_0^2 s / 16\pi V^2 \quad . \quad . \quad . \quad (83)$$

64. Remembering (73), we find that, with the particular set of numerical values adopted in the foregoing table $T_0 = 130$ volts per cm., or in absolute e.m. units 1.3×10^{10} ; $s = 2\pi n = 2\pi \times 1.29 \times 10^{10}$; the average rate at which energy is being dissipated in the conductor being in that case

3×10^8 ergs per sec. per c.c. or about 7.5 calories per sec. per c.c.

This is with the conductivity most favourable to dissipation,

the value of the conductivity being then given by $c=s/4\pi V^2$; so that the specific resistance would be $4\pi V^2/s=1.4 \times 10^{11}$ in absolute measure, which is equivalent to 140 ohms as the resistance of a centimetre cube. If the slab were of much higher or much lower conductivity, the dissipation of energy (corresponding to the same set of assumptions) would of course be much less.

65. Thus with the special assumptions of § 59 the rate of development of heat, through the action of the primary waves on matter, though excessively minute compared with the rate at which we have supposed energy to be propagated in those waves, is nevertheless, at its maximum, much greater than could be reconciled with experience. But in obtaining the estimates of the previous paragraph, it was assumed that the "extrusion" of the æther per unit mass of matter* was entirely due to the negative electrons, and that the inertia of matter was almost entirely due to the positive electrons. Unless we were to suppose the extrusions due to positive and negative electrons to be of opposite signs, this would be the assumption corresponding with the greatest electromotive effects from the action of the primary waves on atomic matter, and with the greatest consequent dissipation of energy; the inference being that, if we are to retain our tabulated estimates, or any others not wholly different from these in order of magnitude, we must adopt some more favourable assumption as to the relative "extrusions" and masses of positive and negative electrons. Now it is evident from (71) that if the "extrusions" of positive and negative electrons were proportional to their masses (that is $F_1=F=F_2$), the electromotive effects of the primary waves would vanish. In that case there would be no heating of matter through dissipation of the energy of the primary disturbance, nor would there be any corresponding production of electromagnetic waves. Experience suggests, then, that the relation

$$F_1=F=F_2 \quad . \quad . \quad . \quad . \quad . \quad (84)$$

is exactly or very nearly fulfilled.

66. If indeed the æther is traversed by waves of other than electromagnetic type, to the penetration of which no sort of matter forms a barrier, it would be interesting to test for any residual heating effect due to the interaction of such waves with atomic matter. To devise the most sensitive arrangement within given limits of size would need some

* Cf. § 7.

thought, but evidently a body of large bulk would be best. Such a body (perhaps hollow) of suitable conductivity, well jacketed with a material of the highest possible thermal and electrical resistance, and further enveloped in a double-walled vessel containing ice, might be examined for a gradual rise of temperature. Even if all laboratory experiments on these lines should give negative results, it might still be held possible that some part of the earth's heat was due to the electromotive effect of compressional ætherial waves. This is only mentioned as a possibility suggested by the theory of this paper, and not as tending to remove any existing difficulty in the domain of geophysics.

67. Some attempt having already been made in Appendix A to trace the possible electromotive effects of a gravitational field, the modifications which must be made in the equations of § 17 above, when we wish to discriminate between positive and negative electrons, will now be very briefly indicated. With the notation explained in § 58, the source whose strength is expressed by (10) must now be represented by the more complete expression

$$-(H_1\mu_1 + H_2\mu_2) \frac{\partial p}{\partial t} \sigma dx dy dz, \quad \dots \quad (85)$$

which is to be equated to $\nabla^2 \phi dx dy dz$; hence (11) becomes

$$\phi = \frac{H_1\mu_1 + H_2\mu_2}{4\pi} \frac{\partial p}{\partial t} \iiint \frac{\sigma'}{r} dx' dy' dz'; \quad \dots \quad (86)$$

while (16) is replaced by

$$\frac{\partial \pi}{\partial x} = \frac{(H_1\mu_1 + H_2\mu_2)\rho}{4\pi} \frac{\partial^2 p}{\partial t^2} I_x, \quad \dots \quad (87)$$

where I_x , as before, has the meaning indicated by (17).

In place of (18) we have now two equations, expressing the forces acting on positive and negative electrons respectively. The force on the positive electrons in unit mass of matter

$$= -\mu_1 F_1 \frac{\partial \pi}{\partial x} = -\mu_1 F_1 \frac{(H_1\mu_1 + H_2\mu_2)\rho}{4\pi} \frac{\partial^2 p}{\partial t^2} I_x, \quad \dots \quad (88)$$

the force on the negative electrons being similarly

$$= -\mu_2 F_2 \frac{\partial \pi}{\partial x} = -\mu_2 F_2 \frac{(H_1\mu_1 + H_2\mu_2)\rho}{4\pi} \frac{\partial^2 p}{\partial t^2} I_x; \quad \dots \quad (89)$$

(19) being replaced by

$$\left. \begin{aligned} F_1 &= \overline{F}_1 + \delta F_1 = \overline{F}_1 - H_1 p \\ F_2 &= \overline{F}_2 + \delta F_2 = \overline{F}_2 - H_2 p \end{aligned} \right\} \quad (90)$$

Hence the gravitational force acting on the positive electrons in unit mass of matter is

$$- \frac{\rho}{4\pi} \mu_1 (\overline{F}_1 - H_1 p) (H_1 \mu_1 + H_2 \mu_2) \frac{\partial^2 p}{\partial t^2} I_x; \quad (91)$$

and since the average value of $\partial^2 p / \partial t^2$ is zero, the only part of this expression which does not disappear on averaging is

$$\text{average of } \left\{ \frac{\rho}{4\pi} H_1 \mu_1 (H_1 \mu_1 + H_2 \mu_2) p \frac{\partial^2 p}{\partial t^2} I_x \right\}. \quad (92)$$

Similarly, the average force on the negative electrons in unit mass of matter is

$$\text{average of } \left\{ \frac{\rho}{4\pi} H_2 \mu_2 (H_1 \mu_1 + H_2 \mu_2) p \frac{\partial^2 p}{\partial t^2} I_x \right\}. \quad (93)$$

On adding (92), (93) and comparing with (21), it is seen that $(H_1 \mu_1 + H_2 \mu_2)^2 = H^2$, which is otherwise immediately evident.

68. Now we have seen that the only escape from inadmissibly great heating effects of the primary waves lies in supposing the relation (84) to be at least very approximately true; and unless we make the very special assumption that the relation in question is only fulfilled for one particular value of the general ætherial pressure, we must likewise conclude that

$$dF_1/dp = dF/dp = dF_2/dp; \quad (94)$$

or, remembering (68), that

$$H_1 = H = H_2. \quad (95)$$

In this case the expressions (92), (93) for the forces exerted in a given gravitational field on the positive and the negative electrons in unit mass of matter, become proportional respectively to the aggregate masses of the electrons in question. As was pointed out in § 51 (Appendix A), this is the condition that no electromotive effects shall be experienced by a body moving freely under a sensibly uniform field of gravity.

The relation between the notation of this Appendix and that of Appendix A is evident: μ_1 and μ_2 having the same meaning in each case, while H_1 , H_2 are respectively equivalent to $\gamma_1 H$, $\gamma_2 H$.

APPENDIX C.

A Non-Electromagnetic Term in the Inertia of an Electron.

69. It will readily be realised that, just as an electron possesses mass or inertia of electromagnetic origin, so on our theory the total effective mass of the electron comprises what may be called a gravitational term. The motions which are, on this view, to be classed as of gravitational type are wholly of the nature of irrotational bodily movements of the volume-elements of æther. What has here been called the primary disturbance is assumed to consist of compressional waves, while the secondary disturbance—the supposed cause of gravitation—takes the form of a pulsatory movement whereof every particle of matter may be regarded as a centre. Finally, the motion of matter with respect to the æther appears in general to involve (over and above electromagnetic phenomena) an irrotational distribution of ætherial motion. This necessarily implies the addition of a corresponding term to the inertia of the matter in question; but as will now appear this term may, on our assumptions, be relatively insignificant. By way of illustration, expressions for the gravitational inertia of an electron, obtained on two further alternative assumptions, will now be given.

70. In the first place, let us suppose that the free æther is a strictly continuous medium, and that the constitution of an electron involves a modification of ætherial density expressible as a continuous function of coordinates. Let the modification of density be such as corresponds with a radial displacement δR (measured outwards) of any element distant R from the centre of the electron. It is understood, of course, that the unmodified state of the æther, expressed by $\delta R = 0$, is one of uniform density. In particular, let *

$$\left. \begin{aligned} \delta R &= CR \text{ from } R=0 \text{ to } R=R_1 \\ \delta R &= CR_1^3/R^2 \text{ from } R=R_1 \text{ to } R=\infty \end{aligned} \right\}; \therefore (96)$$

then it can be shown that the corresponding term in the inertia of the electron is

$$\frac{20}{3} \pi \rho C^2 R_1^3, \quad \dots \dots \dots (97)$$

where ρ as before is the density of the æther.

* This distribution of δR suffers from the disadvantage that motion of the electron through the æther involves impulsive changes of velocity of those æther elements for which, instantaneously, $R = R_1$. A slight modification of (96) would remove the objection, but at a great sacrifice of that analytical simplicity which must always be a leading consideration in the choice of purely illustrative examples.

71. Now the mass of æther "extruded" by the electron is $4\pi\rho CR_1^3$ and the expression (97) bears to this the ratio $\frac{5}{3}C$, which may be very small provided C is small enough. Moreover, in the table given above, the mass of æther "extruded" by one gram of neutral matter is estimated at 2.9×10^{-4} gram; and it therefore seems possible that the theory proposed may involve only a very minute non-electromagnetic term in the total inertia of a body.

72. Alternatively let the æther be regarded (for present purposes of illustration) as made up of thin coreless vortex-filaments pervading a frictionless liquid* of density ρ ; the sum total of the vacuous cores comprised in any considerable volume of æther being so small a proportion of that volume that ρ may sensibly be identified with the average density of the æther. Let us further suppose, as in § 70, that the constitution of an electron involves ætherial displacements expressible by (96), and that such differences of density as exist from one volume-element of æther to another are due solely to altered diameter of the vacuous cores, no alteration in the total length of cores per unit volume being involved. It then appears that what we have called the gravitational part of the inertia of an electron, in addition to the quantity (97) contains a term

$$\frac{2}{3}\rho \times \text{volume of electron} \times \text{average value of} \left\{ \text{volume of vacua} \right. \\ \left. \text{per unit volume of æther} \times \left(\frac{d\delta r_0}{dR} \right)^2 \log \frac{r_1}{r_0} \right\}; \quad \therefore \quad (98)$$

where $r_0 + \delta r_0$ is the radius of a vacuous core at a distance R from the centre of the electron, and r_1 is a length corresponding in order of magnitude with the scale of structure of the æther. In (98) "volume of electron" must be understood as extending to all that region wherein $d\delta r_0/dR$ differs sensibly from zero. The addition of the term (98) does not disturb the conclusion that the gravitational part of the inertia of an electron may be very small compared with the mass of æther which the electron "extrudes."

APPENDIX D.

A Kinetic Model of a Slightly Compressible Medium.

73. The representation of the electromagnetic properties even of free æther by means of a turbulently moving liquid is beset by difficulties which may well prove to be insuperable; but it is nevertheless interesting to remark that [a

* Cf. Appendix D.

perfectly incompressible liquid, through which vortices are distributed, constitutes a kinetic model of a slightly compressible medium such as we have assumed the æther to be, provided some or all of the vortices are coreless. One way in which a very minute degree of compressibility may be represented, is by supposing only a small proportion of the vortices present to be coreless. When such a medium is subjected to increased pressure, so as to diminish the volume occupied by the vacuous cores, the circulation around each vortex remaining unaltered, the energy of the turbulence will be increased by an amount equal to the work done by the pressure, and this additional energy is to be regarded as potential when the turbulently moving liquid is treated as a continuous medium.

74. If we suppose each core, whether vacuous or consisting of rotationally moving liquid, to be of very small diameter in relation to the radius of curvature of its "curved axis," and very small also in comparison with the distance between neighbouring vortices, an expression may readily be obtained for the compressional elasticity of the medium. If p is the mean pressure, we shall have, for any point in the liquid sufficiently remote from vortices,

$$p = \frac{\Omega_1^2 \rho}{8\pi^2 r_1^2} = \frac{\Omega_2^2 \rho}{8\pi^2 r_2^2} = \dots; \dots \quad (99)$$

where r_1, r_2, \dots are the radii of the various vacuous cores, $\Omega_1, \Omega_2, \dots$ the circulations around them, and ρ the density of the liquid. Hence, extending the summation to the unit of volume, and calling l_1, l_2, \dots the lengths of the respective coreless vortices comprised in that volume,

$$\frac{\rho \Sigma l \Omega^2}{8\pi p} = \Sigma l \pi r^2 = U, \text{ say}; \dots \quad (100)$$

so that U is the total volume of vacuous cores comprised in unit volume of the total turbulent liquid. Also, assuming the l 's to be invariable,

$$-\frac{dU}{dp} = \frac{\rho}{8\pi p^2} \Sigma l \Omega^2 = \frac{8\pi^2}{\rho} \Sigma \frac{v}{\Omega^2}; \dots \quad (101)$$

where, corresponding with any one vortex of strength Ω , there is a volume v of vacuous core comprised within the particular unit volume now considered. The left-hand member of (101) represents evidently the reciprocal of the bulk-modulus of elasticity appertaining to the medium as

a whole, while the velocity \mathbf{V} of compressional waves is given by

$$\mathbf{V}^2 = \frac{1}{8\pi^2 \Sigma(v/\Omega^2)} \cdot \cdot \cdot \cdot (102)$$

These expressions are given merely by way of illustration; they are of course far from being the most general, in the first place because of our assumption that all the vortices may be treated as linear, and in the second place because it has been supposed that the core of any one vortex either consists wholly of rotationally moving liquid or is wholly vacuous.

V. *The Double Nature of Nabla.* By FRANK LAUREN HITCHCOCK, *Kenyon College, Gambier, Ohio, U.S.A.**

THE importance of the operator nabla in Physics must in the nature of things continue to increase. The more we come to regard a force between two material bodies as a manifestation of the properties of the intervening medium, the more any attempt to express facts in mathematical form leads to the use of the potential, and thence to nabla or some equivalent of nabla. Every actual distribution of force has a potential, which is sometimes a scalar, sometimes a vector, in general a quaternion, but in any case yields the force again when acted on by nabla. The same is true of every distribution of velocity in a fluid. Nabla might even be defined, from the point of view of a physicist, as that operation by which any quantity is derived from its potential†.

It is therefore of interest to inquire what are the most convenient methods of working with this operator and applying it to various kinds of functions. As is well known, nabla is a differentiator, which we may, if we wish, express in terms of three ordinary differentiations; and is also a vector, in a mathematical sense, obeying all the laws of vector algebra. We arrive at very direct methods of handling nabla as soon as we distinguish clearly these two properties, and take full advantage of the transformations to which they lead.

For, in the first place, nabla, in its capacity of differentiator, conforms to all the familiar rules for differentiation so long as we do not move it from its place relative to other vectors. Vector multiplication is not commutative, but it is distributive, and, in its original form, it is associative, obeying, therefore,

* Communicated by the Author.

† For a proof by the late Prof. C. J. Joly that every distribution of a quantity has a definite potential, see his appendix to Sir W. R. Hamilton's 'Elements of Quaternions,' 2nd ed., p. 451.

ordinary algebraic principles if the order of vectors in any term is not changed. The expressions thus obtainable with nabla are noteworthy, both for their simplicity and their limitless variety; a few are given below, but are to be regarded as exemplifying the method rather than as constituting any attempt at completeness.

In the second place, as a vector, nabla satisfies the formulæ proved by Sir W. R. Hamilton for vectors and quaternions, so that by using these we may change the order of multipliers and bring nabla next to any desired operand. The variety of results obtainable by this method is even greater than by the other. The transformations which take the place of the commutative law of ordinary algebra are wonderfully many-sided in their application.

2. As a significant example of these two ways of working with nabla, we may take the problem of applying nabla to the product of any two functions. Call these functions q and r . They may be either scalars or vectors; in general they are quaternions. They are assumed to be finite and continuous in the portion of space considered. We then have immediately, by ordinary differentiation,

$$\nabla(qr) = \nabla'q'r + \nabla'qr', \quad . \quad . \quad . \quad (1A)$$

where, on the left, nabla acts on both q and r , but on the right is distributed, as indicated by the accents, according to the familiar rule for the differentiation of a product; that is, in the first term on the right, nabla acts on q but not on r , and in the last term it acts on r but not on q ; the order of multiplication is not changed in either term.

The first term on the right may, by the associative law of multiplication, be rewritten as $\nabla q \cdot r$; this gives

$$\nabla(qr) = \nabla q \cdot r + \nabla'qr', \quad . \quad . \quad . \quad (1B)$$

a convenient equation for some purposes (to which I shall recur below). If, now, we wish to bring ∇ , in the last term, next to the operand r , we may do so by means of the formula

$$K(q_1q) = KqKq_1 \quad . \quad . \quad . \quad (2)$$

true for any two quaternions q_1 and q^* . For this, by operating on both sides with K , becomes

$$q_1q = K(KqKq_1), \quad . \quad . \quad . \quad (3)$$

because $KK = 1$ †. This formula shows that we may invert the order of any two quaternions by introducing the symbol K . With regard to notation, the parentheses on the right may well be omitted, for the first K obviously applies to all that follows it, otherwise the two adjacent K 's would together be

* Hamilton, 'Elements of Quaternions,' 2nd ed., Art. 192.

† *Ibid.* Art. 145.

equal to unity and would not be written down. Thus we have

$$q_1 q = K K q K q_1. \quad . \quad . \quad . \quad . \quad . \quad (3 A)$$

We may now write ∇ in place of q_1 in this formula, for ∇ satisfies all quaternion transformations. Accordingly

$$\begin{aligned} \nabla q &= K K q K \nabla \\ &= -K K q \nabla, \quad . \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

because the conjugate of a vector is its negative*. This result is true whether ∇ acts on q or not, since it depends on the vectorial character of ∇ and has nothing to do with differentiation. On substituting in (1 B) we may drop accents, with the usual understanding that an operator does not act to the left of itself unless accents are inserted to that effect; then

$$\nabla(qr) = \nabla q \cdot r - K K q \nabla \cdot r, \quad . \quad . \quad . \quad (1 C)$$

the period in the last term indicating that the first K acts only as far as ∇ .

3. Operators such as $K K q \nabla$, occurring in the last term of (1 C), are not always entirely comprehended at first sight, on account of their generality in comparison with ordinary differentiation. A point of view from which all such operators may be directly worked with is indicated below (Art. 5). It will suffice here to give three illustrations of the methods by which we may, if we wish, introduce more elementary operators to suit different purposes.

First, suppose we wish to obtain, instead of $K K q \nabla$, a form which shall exhibit as obviously as possible its own relation to ordinary differentiation. We shall then naturally bear in mind that the notation connecting nabla with ordinary differentiation along a direction in space is $S\sigma\nabla$, where σ is any vector along the direction of differentiation. In the present case this vector will be Vq ; hence we shall so arrange the work as to bring in a term of the form $SVq\nabla$. Thus

$$\begin{aligned} \nabla q &= \nabla S q + \nabla V q, & \text{because } S + V &= 1, \\ &= S q \cdot \nabla + K V q \nabla, & \text{by (3),} \\ &= S q \cdot \nabla + (2S - 1) V q \nabla, & \text{because } K = S - V = 2S - 1, \\ &= (S q - V q) \nabla + 2S V q \nabla, & \text{identically,} \\ &= K q \cdot \nabla + 2S V q \nabla, & . \quad . \quad . \quad . \quad . \quad (4 A) \end{aligned}$$

and by substituting in (1 B),

$$\nabla(qr) = \nabla q \cdot r + K q \cdot \nabla r + 2S V q \nabla \cdot r, \quad . \quad (1 D)$$

a useful expansion which is in the form we started to obtain;

* *Ibid.* Art. 144.

the only kind of differentiation implied in the first two terms on the right is the direct application of nabla; the other term implies ordinary differentiation along the direction Vq , or, if we prefer to put $dr = f d\rho$, so that f is the linear function obtained by differentiating r completely, we may rewrite the last term as $-2fVq^*$.

Second, suppose we wish to obtain, instead of the operator $SVq\nabla$, which is the scalar product of a vector and nabla, the corresponding vector operator $VVq\nabla$, useful in the transformation of line and surface integrals†. Arranging the work so as to bring in such a term,

$$\begin{aligned} \nabla q &= Sq \cdot \nabla + KVq\nabla, && \text{as before, second line of (4 A),} \\ &= Sq \cdot \nabla + (1 - 2V)Vq\nabla, && \text{because } K = S - V = 1 - 2V, \\ &= (Sq + Vq)\nabla - 2VVq\nabla, && \text{identically,} \\ &= q\nabla - 2VVq\nabla, && \dots \dots \dots (4 B) \end{aligned}$$

and by substituting in (1 B),

$$\nabla(qr) = \nabla q \cdot r + q\nabla r - 2VVq\nabla \cdot r, \quad \dots \quad (1 E)$$

which is in the form we started to obtain.

Third, suppose we wish to obtain that form of expansion which shall most clearly exhibit the quaternion equation as a generalization for three dimensions of the theory of ordinary complex quantities represented on a plane. We shall then keep nabla always at the left, and, instead of interchanging ∇ and q , in the last term of (1 B), shall invert the order of the two quaternions q and r ; the steps are not unlike those in (4 B):—

$$\begin{aligned} qr &= (Sq + Vq)(Sr + Vr) \\ &= SrSq + Vr \cdot Sq + KVrVq, && \text{scalars being commutative,} \\ &= Sr \cdot q + Vr \cdot Sq + (1 - 2V)VrVq, && \text{because } K = 1 - 2V, \\ &= Sr \cdot q + Vr \cdot (Sq + Vq) - 2VVrVq, && \text{identically,} \\ &= rq - 2VVrVq, && \dots \dots \dots (3 B) \end{aligned}$$

another formula for inverting two quaternions, giving by (1B)

$$\nabla(qr) = \nabla q \cdot r + \nabla r \cdot q - 2\nabla'VVr'Vq, \quad \dots \quad (1 F)$$

which is in the form we started to obtain, for if q and r are

* By putting two vectors for q and r , and by operating with S or V either before or after such a substitution, we may obtain a number of special cases very useful in practice. For another quite different method of obtaining them see Phil. Mag. [6] No. 18, June 1902, p. 580.

† See, for instance, the physical applications in Tait's 'Quaternions,' Art. 497 *et seq.*

in a fixed plane, their axes, or vector parts, are parallel to each other and to the vector parts of their differentials and the last term vanishes. The expansion then reduces to what it would be if ∇ were the symbol of ordinary differentiation and q and r were scalars. That coplanar quaternions obey all the laws of ordinary algebra is well known.

These results may be extended. Thus for three quaternions, q , r , and s , by writing qr for q in (1 F), and s for r ,

$$\begin{aligned}\nabla(qrs) &= \nabla(qr) \cdot s + \nabla s \cdot qr - 2\nabla'VV s'Vqr \\ &= \nabla q \cdot rs + \nabla r \cdot qs + \nabla s \cdot qr - 2\nabla'VV r'Vq \cdot s \\ &\quad - 2\nabla'VV s'Vqr, \quad . \quad (5)\end{aligned}$$

which may be transformed in a multiplicity of ways.

As a special case of (1 F) let $q = r$, then

$$\nabla q^2 = 2\nabla q \cdot q - 2\nabla'VV q'Vq, \quad . \quad . \quad . \quad (6)$$

where the last term does not vanish unless q and dq are coplanar.

4. The result just obtained for the effect of nabla on the square of any quaternion suggests the question whether a simple general method exists for applying nabla to any function of a function. The answer is that, as has already been illustrated, the accents which we apply to ∇ and to the operand may be treated in any way that the symbol d of ordinary differentiation can be treated. Thus (1 A) is true because

$$d(qr) = dq \cdot r + qdr,$$

and we may similarly, in any differential identity, replace d by an accent applied to the letter before which d stands, and write ∇ at the left of the whole: this is the first of the two characteristic properties of nabla mentioned in Art. 1. Therefore, if Fq be any function of q and we write

$$dFq = f dq, \quad . \quad . \quad . \quad . \quad (7)$$

where f is the linear function obtained by differentiating Fq completely, it at once follows that

$$\nabla Fq = \nabla' f q', \quad . \quad . \quad . \quad . \quad (8)$$

which is the fundamental formula for applying nabla to a function of a function. The constituents of f will in general contain unaccented q as well as constants.

As an illustration, let us obtain (6) by this method.

We have

$$\begin{aligned} dq^2 &= dq \cdot q + q dq \\ &= 2dq \cdot q - 2VVdqVq, \end{aligned}$$

by putting $r = dq$ in (3 B), so that by (8)

$$\nabla q^2 = 2\nabla'q'q - 2\nabla'VVq'Vq,$$

which is the same as (6). The method, we repeat, consists in differentiating the given function, arranging the differential in any way that may suit our purpose, and then replacing dq by an accented q' , while ∇ is written at the left of each term.

To obtain the important generalization of (6) where the exponent shall be any constant scalar whatever, we have therefore to differentiate q^n . To do so, we may think of the differentiation as conducted in two steps: first, as if the plane of q were fixed, and UVq constant, second, as if the rest of the quaternion were constant. The sum of the two results will be the complete differential*. Call the first step d_0 and the second d_1 , then in general

$$d = d_0 + d_1. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The result of d_0 on q^n will be the same in form as if q were a scalar, since quaternions of constant plane act like scalars; that is

$$d_0 q^n = d_0 q \cdot nq^{n-1}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The result of d_1 implies no differentiation except $dUVq$, for q^n is coplanar with q itself, UVq^n is the same as $\pm UVq$, and we may write †

$$q^n = Sq^n \pm TVq^n \cdot UVq,$$

whence, differentiating as if UVq alone were variable,

$$d_1 q^n = \pm TVq^n \cdot dUVq, \quad . \quad . \quad . \quad . \quad (11)$$

and by adding (10) and (11)

$$dq^n = d_0 q \cdot nq^{n-1} \pm TVq^n \cdot dUVq,$$

but by (9)

$$\begin{aligned} d_0 q &= dq - d_1 q \\ &= dq - TVq \cdot dUVq \quad . \quad . \quad . \quad . \quad . \quad (12) \end{aligned}$$

by the definition of d_1 , which substituted in the preceding gives

$$dq^n = dq \cdot nq^{n-1} + dUVq \cdot (\pm TVq^n - TVq \cdot nq^{n-1}), \quad . \quad (13)$$

* Hamilton, Art. 329.

† The double sign is of course ambiguous in appearance only. We may, if we prefer, write $+\frac{Vq^n}{UVq}$ instead of $\pm TVq^n$ in equations (11)–(14).

and as the differentials stand at the left of the terms we have immediately, by (8),

$$\nabla q^n = \nabla q \cdot nq^{n-1} + \nabla UVq \cdot (\pm TVq^n - TVq \cdot nq^{n-1}). \quad (14)$$

There is nothing in the reasoning employed which will not apply equally well if, instead of q^n , we take any other function of q containing neither quaternion constants nor any of the selective symbols S, V, &c. ; in other words, if the functional form be any of those contemplated in the Newtonian, as distinguished from the Hamiltonian, differential calculus. For if we do not introduce any vector except Vq into the functional operation, the result can contain no other vector, or scalar multiples, and the vector part, axis, and plane of the function will be parallel to those of q itself. All such functions of a single quaternion q must therefore be commutative with one another. They will also be commutative with the result of d_0 upon them, since that operator also can yield no other vector except scalar multiples of UVq .

It will be convenient to call any such function, agreeing in form with those treated in the usual text-books on differentiation, or, better, any function which we can construct by the employment of scalars only, an *ordinary* function. For example, $\sin(nq)$ is an ordinary function of the quaternion q if n is a scalar, but not if n is a vector or a quaternion.

It will also be convenient to represent such a function by the symbol Q , and to write DQ for the result of differentiating it with regard to q as if q were a scalar. Thus if Q be any ordinary function of q we shall always have

$$\begin{aligned} dQ &= d_0Q + d_1Q \\ &= d_0q \cdot DQ \pm TVQ \cdot dUVq \\ &= dq \cdot DQ + dUVq \cdot (\pm TVQ - TVq \cdot DQ), \end{aligned} \quad (15)$$

of which (14) is evidently a particular case*. We note that the only differentiation to be performed upon Q is that denoted by D , namely ordinary differentiation as if q were a scalar. The differentials dq and $dUVq$ pertain to q only, depend therefore on the distribution of q with respect to the variables of which it may be a function, and have nothing to do with the relation of Q to q . Thus we may, by (15), at once write down the differentials of any of the functions which Hamilton has defined in the *Elements*, or any which we might ourselves define in case we use forms not treated by him (*e.g.* elliptic

* We may write $+\frac{VQ}{UVq}$ instead of $\pm TVQ$ in (15). A similar remark applies to equations (16)–(26).

functions), provided such forms are what were just above called ordinary functions. For example,

$$de^q = dq \cdot e^q + dUVq \cdot (\pm TVe^q - TVq \cdot e^q), \quad . \quad . \quad (16)$$

and similarly

$$d \sin q = dq \cdot \cos q + dUVq \cdot (\pm TV \sin q - TVq \cdot \cos q), \quad (17)$$

and so on. If the plane of q is fixed, so that UVq is constant, it is evident that the right side of (15), or of any special case of it, reduces to its first term. The presence of the term, or terms, in $dUVq$ is due to the fact that the quaternion expansion is valid for space of three dimensions, so that the ordinary formulæ have to be generalized. This is an illustration of the truth referred to in Art. 3 that the Quaternion Calculus is an extension of the system, due to Bueé, Argand, and others, by which a complex quantity is represented on a plane. It is the only such extension which preserves both the distributive and the associative laws.

In (15), or in any special case of it, we may replace d by ∇ , thus,

$$\nabla Q = \nabla q \cdot DQ + \nabla UVq \cdot (\pm TVQ - TVq \cdot DQ), \quad . \quad (18)$$

Q being any ordinary algebraic or transcendental function of the quaternion q . As an instance,

$$\nabla \log q = \nabla q \cdot q^{-1} + \nabla UVq \cdot (\pm TV \log q - TVq \cdot q^{-1}); \quad (19)$$

$TV \log q$ as defined by Hamilton is the angle, or the amplitude, of q .

More generally, if FQ be any quaternionic function of Q , we have

$$dFQ = f dQ,$$

where f is the linear function obtained by differentiating Q . Then by (15)

$$dFQ = f[dq \cdot DQ + dUVq \cdot (\pm TVQ - TVq \cdot DQ)], \quad (20)$$

and by (8)

$$\nabla FQ = \nabla' f[q'DQ + UVq' \cdot (\pm TVQ - TVq \cdot DQ)]. \quad (21)$$

As an example under the last two results, we have, if a and b are any two constant quaternions,

$$d(a \sin bq) = abdq \cdot \cos bq + adUVbq \cdot (\pm TV \sin bq - TVbq \cdot \cos bq), \quad (22)$$

obtained from (20) by putting a for f and bq for q . And by (21),

$$\nabla' a \sin bq' = \nabla' abq' \cos bq + \nabla' aUVbq' \cdot (\pm TV \sin bq - TVbq \cdot \cos bq). \quad (23)$$

In the most general case of a function of a single quaternion q , we may have under F several ordinary functions; call them Q_1, Q_2 , &c., to which will correspond f_1, f_2, \dots , giving

$$dF(Q_1, Q_2, \dots) = f_1 dQ_1 + f_2 dQ_2 + \dots; \quad (24)$$

each of the ' f 's being obtained by differentiating the given function as if only the corresponding Q were variable*. On the right we may substitute the value of each of the differentials from (15).

If we have a function of several quaternions, q, r , &c., the formulæ (15) and (24) will still serve, but the Q 's and the f 's will contain any or all of these quaternions. When we change d for ∇ , we shall accent only the variables which we write in place of the differentials. As an example,

$$d \log qr = d(qr) \cdot (qr)^{-1} + dUVqr \cdot (\pm TV \log qr - TVqr \cdot r^{-1}q^{-1}), \quad (25)$$

and changing to ∇ ,

$$\nabla \log qr = \nabla(qr) \cdot (qr)^{-1} + \nabla' UVq'r'. (\pm TV \log qr - TVqr \cdot r^{-1}q^{-1}), \quad (26)$$

in which we may, if we wish, transform $UVqr$ by the known methods for differentiating a unit vector.

Most of the formulæ of the present section contain, on the right, the differential, or the nabla, of some unit vector. The differential of any unit vector is always at right angles to the unit vector itself†. If the unit vector is normal to a family of surfaces, it is also at right angles to its vector nabla; in which case the unit vector, its nabla, and the new vector obtained by differentiating the unit vector along its own direction, form a rectangular system‡. Thus the term by which the differential, or the nabla, of a quaternion function differs in form from that of a scalar may receive a geometrical interpretation.

5. In Art. 2, above, we had terms containing the operators $SVq\nabla$, $VVq\nabla$, and $KKq\nabla$; and the terms on the right of (23) can, by the methods of Art. 2, be thrown into similar form. These operators are special cases of the more general expression $f\nabla$, where f stands for any linear function; such functions of nabla are of great use, for example in hydrodynamics, and in the study of an anisotropic medium§.

* Hamilton, Art. 329.

† Hamilton, Art. 335.

‡ For the proof of this, with other properties of the rectangular system, see Phil. Mag. [6] vol. iii. p. 581, and iv. p. 187.

§ Maxwell, 'Electricity and Magnetism,' 3rd ed., Art. 101, *h*.

Since in Art. 4 use was made of nabla in its capacity as differentiator alone, we may, wherever nabla occurs in any of the formulæ of that article, write a linear function of nabla instead of nabla itself. Thus instead of (8) we may write

$$f\nabla \cdot Fq = f\nabla'f_1q', \quad . \quad . \quad . \quad . \quad (27)$$

where f_1 is the linear function obtained by differentiating Fq , but f is any linear function whatever. The operator $f\nabla$, since it includes scalar functions of ∇ , includes ordinary differentiation as a very special case. Therefore (27) virtually includes all the results of the last article. The expansions (15) and (24) may be used whenever one or more ordinary algebraic or transcendental functions enter into Fq , which may be any function of q whatever.

The properties of linear functions afford many ways of transforming the right side of (27). For example, Hamilton showed that any linear function of a quaternion q may be expressed as the sum of several terms of the form tqs , where t and s are quaternions*. Therefore the right of (27) may be expressed as the sum of several terms of the form $r\nabla'sq't$, where r , s , and t are quaternion functions which will in general contain the variable q without accent. Each of these terms may then be transformed so as to bring ∇ next to the operand, as in (1 D) and (1 E), or if we prefer, so as to bring ∇ to the left of the whole, as in (1 F). Such methods are still possible even when the expression to be transformed contains the selective operators S , V , or K , for if α and β are any two vectors we might put $2S\alpha\beta = \alpha\beta + \beta\alpha$, and $2V\alpha\beta = \alpha\beta - \beta\alpha$; so that by expanding sufficiently we might always arrive at an expression free from these selective symbols. We have, however, direct methods for handling S and V too fully illustrated by Hamilton and Tait to need exemplification here. No matter what the form of the original term, we may always transform so as to bring ∇ next to any desired factor of the term.

Another general method is to put $dq = f_2d\rho$, where f_2 is the linear function obtained by differentiating q . Instead of (27) we then have

$$f\nabla \cdot Fq = f\nabla'f_1f_2\rho'; \quad . \quad . \quad . \quad . \quad (28)$$

and since it is no longer necessary to separate f_1 and f_2 we may more briefly write

$$f\nabla \cdot Fq = f\nabla'f_3\rho', \quad . \quad . \quad . \quad . \quad (29)$$

in which f_3 is the linear function obtained by differentiating

* 'Elements,' Art. 364.

Fq as a function of ρ ; or it is the product of f_1 and f_2 . We may then, as in the last paragraph, arrive at a sum of terms of the form $r\nabla's\rho't$, but by (4 B)

$$\nabla s = s\nabla - 2VV_s\nabla,$$

whence we bring ∇ next to ρ and have a sum of expressions of the form

$$rs\nabla\rho \cdot t - 2rVV_s\nabla \cdot \rho \cdot t,$$

and because $\nabla\rho = -3$ and $VV_s\nabla \cdot \rho = -2Vs^*$, we obtain a sum of expressions of the form

$$-3rst + 4rVs \cdot t,$$

and may write in general

$$f\nabla \cdot Fq = \Sigma(-3rst + 4rVs \cdot t) \dots (30)$$

in which r, s , and t are the quaternions obtained by expanding the right side of (28) or (29). The result (30), apart from practical utility, has a certain interest by showing (what is well known), that we may find the nabla of any quantity (and the more general $f\nabla$), without ever introducing axes of reference, whether rectangular or not; and by means of one differentiation, not three,—a fact which might be commended to the attention of those who think that quaternions are mere abbreviations for Cartesian expressions.

We may, however, if we choose, introduce i, j , and k into the work. The original definition of ∇ gives, by (29),

$$f\nabla \cdot Fq = f_i \cdot f_3 i + f_j \cdot f_3 j + f_k \cdot f_3 k, \dots (31)$$

and the right side is an invariant of the two linear functions f and f_3 . The late C. J. Joly showed how such invariants may be calculated, given the functions f and f_3 †.

A fourth method, sometimes the shortest in practice, is to write the expression $f\nabla' \cdot f_3\rho'$ as a sum of terms in whatever way may be most convenient, then calculate the invariants of each term separately, and add. The advantage of this method is that it is always possible to express a linear function as the sum of a small number of standard forms, whose invariants are easily remembered; as, for example, that used above, $VV_s\nabla \cdot \rho = -2Vs$, or the very common form $\nabla'\beta S\alpha\rho' = -\alpha\beta$.

To sum up, we may find the result of applying any linear function of nabla to any given function by performing a

* Tait, 'Quaternions,' Art. 145-6.

† Appendix to Hamilton's 'Elements,' 2nd ed., chapter v.; and Trans. R. I. A. xxx. p. 709.

single differentiation upon the given function, and transforming to suit the purpose in view. The fundamental formula for the work is (27), which contains the others.

6. A full discussion of the general case, when $f\nabla$ is any function of ∇ , not necessarily linear, would exceed the limits of this paper. A few illustrations of the meaning and use of such operators must suffice.

The simplest function of the second order in nabla is the familiar ∇^2 . Another is $S\sigma\nabla S\tau\nabla$, which implies successive differentiation of the operand along two different directions σ and τ . The most general scalar function homogeneous and of the second order in nabla is of the form $S\nabla\phi\nabla$, where ϕ is any self-conjugate linear vector function.

Vector functions of nabla depend on the properties of vector functions of a vector. No general theory has been worked out except for linear functions. In a future paper I hope to show that a vector function homogeneous of the second degree in a vector λ may be put in the form

$$V\phi\lambda\theta\lambda + \lambda S\alpha\lambda$$

where ϕ and θ are linear vector functions and α is a vector. Then the most general vector function homogeneous of the second order in nabla would be of the same form, writing ∇ for λ .

We may, however, write down at once equations like (27) for any differential operators whatever. Thus if d' and d'' are symbols of two independent differentiations, we shall have

$$d'd''Fq = f_1(d'q, d''q), \quad . \quad . \quad . \quad (32)$$

in which f_1 will be a function linear in each of the two differentials. Then if $f(\nabla', \nabla'')$ is any operator homogeneous and of the second order in nabla, the differential nature of nabla enables us to write

$$f(\nabla', \nabla'') \cdot Fq = f(\nabla', \nabla'') \cdot f_1(q', q''), \quad . \quad . \quad (33)$$

in which ∇' acts on q' and ∇'' on q'' . Since d' and d'' are independent, f_1 is symmetrical in its two operands, a property not in general possessed by f in the two nablas. One or both nablas will usually act on the constituents of f .

7. The commutation of these differential operators is a matter of importance both in Physics and in Mathematics. Tait gives a proof of the celebrated theorem that vortex motion cannot originate in a frictionless medium, by commuting $\frac{\delta}{\delta t}$ with $V\nabla^*$. Commuting two ordinary differentiations along different directions gives rise to the so-called "Klammerausdruck," due to Jacobi, and important

* 'Quaternions,' 3rd ed., Art. 513.

in Lie's Theory of Transformation Groups. If we have any two linear functions of ∇ , which we may call $f\nabla$ and $g\nabla$, they will be amenable to commutation by (3 B) whether or not either of them acts on the constituents of the other, but usually each will act on all that follows it; if so, we have, by Hamilton, Art. 329,

$$f\nabla.g\nabla.q = f\nabla'.g'\nabla.q + f\nabla'.g\nabla'.q', \quad . \quad . \quad (34)$$

and with the operators in inverted order,

$$g\nabla.f\nabla.q = g\nabla'.f'\nabla.q + g\nabla'.f\nabla'.q', \quad . \quad . \quad (35)$$

in both of which identities ∇ , if without accent, acts on all that follows it in the same term, but, if accented, acts only on the accented factor. The terms last on the right of (34) and (35) are amenable to commutation by (3 B), whence the difference of these two terms is $2VVf\nabla'Vg\nabla'.q'$ and by subtracting (35) from (34),

$$(f\nabla.g\nabla.q - g\nabla.f\nabla.q) = f\nabla'.g'\nabla.q - g\nabla'.f'\nabla.q + 2VVf\nabla'Vg\nabla'.q'; \quad . \quad (36)$$

in the last term both ∇ 's act on q only, but they both act on it, making this term one of the second order. The last term vanishes (a) if one or both of the functions f and g is only a scalar function, or (b) if the vector parts of these two functions are parallel. The "Klammerausdruck" of Jacobi comes, of course, under (a). We might, by using three operators, work out an analogue of the "Jacobian identity."

8. Finally, we may, in the general formulæ (33) and (36), and in the whole of Art. 4, write $\nabla + \frac{d}{dt}$ instead of ∇ , where $\frac{d}{dt}$ means taking the partial derivative with regard to some scalar variable, usually the time t . For these results are consequences of the differential character of nabla, and are equally valid if we put the quaternion $\nabla + \frac{d}{dt}$ for the vector ∇ . If we do so, we shall best follow Prof. Kimura* in regarding $\nabla + \frac{d}{dt}$ as an extended nabla, and represent it by a single symbol. Its quaternion character makes it amenable to any of the processes used in Art. 3 and Art. 5 of this paper, and the results will differ only slightly from those there obtained. Naturally, in using the extended operator, we imply that the operand is variable both in space and in time, or in some other scalar.

* Annals of Math. x. p. 127.

VI. *The Absorption of the Different Types of Beta Rays, together with a Study of the Secondary Rays excited by them.*
By V. E. POUND, M.A., University of Toronto*.

[Plate II.]

I. INTRODUCTION.

IN a paper in the Phil. Mag. of July 1907, Prof. Mackenzie gives an account of some observations which he made on the secondary radiation issuing from each side of plates of lead upon which a pencil of β rays was allowed to fall. Using plates of increasing thickness, he found that the secondary radiation issuing from the side of the plate upon which the β rays fell, gradually increased in intensity and reached a maximum value when a plate 2 mm. in thickness was used. With plates of still greater thickness, this secondary radiation remained constant in intensity. He obtained, however, an entirely different result on investigating the secondary radiation from the back of the plate upon which the β radiation was allowed to fall. Under these conditions, the secondary radiation fell off very slowly as the thickness of the plates increased, and was still quite measurable with plates of lead 15 mms. in thickness.

In arriving at these results Mackenzie† investigated the secondary radiation issuing from each side of the plates, first, when both β and γ rays were allowed to fall on them, and, second, when γ rays alone were allowed to fall on the plates; and the results quoted by him, and ascribed by him to the action of the β rays, were obtained by subtracting the effects due to the γ rays alone from those due to the combined β and γ radiations.

With the arrangement he adopted, it was possible that in cutting off the β rays in order to study the effect of the γ radiation alone, he also cut off a greater proportion of the latter than he estimated. If this were so, it would result in ascribing to the β radiation a part of the secondary radiation which properly should have been ascribed to the γ rays.

In view of the importance of his results in their relation to theories of secondary radiation now being put forward by Bragg‡ and others, it was thought well to make a more extended examination of the secondary radiations excited by both β and by γ rays; and in the following paper an account

* Communicated by Prof. J. C. McLennan. Read before the Royal Society of Canada, May 16, 1908.

† Phil. Mag. July 1907.

‡ Phil. Mag. May 1908.

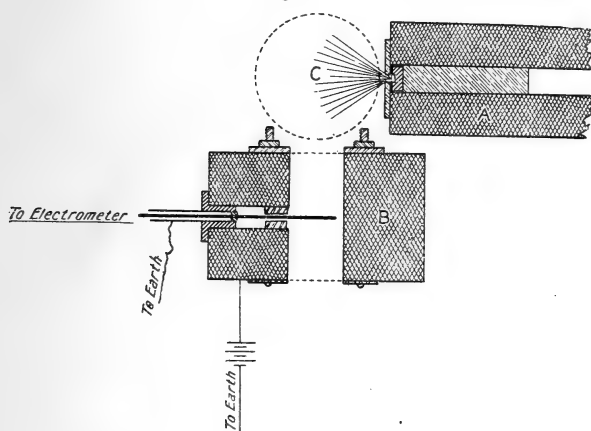
is given of some experiments in which the secondary radiation both from the back and front of metal plates was studied when these were traversed by γ rays alone, and also when pencils of β rays of different types were allowed to fall on them.

In differentiating the effects due to the various types of rays, the action of each was ascertained by deflecting pencils of each type into an ionization-chamber away from the others by means of a magnetic field.

II. APPARATUS.

The arrangement adopted is shown in fig. 1. The receptacle for holding the radium was a lead cylinder A with walls 4 cms. thick. One end of the cylinder was covered by

Fig. 1.



a brass plate 2.5 mms. thick, in the centre of which was a hole 4.5 mms. in diameter. The capsule holding the radium was held close against this plate in such a position that the rays from the radium on issuing, passed between the poles of a powerful electromagnet. Immediately beneath the pole-pieces of the electromagnet was the ionization-chamber B. It also was made of lead in the form of a cylinder 6.7 cms. long, with walls 4.6 cms. in thickness, the ionization-chamber proper being 4.7 cms. in diameter. At the top and bottom were brass rings to hold different thicknesses of selected absorbing materials over the ends of the chamber.

A properly screened and insulated electrode was inserted

into the ionizing-chamber and connected to a Dolezalek electrometer by means of which saturation currents were measured. The sensibility of the electrometer was such that a potential-difference of one volt between the quadrants produced a deflexion of 625 mms. on a scale about one metre distant from the needle. It was found that a potential of 240 volts applied to the ionization-chamber was always amply sufficient to give the saturation current.

III. EXPERIMENTS ON THE ABSORPTION AND REFLEXION OF β RAYS BY TINFOIL.

A. Measurements on Transmitted Rays.

In these experiments the β rays from the radium, on coming between the poles of the magnet, were deflected either downwards or upwards according to the direction of the field between the poles. As the capsule containing the radium was covered by a thin sheet of mica the α rays were largely absorbed, so that the issuing beam contained only β and γ rays, which could easily be separated by the magnetic field in the manner indicated. Readings were taken of the saturation currents in the ionization-chamber as the current through the electromagnet was changed by small increments from 0 to about 28 amperes.

A series of measurements was made with a number of different thicknesses of absorbing layers of tinfoil over the top of the ionization-chamber, and with the bottom of the chamber closed by a thin sheet of aluminium-foil 0.0065 mm. in thickness.

Before making these, however, a set of readings was taken without any metallic covering over the opening at the top of the ionizing-chamber, and with only the single sheet of aluminium-foil over the opening at the bottom. In taking these the magnetic field was first applied in such a direction as to deflect the β rays down into the chamber, and observations were made on the saturation currents corresponding to various field-strengths. The field was then reversed, and a second set of readings taken as the β rays were gradually deflected upwards and away from the chamber. Both sets of readings are given in column I. of Tables I. and II. respectively, and curves representing them are shown in fig. 2 (Pl. II.). From these curves it is seen that as the β rays were deflected down into the chamber by the magnetic field, the ionization in the chamber rapidly increased to a maximum value, and then decreased as the different pencils of rays were swept past by the increasing magnetic fields. It is seen,

TABLE I.—Ionization by β transmitted Secondary Rays from Tinfoil with Primary Rays reflected down into Chamber.

Current in magnet (amperes).	Saturation Currents (arbitrary scale) with different thicknesses of Tinfoil.							
	I. mm. 0.00	II. mm. 0.0196	III. mm. 0.0784	IV. mm. 0.1568	V. mm. 0.3136	VI. mm. 1.254	VII. mm. 1.96	VIII. mm. 3.136
0.0	46.6	42.3	27.3	23.3	19.4	16.25	14.59	13.08
0.2	51.5	53.9	29.7					
0.45	128.4	84.2						
0.85	292.0	196.5	52.0	27.0	20.3	...	14.51	12.87
1.15	348.5	294.8	86.0					
1.65	360.0	343.2	152.0	62.6	25.0	12.73
2.2	...	327.3	221.1	115	...	16.19		
2.5	305.0	45.8	16.19	...	12.66
3.2	170.2	...	161.9	14.44	
4.0	...	149.3	...	97.4	62.8	16.13	...	12.61
5.0	96.9	79.0	57.0	16.48	14.35	12.45
6.6	...	81.1	69.1	...	46.7	...	14.21	
7.0	62.5	73.4	...	58.5	...	16.43	...	12.29
10.0	34.0	50.7	47.1	45.2	34.5	16.22	14.02	12.28
15.0	20.8	35.4	29.8	31.5	25.7	15.9	13.96	12.22
20.0	16.0	27.5	26.5	25.4	22.5	15.64	13.85	
25.0	20.0	15.51	13.82	
27.5	...	23.0	22.2	22.0	12.1

TABLE II.—Ionization by β transmitted Secondary Rays from Tinfoil with Primary Rays deflected up from Chamber.

Current in magnet.	Saturation Current (arbitrary scale) for different thicknesses of Tinfoil.							
	I. mm. 0.0	II. mm. 0.0196	III. mm. 0.0784	IV. mm. 0.1568	V. mm. 0.3136	VI. mm. 1.254	VII. mm. 1.96	VIII. mm. 3.136
0.0	46.6	41.7	26.4	22.1	19.1	16.25	14.55	12.82
0.7	14.7							
0.85	...	23.2	20.9	20.1	18.3	...	14.34	
1.45	10.5							
1.70	...		19.5	19.0	18.0	12.56
1.80	...	18.4	15.84		
2.5	17.9	15.84	...	12.52
3.8	7.6	17.3	17.9	18.2	17.7	15.73	...	12.44
5.0	17.8	17.9	17.6	15.68	...	12.4
6.6	7.41	...	17.5	17.8	17.3	15.58	14.15	12.36
7.0	...	16.9						
10.0	...	16.8	17.2	15.21	14.09	12.26
12.0	17.9	17.6				
15.0	7.23	16.6	17.9	...	17.0	15.29	13.9	12.10
17.5								
21.0	7.14	16.3	...	17.3	16.9	15.17	13.87	12.06
26.5	...	16.2	17.6	17.2	16.8	...	13.79	

also, that when the β rays were deflected upwards and away from the chamber by gradually increasing magnetic fields, the corresponding saturation currents decreased rapidly until a constant limiting value was reached.

As already stated, similar sets of readings were taken for different thicknesses of absorbing layers of tinfoil over the top of the chamber. In columns II., III., IV., and V. of Table I. are given the results obtained with layers 0.0196 mm., 0.0784 mm., 0.1568 mm., and 0.3136 mm. in thickness respectively, and curves A, B, C, and D, corresponding to the results given in these columns, are shown in fig. 3 (Pl. II.).

Here again, it will be seen, when the β rays were deflected downwards, that with each absorbing layer the saturation current passed through a maximum value. It will be seen, too, that the maximum saturation current fell away as the absorbing layer was increased, and further, that as the thickness of the layers was increased it required a stronger and stronger field to produce the maximum ionization.

The explanation of these results is found in the fact that the β rays issued from the radium in a number of approximately homogeneous sheaves or pencils possessing a maximum intensity in a direction at right angles to the axis of the ionizing-chamber. On applying the magnetic fields, these sheaves or pencils would undergo different degrees of deflexion, those of high velocity being less affected by the field than the more slowly moving ones.

As the rays from a sheaf of low velocity would enter the chamber first, the ionization would increase and reach a maximum when the axis of this sheaf of rays coincided with the axis of the ionizing-chamber. Still higher fields would deflect the slow moving rays past the opening of the ionizing-chamber and introduce others possessing still higher velocities. Inasmuch as Bragg* and others have shown that the ionizing power of β rays of high velocity is not so great as that of those moving more slowly, it follows that with increasing magnetic fields the ionizing power of the rays introduced would be less than that of the rays cut out, and hence a drop in the ionization values would occur. This drop in the conductivity would continue until ultimately all the deflectable β rays were swept past the chamber. As the layers of tinfoil were gradually increased in thickness, the more slowly moving β rays would be absorbed, and the first effective sheaves transmitted would consist of rays possessing higher and higher penetrability, and consequently of rays with less and less ionizing power. It follows, then, that while a maximum

* Bragg, Phil. Mag. Oct. 1907.

conductivity would be obtained with each thickness of tin-foil, the value of the maximum would decrease with the thickness of the absorbing layer. It is evident, too, since with increasing thicknesses the first effective sheaves of transmitted rays would possess higher and higher velocities, that the field required to deflect the axes of these different sheaves into coincidence with the axis of the chamber would increase. Hence the maximum conductivities, when absorbing layers of increasing thicknesses were used, would be obtained by fields excited by currents of greater and greater intensity, and this, as the curves A, B, C, and D show, is actually what happened.

The numbers corresponding to the saturation currents obtained with different absorbing layers when the rays were deflected upwards and away from the chamber by the magnetic fields, are given in columns II., III., IV., and V. of Table II., and curves representing them are shown in fig. 4. From these it will be seen that with each absorbing layer the ionization fell away as the rays were deflected upwards, and soon reached a value which was constant, and which represented the natural conductivity of the air in the chamber, together with that impressed upon it by the undeviable rays from the radium and by the secondary rays which they excited.

These limiting curves, it will be seen, exhibit an effect already pointed out and emphasized by Mackenzie * and others, that when the thickness of a plate or wall upon which γ rays are allowed to fall is gradually increased, the gain in ionization at the back of the plate from the secondary radiation is at first greater than the loss produced by the absorption of the primary rays. This result is well exemplified by the curves A', B', C', and D', which correspond to absorbing layers of increasing thicknesses, and which show that the limiting value of the ordinate of B' is greater than that of A', that of C' is equal to that of B', and that of D' is again less than that of C'.

In addition to the measurements just described, others were taken for magnetic fields in both directions with absorbing layers 1.254 mms., 1.96 mms., and 3.136 mms. in thickness, and the results of these are recorded in columns VI., VII., and VIII. of Table I. The curves E, F, and G, fig. 5, were plotted from the numbers in these columns, and represent the conductivities obtained with fields which deflected the β rays down towards the chamber. The numbers corresponding to the saturation currents obtained with

* Mackenzie, *Phil. Mag.* July 1907.

different absorbing layers when the rays were deflected upwards and away from the chamber, are given in columns VI., VII., and VIII. of Table II., and curves E', F', and G', which were drawn from these numbers, are shown in fig. 5, and represent the conductivity when the rays were deflected in the opposite direction.

The short sharp rise in the curve E shows that with tinfoil 1.254 mms. in thickness the β rays were still able to penetrate the absorbing layer. A slight rise, as can be seen from the figure, occurs in the curve F, but with the curve G no evidence exists of any rise in conductivity. This curve, moreover, coincides with the curve G', which is drawn from values of the ionization obtained when the rays were deflected upwards; and this coincidence of the two curves G and G' shows that with the absorbing layer with which the results illustrated by those curves were obtained, a thickness of tinfoil was finally reached which could not be penetrated by the β rays, and by the secondary rays which were produced by them in the metal.

In order to find the precise thickness of tinfoil necessary to stop all the β and β secondary radiations, a curve shown in fig. 6 was plotted, taking as ordinates the ionization in the chamber due to the maximum β and β secondary rays for each thickness of tinfoil, and as abscissæ the thickness of the tinfoil screen with which each maximum was obtained. The maximum β and β secondary ionization for each thickness was determined in the following manner. Taking the results for a particular thickness, the limiting value of the ordinate of the curve drawn for a deflexion of the β rays upward was subtracted from the maximum value of the ordinate of the curve drawn for deflexions of the β rays downwards. Inasmuch as the limiting value of the ordinate of the former curve represented the ionization in the chamber due to γ rays and γ secondary together with that due to natural causes, and the maximum value of the ordinate of the latter, the ionization due to the maximum β and β secondary, γ and γ secondary radiations with that due to natural causes, the difference gave the ionization due to maximum β and β secondary ionization for the particular thickness of tinfoil over the top of the ionization-chamber.

In Table III. there is collected and given in row I. the maximum saturation currents in the chamber due to the β and β secondary, γ and γ secondary radiations, and that due to natural causes, for the screens of different thicknesses of tinfoil; in row II. the saturation-currents due to the γ and γ secondary radiations and that due to natural causes, and

in row III. the deduced maximum ionization due to the β and β secondary radiations for the same screens of tinfoil sheets.

TABLE III.

Remarks.	Saturation Current.									
Thickness of tin over the top of chamber. }	mm. 0.0	mm. 0.0196	mm. 0.039	mm. 0.0784	mm. 0.1568	mm. 0.3136	mm. 1.254	mm. 1.96	mm. 3.136	
I. Max. $\beta + \beta$ secondary + $\gamma + \gamma$ secondary + natural ionization. }	360.0	343.2	287.5	230.0	113.8	67.8	17.2	14.4	12.37	
II. $\gamma + \gamma$ sec- ondary + natu- ral ionization. }	7.37	16.9	17.8	17.7	17.7	17.6	15.4	14.1	12.37	
III. $\beta + \beta$ secondary ionization. }	352.63	326.3	269.7	212.3	96.1	50.2	1.8	0.3	0.0	

The ordinates of the curve in fig. 6 are the values of the maximum β and β secondary ionizations taken from row III. of the above table. The curve, as will be seen, is drawn with an initial rise, although no values were obtained from which the position of the highest point could be determined. Some observations to be given later, however, on the determination of the maximum thickness of aluminium necessary to absorb all the β and β secondary radiations, show clearly that for aluminium the ionization in the chamber due to the β rays rose and fell as the thickness of aluminium was increased. The inference was therefore drawn that for all metals this rise and fall in the conductivity due to β rays striking a wall of the chamber would occur, and would be made manifest if thin enough sheets of the metal were used.

From the regular manner in which the curve in fig. 6 falls away, it is clear that in the experiments with tinfoil the thickest screen used was amply sufficient to absorb all the β rays, and the secondary rays excited by them. An examination of the curve, moreover, makes it evident that even a thickness of 2.5 millimetres of tinfoil was amply sufficient for that purpose.

B. Measurements on Reflected Rays.

A series of measurements was also made on the secondary radiation produced at the front side of sheets of tinfoil when β and γ rays were allowed to fall on them. One layer of aluminium-foil $\cdot 0065$ mm. in thickness was placed over the opening at the top of the chamber, and layers of tinfoil of increasing thickness were placed over the opening at the bottom. In these experiments, the capsule containing the radium bromide was placed vertically above the ionization-chamber, so that the rays, after passing between the poles of the magnet, impinged directly on the thin sheet of aluminium-foil forming the upper wall of the ionization-chamber, and after passing through it, traversed the air in the chamber and then impinged on the tinfoil at the bottom. As the magnet was excited by increasing currents, the β rays were more and more deflected until all were swept aside by the field and γ rays alone entered the chamber.

TABLE IV.

Ionization by β reflected Secondary Rays from Tinfoil.

Current in magnet.	Saturation Currents (arbitrary scale). Different thicknesses of Tinfoil.							
	I. mm. 0.0	II. mm. 0.0081	III. mm. 0.0196	IV. mm. 0.039	V. mm. 0.0784	VI. mm. 0.1568	VII. mm. 0.6272	VIII. mm. 1.96
0.0	1086	1121.9	1165	1198	1225	1230	1234	1230
0.85	...	826.4	858.2	864.6	881.2	891.4	898.4	920
0.9	821.4							
2.5	267.4	230.3	224.1	294.8	225.7	226.3	232.1	250.2
4.5	103.5	104.5	95.8	98.2	91.2	95	95.0	86.4
6.75	68.0	72.3	64.9	64.9	57.5	61.8	62.5	57.3
10.0	58.9	61.4	53.9	55.4	47.6	51.5	50.8	47.7
15.0	54.2	54.7	49.1	48.7	43.9	47.3	46.7	43.5
22.0	51.8	50.4	46.3	...	43.4	44.9	44.9	41.0
23.5	46.0
28.5	...	48.5	44.7	...	42.6	43.2	43.0	...
29.5	44.7

In column II. Table IV. is given a typical set of these measurements, and a typical curve plotted from them is shown in fig. 7. The values obtained with the complete set of reflectors used are recorded in columns I. to VIII. of Table IV.: and it may be seen from the curve in fig. 7 that after a field corresponding to ten amperes was exceeded the ionization approached a limiting value which indicated that

for magnetic fields excited by currents of ten amperes and greater, the β rays were all turned aside, and the γ rays alone were left to enter the chamber. The maximum ionization due to the β and β secondary, γ and γ secondary, and that due to natural causes is given by the ordinate of the initial point of this curve. With the interpretation given above the ionization due to the γ , γ secondary, and to any radiations from the metal forming the walls of the chamber, may be taken to be represented by the point on the curve corresponding to the highest field. The difference between the values of these two ionizations gives a value for the maximum conductivity impressed upon the air by the β rays, and by the secondary radiations excited by them in the tinfoil.

TABLE V.

Thickness of Tinfoil over bottom of chamber.	Max. $\beta + \beta$ secondary + $\gamma + \gamma$ secondary + natural ionization.	$\gamma + \gamma$ secondary + natural ionization.	Max. $\beta + \beta$ secondary ionization.
mm.			
0.0000	1086	51.8	1034.2
0.0081	1121	47.5	1073.5
0.0196	1165	44.7	1116.3
0.039	1193	44.7	1153.3
0.0784	1225	42.6	1182.4
0.1568	1230	43.2	1186.8
0.6272	1234	43.2	1190.8
1.96	1230	41.5	1188.5

In Table V. is given the deduced values of the maximum ionizations which were due to β and β secondary rays from similar sets of measurements for different thicknesses of tinfoil at the bottom of the chamber. The curve drawn in fig. 8 is plotted with ordinates representing the values of these maximum β and β secondary ionizations, as recorded in the fourth column of this table, and with abscissæ representing the corresponding thicknesses of tinfoil. From this curve it is clear that the maximum conductivities produced by the β and the reflected β secondary rays reached a limiting value when the tinfoil sheets attained a thickness of 0.24 mm., and for still greater thicknesses remained constant.

Summarizing all the results obtained with tinfoil it would then appear:—That when β rays from radium are allowed to impinge on sheets of tinfoil, a maximum reflected secondary radiation is obtained when the tinfoil attains a thickness of 0.24 mm.; and further, that a thickness of 2.5 mms. of tinfoil

is sufficient to absorb not only the transmitted secondary rays excited by β rays, but also the whole of the primary radiation itself.

This result, however, while giving definite information regarding a lower limit to the thickness of tin-foil requisite to absorb primary β rays, gives only an upper limit to the thickness necessary to absorb the transmitted secondary radiations produced by such rays. In order to obtain a lower limit to the thickness of tin-foil required to absorb the transmitted β secondary radiation alone which is excited by β rays, it would be necessary to modify considerably the arrangement of the apparatus used in making these measurements.

IV. EXPERIMENTS ON THE ABSORPTION AND DEFLEXION OF β RAYS BY LEAD.

A. Measurements on the Transmitted Rays through Lead.

Experiments were conducted with lead in an exactly similar manner to those on the transmitted rays through tin-foil, in order to find the minimum thickness of lead necessary to prevent the emergence of any β or β secondary radiations from the far side of a plate upon which the primary β rays of radium fell. The radium bromide was placed as in fig. 1, and sets of readings were taken of the ionization in the chamber,—1st, with the top of the chamber open, and, 2nd, with it covered by lead-foil of varying thicknesses, the bottom of the chamber being always closed by a sheet of aluminium-foil .0065 mm. in thickness. As before, these readings were taken as the β rays were deflected downwards into the ionization-chamber, and upwards and away from it by different magnetic fields.

The sets of readings taken with the opening at the top of the chamber uncovered, and also covered with the lead-foil screens 0.241 mm., 0.482 mm., 0.723 mm., and 0.964 mm. in thickness respectively, are given in columns II., III., IV., and V. of Table VI. From the values of the ionizations given in columns III., IV., and V. of this table the curves A, B, and C (fig. 9) were drawn. The curves A', B', and C', also shown in fig. 9, were plotted from the numbers in columns III., IV., and V. of Table VII. Both sets of curves (it will be seen) are of the same type as those for the tin-foil which were fully discussed in Section II. A.

TABLE VI.

Ionization by β transmitted Secondary Rays from Lead-foil, with Primary Rays deflected down into Chamber.

Current in magnet (amperes).	Saturation Current (arbitrary scale), with different thicknesses of Lead-foil.				
	I. mm. 0.0	II. mm. 0.241	III. mm. 0.482	IV. mm. 0.723	V. mm. 0.964
0.0	56.5	21.1	19.6	16.9	16.5
0.2	95.8				
0.85	294.8	21.9	16.3
1.35	383.3				
1.70	359.3				
2.5	302.6	30.0	20.4	16.9	16.2
4.0	...	48.7	22.6	...	15.9
4.5	118.5	17.0	
5.0	...	50.4	24.8	17.4	
6.5	68.0	17.34	
7.0	25.4	...	15.8
10.0	41.5	35.4	24.0	17.03	15.7
15.0	26.4	27.9	21.6	16.45	15.4
22.5	18.0	23.3	19.5	16.25	15.4

TABLE VII.

Ionization by β transmitted Secondary Rays from Lead-foil, with Primary Rays deflected up from Chamber.

Current in magnet (amperes).	Saturation Current (arbitrary scale) for different thicknesses of Lead-foil.				
	I. mm. 0.0	II. mm. 0.241	III. mm. 0.482	IV. mm. 0.723	V. mm. 0.964
0	56.5	21.1	19.5	16.9	16.5
0.8	18.5	20.1	16.3
2.5	9.4	20.0	19.0	16.7	16.1
4.5	...	19.7	18.8	16.59	16.0
6.5	8.6	19.5			
7.0	18.5	16.71	15.8
10.0	...	19.1	18.5	...	15.7
15.0	8.1	19.0	18.1	16.15	15.5
22.5	8.1	19.1	18.0	16.28	15.4

The curve B, drawn for a thickness of .723 mm. of lead, indicates that β rays which were deflected by a field corresponding to about 6 amperes, penetrated this thickness of lead, while the coincidence of the curves C and C' drawn from the values corresponding both to the upward and downward deflexions of the β rays when .964 mm. was the thickness of the lead screen, shows clearly that the β and also the β secondary rays could not pass through this thickness of lead.

As is fully explained in Section III. A, the maximum value of the conductivities in the chamber due to β and β secondary rays for the different thicknesses of the screens can readily be deduced from the tables given above. These deduced values are given in row III. of Table VIII., and a curve representing them is shown in fig. 10. From the

TABLE VIII.

Remarks.	Saturation Current.				
Thickness of lead over } the top of chamber. }	mm. 0	mm. .241	mm. .482	mm. .723	mm. .964
I. Max. $\beta + \beta$ secondary + $\gamma + \gamma$ secondary + natural ionization. }	383.2	50.9	25.8	17.5	15.6
II. $\gamma + \gamma$ secondary + natural ionization. }	8.4	19.4	17.7	16.5	15.6
III. Max. $\beta + \beta$ secon- dary ionization. }	374.8	31.5	8.1	1.0	0

curve it is evident that a screen of lead .9 mm. in thickness completely absorbed all of the β and the β secondary radiations excited in the lead, including the most penetrating.

B. Measurements on Reflected Rays from Lead.

In this set of measurements the arrangement of the apparatus was the same as when the measurements on the reflected rays from tin were taken, the radium being placed vertically above the ionization-chamber. Different thicknesses of lead were placed over the opening at the bottom, while the single sheet of aluminium-foil, .0065 mm. in thickness, covered the top. As before, the saturation-currents in the chamber were taken as the magnetic field deflected the β rays farther and farther from the opening of

the chamber ; and the values of the saturation-currents are given in columns I., II., III., IV., V., and VI. of Table IX.

TABLE IX.

Ionization by β reflected Secondary Rays from Lead-foil.

Current in magnet (amperes).	Saturation Current (arbitrary scale), with different thicknesses of Lead-foil.					
	I. mm. 0.0	II. mm. 0.066	III. mm. 0.093	IV. mm. 0.116	V. mm. 0.241	VI. mm. 0.964
0.0	1161.0	1315.0	1337.0	1349.0	1349.0	1340.0
0.8	799.0	...	809.8	...	818.5	801.3
2.5	146.0	...	147.8	...	139.8	145.0
4.5	64.9	...	59.5	...	65.3	63.8
6.5	58.9	...	54.7	...	54.2	55.5
10.0	57.8	...	56.3	58.3	56.1	...
15.0	58.9	57.8	58.6	58.0	...	56.6
22.0	57.8	56.9	57.2	56.6	56.9	56.9
28.0	56.9	56.1	56.3	56.0	55.3	55.8

From these it will be seen that with the lead reflectors of different thicknesses, the saturation currents were practically the same when magnetic fields of sufficient strength to deflect all the β rays were applied. In order to ascertain the maximum ionization for the various lead reflectors due to the β ray effect, the mean of the readings obtained with the high fields was taken as representing the conductivity due to γ radiations, that due to the secondary radiations excited by these in the reflectors, and also that due to the so-called natural ionization. This mean was subtracted from the maximum ionization obtained with each of the reflectors before the application of a magnetic field, and the differences, which are recorded in column IV. of Table X.

TABLE X.

Thickness of lead over bottom of chamber.	Max. $\beta + \beta$ secondary + $\gamma + \gamma$ secondary + natural ionization.	$\gamma + \gamma$ secondary + natural ionization.	Max. $\beta + \beta$ secondary ionization.
mm.			
0.0	1161	57.8	1103.2
0.066	1315	56.9	1258.1
0.093	1337	56.7	1280.3
0.116	1349	57.3	1291.7
0.241	1349	55.6	1293.4
0.964	1340	56.2	1283.8

and represented graphically by the curve A in fig. 11, were taken to represent the ionizations produced in the chamber by the primary β rays and by the secondary rays excited by them in the lead reflectors. From a consideration of these values and of the form of the curve in fig. 11, it is evident that a maximum secondary radiation, due to the impact of β rays on the lead reflectors, was obtained with a thickness of .16 mm. of this metal.

From these results, then, it is clear that the secondary radiation emitted by the front side of a lead plate upon which the β rays from radium fall, do not come from a depth of the metal greater than .16 mm. It is also established by the results that a plate of lead .9 mm. in thickness will completely absorb all the primary β rays from radium, as well as all the secondary radiation excited by these rays in the lead plate.

V. EXPERIMENTS ON THE ABSORPTION AND REFLEXION OF β RAYS BY ALUMINIUM-FOIL.

A. Measurements on Transmitted Rays.

A series of readings was also made with a number of different thicknesses of absorbing layers of aluminium-foil over the top of the chamber, in order to find the minimum thickness of aluminium necessary to stop the β rays. The bottom of the chamber was closed by the same sheet of aluminium-foil, .0065 mm. thick, used throughout these experiments. As before, the first series of measurements was taken without any cover over the top of the chamber; and this series is given in column I. of Table XI. The results obtained with layers .0065 mm., .28 mm., 1.184 mm., 3.41 mm., 4.73 mm., and 8.14 mm. in thickness respectively, are given in columns II., III., IV., V., VI., VII. of the same Table.

The curves A, B, C, D, and E, shown in figs. 12 and 13, are plotted from the results given in columns I., II., IV., VI., and VII. of Table XI.; and curves D' and E', fig. 13, from the results given in columns VI. and VII. of Table XII.

On comparing the results obtained when there was no metallic covering over the opening at the top of the ionization-chamber with the results when a covering of 0.0065 mm. of aluminium was used, it is readily seen that the addition of the covering considerably increased the maximum ionization in the chamber as the β rays were deflected into it. This effect is also brought out very clearly by the curves A and B

TABLE XI.

Ionization by β transmitted Secondary Rays from Aluminium-foil with Primary Rays deflected down into Chamber.

Current in magnet (amperes).	Saturation Current (arbitrary scale), with different thicknesses of Aluminium-foil.						
	I. mm. 0.0	II. mm. 0.0065	III. mm. 0.28	IV. mm. 1.184	V. mm. 3.41	VI. mm. 4.73	VII. mm. 8.14
0.0	77.6	65.7	25.7	14.6	13.99	13.12	12.32
0.2	117.3	100.8					
0.45	176.9	157.5	36.2				
0.8	302.6	319.4	63.1	15.7	13.97	13.24	
1.3	365.0	396.5					
1.65	359.3	396.5	280.4	22.2	14.57		
2.5	294.8	319.4	287.5	50.4	15.65	13.15	
4.5	111.6	127.7	111.6	73.2	17.42	13.22	12.24
6.5	63.1	74.2	68.0	53.7	20.31	13.42	12.22
10.0	37.4	43.2	22.7	35.5	...	13.42	12.15
15.0	25.1	18.4	27.9	23.8	16.64	13.17	12.13
22.0	18.6	20.9	20.1	18.1	15.37	13.08	12.09

TABLE XII.

Ionization by β transmitted Secondary Rays from Aluminium-foil with Primary Rays deflected up from Chamber.

Current in magnet (amperes).	Saturation Current (arbitrary scale) for different thicknesses of Aluminium-foil.						
	I. mm. 0.0	II. mm. 0.0065	III. mm. 0.28	IV. mm. 1.184	V. mm. 3.41	VI. mm. 4.73	VII. mm. 8.14
0.0	63.1	71.5	25.9	14.7	13.93	13.12	12.37
0.85	19.8	11.2	16.3	13.9	13.74	13.02	12.27
2.5	11.7	10.3	11.8	13.1	13.65	12.90	12.08
4.5	...	10.4	11.3	12.7	13.41	12.57	
6.5	13.5	10.2	11.0	12.4	12.90	12.30	11.51
15.0	12.6	10.1	10.7	12.0	12.72	11.87	11.11
22.0	12.3	10.1	10.6	11.9	12.46	11.68	10.89

in fig. 12. This increase in ionization in the chamber due to the thin covering of aluminium was interpreted as being due to the action of secondary radiation. The small thickness of aluminium-foil used would only absorb a very small proportion of the primary β rays, and consequently it would be possible for the excited secondary rays to make a contribution

to the ionization in the chamber greater than the loss incurred by the absorption of the primary rays. Of course it is also possible that the increase in ionization observed could be interpreted as being due to a decrease in velocity impressed upon the primary rays by their passage through the foil. It is to be noted, too, in connexion with this explanation that, since the values of the ionization shown by curve A were obtained with the opening at the top of the chamber uncovered, these undoubtedly represented the ionization of a somewhat larger body of air than was used in the experiments when the opening was covered. It follows, therefore, that the real increase in ionization produced by the passage of the β rays through the single sheet of aluminium should have been greater than that indicated by the curves A and B of fig. 12. Some measurements were made with screens of two and of three sheets of aluminium; and as these were found to give maximum ionizations approximately the same as that obtained with a single sheet, it was seen that, in order to investigate more fully this rise in conductivity, it would be necessary to use still thinner sheets of aluminium than the one with which the opening was first closed. As this point was not specially pertinent to the subject under investigation by the writer, its examination was deferred. This rise in conductivity resulting from the passage of β rays through a thin layer of aluminium was not observed in the experiments with lead and tin screens, doubtless because the least thicknesses of these metals absorbed more of the primary β rays than could be compensated for by the excited secondary radiations. This result, it will be remembered, was referred to in Section III. A, and was given as a reason for drawing the curve shown in fig. 6 with an additional rise, although no determinations were made with which it could be confirmed.

Curve C shows that while the more deflectable of the β radiations were absorbed by 1.184 mm. of aluminium-foil, the more penetrating still passed through it. The slight rise in curve D also indicates that some of the β radiation was still able to penetrate 4.73 mm. of aluminium. With a thickness of 8.14 mm. of aluminium, however, no rise in the conductivity occurred, and, as curve E (fig. 13) shows, this thickness was sufficient to cut off all the β -ray effect.

It will be seen that the curves which are drawn on a large scale for deflexions of β rays downwards, and for deflexions of these rays upwards, corresponding to a thickness of 8.14 mms. of aluminium over the opening at the top of the chamber and denoted by E and E', do not coincide. It will be recalled, further, that the curves drawn for the limiting thicknesses

of tin and lead under the same conditions showed an exact coincidence. This peculiarity in the behaviour of the aluminium screen was investigated at considerable length, and was finally shown by some experiments which are described later in Section VI. to be due to the action of the secondary rays excited on the far side of the thicker aluminium screens by the γ rays entering the chamber.

TABLE XIII.

Remarks.	Saturation Current.						
Thickness of aluminium } over the top of chamber }	mm. 0.0	mm. 0.0065	mm. 0.28	mm. 1.184	mm. 3.41	mm. 4.73	mm. 8.14
I. Max. $\beta + \beta$ secondary } + $\gamma + \gamma$ secondary + } natural ionization..... }	365	396.5	287.5	79.3	20.3	13.5	12.2
II. $\gamma + \gamma$ secondary + } natural ionization..... }	12.6	10.2	11	12.3	13.1	12.3	11.7
III. Max $\beta + \beta$ secondary } ionization	352.4	386.3	276.5	67.0	7.2	1.2	0.5
IV.	332.4	386.3	276.5	67.0	6.7	0.7	0.0

In Table XIII. there is given in row I. the maximum saturation-currents in the chamber due to the β and β secondary, γ and γ secondary radiations, and that due to natural causes for the different thicknesses of aluminium-foil ; in row II. saturation-currents due to the γ and γ secondary radiation and that due to natural causes ; and in row III. the maximum ionizations due to the β and β secondary radiations deduced, as explained in Section III. A, from the Tables above and their corresponding curves. On looking at the figures given in row III. of this table, it is seen that there is apparently a β -ray ionization of .5, or about one-seventh of one per cent. of the greatest β -ray ionization in the chamber when the top of the chamber is covered by 8.14 mm. of aluminium. This conductivity, however, represents really a γ -ray effect, due, as said before, to the thickness of the aluminium used, and should be deducted from the last three of the numbers given in row III. of the table. These corrected values of the maximum β and β secondary ionizations are given in row IV., and a curve (fig. 14) is plotted from these values. An examination of this curve makes it evident that a thickness

of approximately 7 mm. of aluminium-foil was amply sufficient to absorb all the β rays and the secondary rays excited by them.

B. Measurements on Reflected Rays from Aluminium.

A series of measurements was also made on the secondary radiation produced at the front side of sheets of aluminium-foil when β and γ rays were allowed to fall on them; and from these the critical depth of the β -ray effect has been determined. The arrangement of the apparatus was the same as for the measurements on the reflected radiations from tin and lead. The values of the saturation-currents in the chamber found for the different thicknesses of aluminium-foil at the bottom are given in Table XIV., and the maximum ionizations due to the β rays have been deduced from these tables and their corresponding curves. These maximum currents are given in Table XV., and the curve in fig. 15 plotted from

TABLE XIV.

Ionization by β reflected Secondary Rays from Aluminium.

Current in magnet (amperes).	Saturation Current (arbitrary scale), with different thicknesses of Aluminium-foil.				
	I. mm. 0.0065	II. mm. 0.026	III. mm. 0.065	IV. mm. 0.280	V. mm. 0.963
0.0	1106.0	1126.0	1140.0	1176.0	1181.0
0.8	709.8	716.5			
0.85	684.3	710.0	723.9
2.5	127.1	128.9	116.5	125.8	170.4
4.5	66.8	58.6	65.7	59.2	62.1
6.5	57.4	52.0	55.5	49.5	52.9
15.0	57.7	55.2	58.0	54.3	53.2
23.0	56.1	54.5	56.6	53.4	52.5
30.0	...	53.4	54.7	52.0	51.5

TABLE XV.

Thickness of aluminium-foil over bottom of chamber.	Max. $\beta + \beta$ secondary + $\gamma + \gamma$ secondary + natural ionization.	$\gamma + \gamma$ secondary + natural ionization.	Max. $\beta + \beta$ secondary ionization.
mm. 0.0065	1106	57.1	1048.9
0.026	1126	53.8	1072.2
0.065	1140	56.2	1083.8
0.280	1176	52.3	1123.7
0.963	1181	52.5	1128.5

them shows the manner in which the intensity of the secondary radiation excited by the β rays rose as the thickness of the aluminium was increased. From this curve it is evident that the maximum conductivity produced by the β and the reflected β secondary rays attained a limiting value when the aluminium-foil sheets reached a thickness of .4 mm.

It follows, then, from these results that a thickness of 7 mm. of aluminium will completely absorb all the β rays from radium and the secondary rays which they excite in the metal. It follows, too, that the secondary rays emitted by the front side of a plate of the metal when bombarded by the β rays from radium do not come from a depth in the metal greater than .4 mm.

VI. EXPERIMENTS ON THE SECONDARY RAYS EXCITED IN ALUMINIUM BY γ RAYS.

It has been stated in Section V. A, that when a sheet of aluminium 8.1 mm. in thickness, which was sufficient to absorb all the β rays and the secondary rays excited by them, was placed over the opening at the top of the ionization-chamber, the saturation-currents were not the same with a magnetic field applied in one direction as those obtained with the same field reversed. This lack of symmetry in the values of the saturation-currents obtained when screens of aluminium were used, is illustrated by curves E and E' in fig. 13. In the experiments with lead and tin screens no effect of this kind was observed; and in order to clear up the matter, an additional series of experiments was carried out to ascertain, if possible, the cause of it in the case of aluminium.

(1) In the first experiment a thickness of 4.73 mms. of aluminium was placed over the opening at the top of the chamber, and above this a thickness of .964 mm. of lead. This thickness of lead, it will be remembered, was found in the earlier experiments sufficient to absorb all the β and the β secondary radiations. It follows, then, that with this screen none but the γ rays of radium could enter the ionization-chamber when this double thickness of lead and aluminium was placed over the top. The conductivities in the chamber for gradually increasing fields in both directions were taken, and these are given in Table XVI. The second column of

TABLE XVI.

Thickness of lead over the top of chamber = .964 mm.

Thickness of aluminium over the top of chamber = 4.73 mms.

Lead above aluminium.

β rays deflected towards chamber.		β rays deflected away from chamber.	
Current through magnet (amperes).	Saturation Current.	Current through magnet (amperes).	Saturation Current.
0.0	10.85	0.0	10.68
2.5	10.94	2.5	10.74
4.5	10.87	4.5	10.37
6.5	10.89	6.5	10.10
10.0	10.70	10.0	9.88
15.0	10.67	15.0	9.74
22.0	10.65	21.0	9.60

this table shows a slight gradual decrease in ionization as the β rays were deflected downwards toward the chamber, and the fourth column shows a greater decrease as the β rays were deflected away from it. Here, again, it will be seen that the difference in the ionizations for the directions of the magnetic field was approximately of the same magnitude as when there was 8.4 mm. of aluminium over the ionization-chamber. Since none but γ rays could enter the chamber, this difference in ionization must have been due to the action of the magnetic field in the chamber on the secondary radiation issuing from the back of the aluminium screen under the excitation of the γ rays.

(2) The next experiment was to place the radium protected by the lead cylinder on the side of the ionization-chamber directly opposite to its former position. The same aluminium screen, 8.14 mm. in thickness, was placed over the chamber as before, and the β rays were again deflected down into the chamber by a suitably directed magnetic field and afterwards upward and away from it with the field reversed. The results are given in Table XVII. The numbers thus recorded show the same characteristics as when the radium was in the first position. When the magnetic fields were such as to deflect the β rays downwards into the chamber, the ionization decreased but slightly. On the other hand, a considerably greater decrease took place when the β rays were deflected in the opposite direction.

TABLE XVII.

Thickness of aluminium over the top of chamber = 8.14 mms.
Radium on opposite side of chamber from its usual position.

β rays deflected towards chamber.		β rays deflected away from chamber.	
Current through magnet (amperes).	Saturation Current.	Current through magnet (amperes).	Saturation Current.
0.0	12.50	0.0	12.56
4.5	12.46	0.85	12.43
10.0	12.43	2.5	12.32
15.0	12.41	4.5	12.06
22.5	12.40	7.0	11.76
		10.0	11.43
		15.0	11.11
		22.5	10.91

(3) A third experiment was carried out with the radium and its lead protection placed back in the original position. One sheet of tinfoil, .0196 mm. in thickness, was inserted over the top of the ionization-chamber, and 8.14 mms. of aluminium were then placed over the tin. Readings were then taken of the conductivity in the chamber for the two deflexions. These readings are given in Table XVIII., and

TABLE XVIII.

Thickness of aluminium over the top of chamber = 8.14 mms.
Thickness of tin over the top of chamber = .0196 mm.
Aluminium above tin.

β rays deflected towards chamber.		β rays deflected away from chamber.	
Current through magnet (amperes).	Saturation Current.	Current through magnet (amperes).	Saturation Current.
0.0	16.31	0.0	16.26
2.5	16.29	2.5	16.06
4.5	16.08	4.5	15.77
6.5	15.86	6.5	15.43
10.0	15.75	10.0	15.38
15.0	15.60	15.0	14.89
23.0	15.43	22.0	14.74

the curves A and A' representing them are drawn in fig. 16. These curves and the curves E and E' drawn in fig. 13 are on the same scale. A comparison of the latter, which correspond to a screen of 8.14 mms. of aluminium alone over the top of the chamber, with the curves A and A' in fig. 16, makes it clear that the insertion of the sheet of tinfoil beneath the aluminium screen brought the curves representing the two deflexions more nearly into coincidence. The natural conclusion would be then, that for a greater thickness of tinfoil below the aluminium, the two curves representing the ionizations for the two deflexions would coincide. To test this conclusion, four sheets of tinfoil, or a thickness of .0784 mm., were placed above the opening of the chamber, and over this the 8.14 mm. of aluminium. The conductivity in the chamber was then measured for different magnetic fields. The results are given in Table XIX., and curve B illustrating them is

TABLE XIX.

Thickness of aluminium over the top of chamber = 8.14 mms.

Thickness of tin over the top of chamber = .0784 mm.

Aluminium above tin.

β rays deflected towards chamber.		β rays deflected away from chamber.	
Current through magnet (amperes).	Saturation Current.	Current through magnet (amperes).	Saturation Current.
0.0	16.00	0.0	16.10
2.5	16.04	4.5	15.69
4.35	15.66	10.0	15.56
6.5	15.66	15.0	15.39
10.0	15.44	22.0	16.17
15.0	15.35		
18.0	15.19		

shown in fig. 16. The numbers in the table and the curve both show that with a screen made up in this way the conductivities in the chamber were identical for magnetic fields of equal intensity in either direction. This experiment showed clearly that the effect under consideration was due to a peculiarity in the secondary radiation emitted by the aluminium.

(4) In the fourth experiment the radium protected by the lead cylinder was placed vertically above the ionization-chamber and also above the poles of the magnet in such

a way that the pencils of rays from the radium were directed straight into the ionizing-chamber. The saturation currents for magnetic fields in both directions, when the 8·14 mms. of aluminium alone covered the chamber, were then measured. From the values of these currents, which are recorded in Table XX., it will be seen that the ionization corresponding

TABLE XX.

Thickness of aluminium over the top of chamber = 8·14 mms.
Radium vertically above chamber.

β rays deflected towards chamber.		β rays deflected away from chamber.	
Current through magnet (amperes).	Saturation Current.	Current through magnet (amperes).	Saturation Current.
0·0	15·69	0·0	15·37
2·5	15·31	3·0	15·31
4·3	15·13	4·5	15·23
6·4	14·97	6·3	14·78
10·0	14·81	10·0	14·66
15·0	14·89	15·0	14·82
22·5	14·72	22·5	14·74

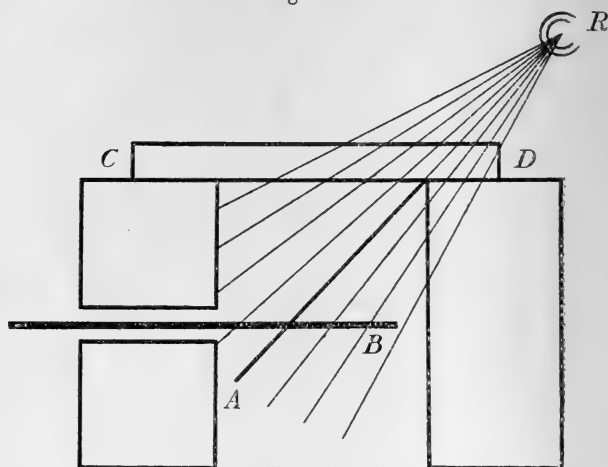
to any selected field-intensity was the same for both directions of the field. Since the disposition of the apparatus in this experiment was symmetrical, it follows that the effect noted with the previous arrangement was not only connected with some special property of the secondary radiation excited in the aluminium by the γ rays, but it also was due evidently to a non-symmetrical configuration of these secondary rays in the ionizing chamber.

The following is offered as an explanation of the foregoing experiments :—

In fig. 17 (p. 150), A represents the ionization-chamber, B the electrode, CD the aluminium screen, and R the position of the radium in the non-symmetrical arrangement. From a consideration of the figure, it is evident that the line RA, which is the axis of a pencil of γ rays entering the chamber, will mark the line of greatest intensity of these rays ; since for all other rays the metal path traversed is longer, and consequently the absorption is greater. It follows then that RA will also represent the direction of the axis of the pencil of secondary rays of greatest intensity issuing from the back of

the aluminium plate. If then the magnetic field were applied in such a direction as to deflect the primary β rays down into the chamber, this field, since the chamber was so situated as to be affected by the field, would deflect the secondary rays

Fig. 17.



issuing from the back of the aluminium screen in the same direction. With the field in the opposite sense, the primary β rays would be deflected upwards and away from the chamber, and the secondary rays in the chamber would also be turned by this field in a similar way, *i. e.*, with one direction of the field the axis of the pencil of secondary rays corresponding to RA would be turned anti-clockwise, while with the field reversed this pencil would undergo a clockwise deflexion. From the diagram shown in fig. 17 it can be seen that when the pencil of maximum intensity RA is given a clockwise rotation, the air-path traversed by it will be lessened, and consequently the ionization produced by it reduced. On the other hand, with the anti-clockwise rotation the length of path traversed by this pencil will be increased; and hence one should not expect the magnitude of the decrease in ionization following the application of the field producing this deflexion to be as great as when the field applied caused the rays to be deflected in the opposite sense. It is evident, too, that the tertiary rays excited on the walls of the chamber by the aluminium secondary rays would be greater in the case of the anti-clockwise rotation of the secondary rays than in experiments when the rotation of these rays was in the opposite direction. One naturally

inquires why this effect did not appear in the experiments when tin and lead were used as coverings for the openings into the chamber, and also when a thickness of 0.0784 mm. of tinfoil was placed below the aluminium cover. The probable explanation is that the transmitted secondary rays from tin and lead are not so effective ionizing agents or so good exciters of tertiary rays as the secondary rays from aluminium. The effect even in the case of aluminium is small, although quite noticeable; and it is probable, therefore, with the weaker secondary rays from the tin and lead, that the effect would be very much less and consequently masked by the other influences present.

The experiments which have just been described are also interesting for the light which they throw on the nature of the transmitted secondary radiation excited in the metals aluminium, tin, and lead by γ rays. According to the argument which has been presented, it follows from Bragg's conclusions, since the secondary rays from aluminium are better ionizers than those from tin and lead, that the particles constituting these secondary rays must be endowed with smaller velocities than those constituting the secondary radiation from the other two metals. The transmitted γ excited secondary rays from aluminium should therefore, from this point of view, be more easily absorbed than those emitted by tin and lead.

This conclusion regarding the character of the transmitted secondary radiation excited in aluminium by γ rays is in accord with the conclusions of McClelland*, Starke†, and others, who have found an exceptionally high coefficient of absorption for the reflected secondary rays excited by β and γ rays in this metal.

VII. A COMPARISON OF THE SECONDARY RADIATIONS

EXCITED IN DIFFERENT METALS BY β RAYS.

Some conclusions of interest can also be drawn from the results of the experiments of the present investigation regarding the secondary rays excited in different metals by β rays. For the purpose of making a comparison, the thicknesses of the limiting absorbing layers of the three metals studied with both reflected and transmitted rays are

* McClelland, Trans. Roy. Dublin Soc. viii. p. 169 (1905).

† Starke, *Le Radium*, Feb. 1908.

collected in Table XXI., and in fig. 18 curves are drawn with the thicknesses of the absorbing layers as abscissæ and the

TABLE XXI.

	Lead.	Tin.	Aluminium.
Thickness required to absorb β and β se- condary radiation. }	mm. 0.9	mm. 2.5	mm. 7.0
Thickness giving maxi- mum reflected se- condary radiation due to β rays }	0.16	0.24	0.4
Density	11.3	7.3	2.6

densities of the absorbing substances as ordinates. The curve A is plotted from the results of the transmitted radiation experiment, while the curve B corresponds to the measurements on the reflected rays. It will be noticed that the scale of abscissæ used for the latter curve is only one-tenth that adopted in laying out the former. From the results in the table and from the form of the curve, it will be seen that the thicknesses of the absorbing material required to stop the β and β secondary rays were not directly proportional to the densities, but that as the densities decreased it required a greater thickness to stop the rays than should have been expected from density considerations alone.

It is highly probable that the maximum depth from which the secondary rays come on the front side of a metal plate when primary β rays impinge on it represents the thickness that the secondary rays excited by the primary ones will penetrate in that metal. Now if the secondary rays excited by the primary in the three metals are all of the same penetrability, one should expect, on the assumption that they are β rays, that numbers representing the maximum penetrability found for these secondary rays would follow the same absorption-law with reference to the density that the numbers representing the maximum penetrabilities of the primary radiation followed. In other words, the two curves A and B should be similar in form if the secondary rays excited in the three metals possess the same penetrability. But it is clear from the manner in which the two curves

intersect in the figure, that they do not typify the same absorption law. It will be seen from the curve B that the maximum penetrabilities of the secondary β rays, as determined by the reflexion experiments, approach very closely to a linear relation which exhibits in a striking manner the important result that secondary rays excited in plates of different metals when β rays are allowed to fall on them are the more penetrative the greater the density of the metal of which the reflector is made.

VIII. SUMMARY OF RESULTS.

(1) The β radiation from radium bromide, which includes the β radiations from all the radium products in the equilibrium state, will not produce any ionization on the far side of a plate of aluminium 7 mms. in thickness, of a plate of tin 2.5 mms. in thickness, or of a plate of lead .9 mm. in thickness.

(2) The maximum secondary radiation emitted from the front side of plates of the metals aluminium, tin, and lead, when bombarded by β rays, are given by the following thicknesses :—

Aluminium	0.4 mm.
Tin	0.24 „
Lead	0.16 „

(3) The transmitted secondary radiations excited by γ rays in lead and tin are more penetrating than the transmitted secondary radiation excited in aluminium by the same rays.

(4) When β rays are allowed to fall in turn on reflectors of different metals, it is found that the greater the density of the metal from which the reflector is made the greater is the penetrability of the reflected secondary rays excited by the β radiation.

(5) From the experiments on the transmission of β rays through sheets of aluminium-foil, it has been shown that when very thin sheets of the metal are used the ionization at first contributed by the transmitted secondary radiation excited by the β rays is greater than that lost through absorption of the primary rays.

In conclusion, I wish to express my best thanks to Prof. McLennan, at whose suggestion the investigation was undertaken, for his help and advice and unfailing kindness throughout the course of the research.

VII. *On the Damping of Long Waves in a Rectangular Trough.* By ROBERT A. HOUSTOUN, Ph.D., D.Sc.,
Lecturer on Physical Optics in the University of Glasgow*.

IN this paper an expression is derived for the rate of damping of stationary long waves in a rectangular trough, and the result is compared with the results of experiment.

§ 1. We shall begin by obtaining the differential equation for long waves, taking viscosity into consideration. In the ordinary treatment of long waves without friction (*cf.* Lamb's 'Hydrodynamics,' p. 239) the horizontal velocity is the same at all points in the same vertical; the bottom is perfectly smooth. We shall suppose that there is no slipping at the bottom and that the velocity there is zero.

Let the motion be in one horizontal dimension in a medium of uniform depth h . Take the axis of x horizontal and in the direction of motion and the axis of z vertically upwards. Let the bottom of the medium be given by $z=0$, and the surface in its undisturbed state be given by $z=h$.

The ordinate of the free surface corresponding to the abscissa x at time t will be denoted by $h+\xi$. Then, on the usual assumption that the vertical acceleration of the fluid particles may be neglected, p the pressure at any point (x, y, z) is given by

$$p=p_0+g\rho(h+\xi-z), \quad . \quad . \quad . \quad . \quad (1)$$

where p_0 is the external pressure and ρ the density of the medium. Hence

$$\frac{\partial^2 p}{\partial t \partial x} = g\rho \frac{\partial^2 \xi}{\partial t \partial x} \quad . \quad . \quad . \quad . \quad (2)$$

To obtain the equation of motion, consider the element $\delta x \delta y \delta z$. The difference of the pressure-thrusts on the ends is

$$-\frac{\partial p}{\partial x} \delta x \delta y \delta z.$$

The difference of the tangential stresses on the upper and lower faces is

$$\mu \frac{\partial^2 u}{\partial z^2} \delta x \delta y \delta z,$$

and on the sides is

$$\mu \frac{\partial^2 u}{\partial y^2} \delta x \delta y \delta z.$$

* Communicated by Professor A. Gray, F.R.S.

Equating the sum of these terms to the mass-acceleration, we get

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \quad (3)$$

To obtain the equation of continuity, consider the rate at which fluid is entering the space bounded by x , $x + \delta x$, and equate it to the rate at which the fluid in this space is increasing. Hence

$$\int_0^h -\frac{\partial u}{\partial x} dz = \frac{\partial \zeta}{\partial t}. \quad (4)$$

Differentiating (4) with respect to x and combining with (2), we obtain

$$\frac{\partial^2 p}{\partial t \partial x} = -g\rho \int_0^h \frac{\partial^2 u}{\partial x^2} dz. \quad (5)$$

Differentiating (3) with respect to t and substituting from (5), we obtain the final equation, writing ν for μ/ρ ,

$$\frac{\partial^2 u}{\partial t^2} - g \int_0^h \frac{\partial^2 u}{\partial x^2} dz - \nu \frac{\partial}{\partial t} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0. \quad (6)$$

This of course reduces to the ordinary equation for long waves if $\nu=0$ and we assume $\partial^2 u / \partial x^2$ to be taken as independent of z .

§ 2. Let us now apply equation (6) to the case of water contained in a rectangular trough, the ends of which are given by $x=0$, $x=a$, the sides by $y=0$, $y=b$, and the bottom by $z=0$. Then, in order that we may apply (6), ζ and $\partial \zeta / \partial t$ must initially be functions of x alone. Let the initial conditions be

$$\left. \begin{aligned} \zeta &= c \left(x - \frac{a}{2} \right) \\ \frac{\partial \zeta}{\partial t} &= 0. \end{aligned} \right\} t=0.$$

The boundary conditions are

$$u=0 \text{ for } x=0, \quad x=a, \quad y=0, \quad y=b, \quad z=0,$$

and $\partial u / \partial z = 0$ for $z=h$,

the latter condition expressing the fact that there is no tangential stress on the free surface.

As a solution we assume

$$u = e^{i\sigma t} Z \sin mx \sin ny,$$

where Z is a function of z alone. Substitution in (6) gives

$$(i\sigma\nu n^2 - \sigma^2)Z + gm^2 \int_0^h Z dz - i\sigma\nu \frac{\partial^2 Z}{\partial z^2} = 0. \quad (7)$$

Z must be zero when $z=0$ and $\partial Z/\partial z$ must be zero when $z=h$. The solution satisfying these conditions is

$$Z = \cosh kh - \cosh k(h-z). \quad (8)$$

On substituting (8) in (7) we find that

$$\sigma^2 - \nu i\sigma n^2 + \nu i\sigma k^2 = 0. \quad (9)$$

and

$$(\sigma^2 - ghm^2) - \nu i\sigma n^2 + \frac{gm^2}{k} \tanh kh = 0. \quad (10)$$

In order to satisfy the boundary conditions we must put $m = q\pi/a$ and $n = r\pi/b$, where q and r are positive integers. From (9) and (10) it is obvious that both k and σ are complex. Put $\sigma = p + i\kappa$; κ is then the reciprocal of the modulus of decay

The complete value of n is given by

$$u = \text{real part } \sum B_{qr} e^{-\kappa t} \sin pt \sin \frac{q\pi x}{a} \sin \frac{r\pi y}{b} [\cosh kh - \cosh k(h-z)]. \quad (11)$$

We shall now investigate the rate of decay of the different terms of the above series. Reserving the justification until later, we shall assume that the term $\nu i\sigma n^2$ may be neglected in equations (9) and (10) and that $\tanh kh$ may be put equal to unity.

In place of (9) and (10) we have then

$$k^2 = \frac{i\sigma}{\nu}, \quad (12)$$

and

$$(\sigma^2 - ghm^2) + \frac{gm^2}{k} = 0. \quad (13)$$

If $p + i\kappa$ be substituted for σ in the above equations, and κ^2 be neglected in comparison with p^2 , we find that κ is given by the equation

$$\kappa^4 - \frac{\nu}{4h^2} \kappa^3 - \left(\frac{\nu p}{8h^2} \right)^2 = 0. \quad (14)$$

It is easy to show that this equation has always one, and never more than one, real positive root. This positive root gives the rate of damping, since that rate must be positive; its reciprocal is the modulus of decay.

If we make the additional assumption that σ/ν is large the expression for κ becomes very much simpler. Then (12) may be written

$$k = \sqrt{\frac{p}{2\nu}} (1+i).$$

Substituting in (13)

$$p^2 + 2ip\kappa - ghm^2 + gm^2 \sqrt{\frac{\nu}{2p}} (1-i) = 0, \quad (15)$$

and equating the imaginary part to zero, we find

$$\kappa = \frac{gm^2}{p} \sqrt{\frac{\nu}{8p}}.$$

To a first approximation it may be assumed that the viscosity has no effect on the period; then $p/m = \sqrt{gh}$ and the wave-length $\lambda = 2\pi/m$. Hence we may write

$$\kappa = \left(\frac{\nu p}{8h^2}\right)^{\frac{1}{2}} = \left(\frac{\nu\pi}{4\lambda}\right)^{\frac{1}{2}} \left(\frac{g}{h^3}\right)^{\frac{1}{4}} \dots \dots (16)^*$$

To find the effect of the viscosity on the period we equate the real part of equation (15) to zero. Then

$$p^2 - ghm^2 + gm^2 \sqrt{\frac{\nu}{2p}} = 0.$$

By substitution from (16), this becomes

$$p^2 - ghm^2 + 2p\kappa = 0.$$

Hence when κ^2 is neglected in comparison with ghm^2

$$p = -\kappa + m \sqrt{gh}. \quad (17)$$

The period $T = 2\pi/p$. T_0 , the value which we obtain when viscosity is neglected, is $2\pi/(m \sqrt{gh})$. Hence substituting in (17)

$$T = T_0 + \frac{\kappa}{2\pi} T_0^2 \dots \dots (18)$$

We shall next determine the values of the coefficients in the expression for u .

* I found, after deriving formula (16), that it had already been obtained by S. Hough for progressive waves in an unlimited medium by a longer method (Proc. Lond. Math. Soc. xxviii. p. 276, 1897); he finds a general expression for the damping of waves in one horizontal dimension, and as a special case gives the formula for long waves.

Differentiating (11) with respect to x and integrating with respect to z from 0 to h , neglecting the term divided by k , we find

$$\int_0^h \frac{\partial u}{\partial x} dz = \Sigma B_{qr} \frac{q\pi h}{a} e^{-\kappa t} \sin pt \cos \frac{q\pi x}{a} \sin \frac{r\pi y}{b} \cosh kh.$$

Integrating this with regard to t and at the same time neglecting the variation of $e^{-\kappa t}$ with t , we obtain by means of (4)

$$\zeta = \Sigma B_{qr} \frac{q\pi h}{pa} e^{-\kappa t} \cos pt \cos \frac{q\pi x}{a} \sin \frac{r\pi y}{b} \cosh kh. \quad (19)$$

Writing

$$B_{qr} \frac{q\pi h}{pa} \cosh kh = A_{qr},$$

and expressing the initial condition, we obtain

$$c\left(x - \frac{a}{2}\right) = \Sigma A_{qr} \cos \frac{q\pi x}{a} \sin \frac{r\pi y}{b}.$$

If we calculate the different values of A_{qr} , we find that the most important ones are as follows:—

$$\begin{aligned} A_{11} &= -\frac{16ac}{\pi^3}, & A_{31} &= -\frac{16ac}{3^2\pi^3}, & A_{51} &= -\frac{16ac}{5^2\pi^3}, \\ A_{13} &= -\frac{16ac}{3\pi^3}, & A_{33} &= -\frac{16ac}{3 \cdot 3^2\pi^3}, & A_{53} &= -\frac{16ac}{3 \cdot 5^2\pi^3}, \\ A_{15} &= -\frac{16ac}{5\pi^3}, & A_{35} &= -\frac{16ac}{5 \cdot 3^2\pi^3}, & A_{55} &= -\frac{16ac}{5 \cdot 5^2\pi^3}. \end{aligned}$$

The values with an even suffix vanish.

For the different terms of (19) q/p is constant and $\cosh kh$ increases rapidly with q ; hence B_{qr} must converge more rapidly than A_{qr} . For a point near the centre of the trough, the terms for which $q=1$ in the expression for u will be at the very least nine times as great as the terms for which $q=3$. Hence the rate of damping of the vibrations there will be sensibly that of the first term.

§ 3. The trough used for the experiments had a slate bottom and plate-glass sides and ends. It was 152.4 cms. long and 20.3 cms. broad inside and the sides were 15.9 cms. high. The waves were excited by raising one end of the trough about 1 cm. by means of a lever. Then the lever was held by a catch until the surface was perfectly still. When

the catch was suddenly released, the end dropped, the surface of the water was represented for an instant by $\zeta = c\left(x - \frac{a}{2}\right)$ and then began to oscillate about its equilibrium position. A scale was held from above over the surface of the water parallel to the sides of the trough at its middle. Any notes that happened to be in the water were brought to the surface there; the turning-points of one of them were read on the scale as long as the motion lasted, and the rate of damping thus obtained.

The following table gives some results :—

h .	Observed period.	T_0 .	Difference.	Difference $\frac{\kappa T_0^2}{2\pi}$ calculated.	κ calculated.	κ observed.
cms.	secs.	secs.	secs.			
1.0	10.1	9.75	.35	.45	.028	.048
2.0	7.12	6.89	.23	.23	.017	.032
3.0	5.80	5.62	.18	.14	.012	.028
5.0	4.49	4.35	.14	.09	.0084	.023
7.0	3.68	3.68	.00	.02	.0066	.022
10.0	3.12	3.08	.04	.01	.0050	.011

The first column gives the depth of the water and the second the observed period. The third gives T_0 , the theoretical value obtained for the period when viscosity was neglected; the fourth gives the difference of the second and third, and the fifth gives the effect of the viscosity of the water on the period calculated according to (18). The seventh column gives the observed values of κ , and the sixth gives values of κ calculated according to (16). If we use (14) slightly better values are obtained.

The actual value of κ is thus much greater than its theoretical value. The difference is due neither to neglecting the other modes, which die away more rapidly, nor to the vibrations not being small enough, nor to the "residual motion" of the water. The same values were obtained for κ when the water had a slight irregular motion in addition to the periodic one. The approximations made in deriving the formula are perfectly justifiable; for, to take a concrete example, for $h = 5$ cms. in equation (10) $\sigma^2 = 2.1$ and $\nu\sigma n^2 = 0.00033$, and in equation (11) the modulus of kh is about 60. If we take

the vertical acceleration into consideration κ becomes smaller, instead of greater.

The difference appears to be due to dissipation of energy in the surface-film. All the other conditions being the same, κ increases with the time. The following figures taken for $h=5$ cms. exemplify this :—

July 2nd, 3 P.M.	$\kappa=.022$
„ 3rd, 10 A.M.036
„ 4th, „040
„ 6th, „049
„ „ „ (surface cleaned)	.025

On July 2nd the trough was cleaned, filled with clean tap-water, and κ determined. The water was then left standing in the trough, and κ determined on the 3rd, 4th, and 6th. Immediately after the first determination of κ on the 6th the surface was “scraped,” the film removed, and κ fell to half its value.

The values of κ determined from a fresh surface were never very consistent; the notes were drawn in different directions owing to the film forming. After the surface had stood some time the values of κ were more regular.

Since κ increases so much with the time—an increase which we can explain only by the formation of a surface film—it seems reasonable to suppose that the difference between the value of κ for a fresh surface and the theoretical value is due to a film which forms with great rapidity. In other words, none of the surfaces were perfectly clean. This view is strengthened by the fact that the addition of minute quantities of lubricating oil or linseed oil to a fresh surface had no appreciable effect on κ . Apparently there was a film already there.

The impurity seemed to come from the atmosphere of the laboratory and from the putty at the corners of the trough. No improvement was obtained on cleaning the trough with acid or on using distilled water. When a smaller trough, which could be kept clean more easily, was used, the waves were not suitable for accurate measurement.

According to the table the effect of viscosity on the period agrees with its calculated values; the fourth and fifth columns agree. Some other measurements of the effect of viscosity on the period are given below. For these h was calculated from the volume of the water, and the surface was not at all so clean.

h .	Observed period.	T_0 .	Difference.	$\frac{\kappa T_0^2}{2\pi}$.
cms. ·5	secs. 16·71	secs. 13·76	secs. 2·95	secs. 1·45
1·0	10·57	9·75	·82	·45
2·0	7·18	6·89	·29	·23
3·0	5·79	5·62	·17	·14
5·0	4·425	4·35	·07	·09
7·0	3·70	3·63	·02	·02
10·0	3·085	3·03	·00	·01

The agreement is not good, but it ought to be stated that (18) is hardly applicable to such small values of h as 1·0 or 0·5 cms.

If we calculate T_0 by the general formula for surface-waves instead of by the formula for long waves, no difference is made, except in the case of $h=10$ cms. The value for the latter becomes 3·10 secs.

The calming effect of oil on a stormy sea is explained by supposing that the oil floats as a membrane on the surface of the water and hampers its motion by offering resistance to extension. The surface-film here seemed to be easily extensible. To explain the dissipation of energy it is necessary to assume that the membrane opposes the extension with a force proportional to the velocity. The tangential stress on the surface of the water is proportional to and opposed to the velocity there.

We thus have to solve (7) with the conditions

$$\begin{aligned}\frac{\partial Z}{\partial z} &= -cZ, & z=h, \\ Z &= 0, & z=0.\end{aligned}$$

We find

$$Z = [\cosh kh - \cosh k(h-z)] - \frac{c}{k} [\sinh kz - \sinh kh + \sinh k(h-z)].$$

Neglecting the term in which n^2 is a factor, we find that (12) still holds ; but in place of (13) we have

$$\begin{aligned}& \left(\sigma^2 - ghm^2 + \frac{2c}{k^2} gm^2 \right) \\ & + \frac{gm^2(1-ch) + c\sigma^2}{k} \tanh kh - \frac{2cgm^2}{k^2} \frac{1}{\cosh kh} = 0. \quad (20)\end{aligned}$$

In the coefficient of $\tanh kh$, gm^2h may be put equal to σ^2 since the whole term is small. $\tanh kh$ is as usual $=1$, and the $1/\cosh kh$ term may be neglected as it is very small.

Substituting from (12) in (20) and writing

$$\sigma = p + i\kappa$$

we get

$$p^2 + i2p\kappa - ghm^2 - i2\frac{cvgm^2}{p} + gm^2\sqrt{\frac{\nu}{2p}}(1-i) = 0. \quad (21)$$

The new term is equal to

$$i2p\frac{cv}{h}.$$

Hence

$$\kappa = \left(\frac{\nu p}{8h^2}\right)^{\frac{1}{2}} + \frac{cv}{h}. \quad (22)$$

The new surface condition is therefore capable of accounting for the rate of damping. It does not affect the period.

For $h = 5$ cms. and a fresh surface c is about 7, *i. e.*, for unit velocity the retarding force/unit area is $\frac{1}{14}$ dyne. We can find the effect of a tangential drag of this magnitude on free waves on deep water very readily by the method of Lamb's 'Hydrodynamics,' 3rd edition, § 331.

It is only necessary to add a new term $-\frac{q^2}{14}$ to the dissipation function. Then using the notation of that section we have instead of equation (8)

$$\frac{d}{dt}\left(\frac{1}{2}\rho k c^2 \alpha^2\right) = -2\mu k^3 c^2 \alpha^2 - \frac{1}{14}k^2 c^2 \alpha^2.$$

Therefore

$$\alpha = \alpha_0 e^{-2\left(\nu k^2 + \frac{k}{28\rho}\right)t},$$

i. e.,

$$\kappa = -\frac{8\pi^2\nu}{\lambda^2} - \frac{\pi}{7\rho\lambda}.$$

For $\lambda = 100$ cms., the first term $= 8 \cdot 10^{-5}$ and the second $4 \cdot 10^{-3}$; hence there cannot be a tangential drag of this magnitude on free waves on deep water.

The damping of the waves in the trough seems in some measure to be due to the film being attached to the sides. According to the theory, if the width of the trough were diminished by half, κ at the centre should not be altered appreciably. I had a wooden partition made which could be

fitted down the middle of the trough dividing it into two. For a fresh surface, $h=5$ cms., on removing this partition κ diminished approximately 10 per cent.

§ 4. It is interesting to note the manner in which the velocity falls off at the bottom of the trough.

$$Z = \text{real part } \{ \cosh kh - \cosh k(h-z) \},$$

$$= \cos \sqrt{\frac{p}{2\nu}} h \cosh \sqrt{\frac{p}{2\nu}} h - \cos \sqrt{\frac{p}{2\nu}} (h-z) \cosh \sqrt{\frac{p}{2\nu}} (h-z).$$

Taking $h=5$ cms. we find that at different depths Z is roughly proportional to the following numbers, the value at the surface being taken as unity :—

z	Z
0.0 mm.	·000
0.5 „	·129
1.0 „	·520
1.5 „	·732
2.0 „	·811
3.0 „	1.03
4.0 „	1.04
5.0 „	1.02
6.0 „	1.01
7.0 „	1.00

Thus the velocity reaches its surface value 3 mm. from the bottom.

§ 5. The case of a circular sheet with symmetry about the centre can be treated in the same way as the rectangular trough.

If u denote the radial velocity the equation of continuity is

$$\frac{\partial \zeta}{\partial t} = - \int_0^h \frac{1}{r} \frac{\partial}{\partial r} (ur) dz. \quad \dots \quad (23)$$

The equation of motion is

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial r} + \mu \frac{\partial^2 u}{\partial z^2}. \quad \dots \quad (24)$$

Eliminating p by means of (1) and (23) we get

$$\frac{\partial^2 u}{\partial t^2} = g \int_0^h \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial (ur)}{\partial r} dz + \nu \frac{\partial^3 u}{\partial t \partial z^2}. \quad \dots \quad (25)$$

Multiply by r , differentiate with regard to r , and divide by r , and (25) becomes

$$\frac{\partial^2}{\partial t^2} \frac{1}{r} \frac{\partial(ur)}{\partial r} = g \int_0^h \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(ur)}{\partial r} dz + v \frac{\partial^3}{\partial t \partial z^2} \frac{1}{r} \frac{\partial(ur)}{\partial r}. \quad (26)$$

Substitute

$$\frac{1}{r} \frac{\partial(ur)}{\partial r} = J_0(mr)Z,$$

where Z is a function of z alone. Then

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(ur)}{\partial r} &= \frac{\partial^2}{\partial r^2} \frac{1}{r} \frac{\partial(ur)}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial(ur)}{\partial r}, \\ &= Z \left(\frac{\partial^2 J_0(mr)}{\partial r^2} + \frac{1}{r} \frac{\partial J_0(mr)}{\partial r} \right), \\ &= -m^2 J_0(mr)Z. \end{aligned}$$

Hence $J_0(mr)$ cancels out of equation (26). If we assume a time-factor $e^{i\sigma t}$ we have for Z

$$\sigma^2 Z - g m^2 \int_0^h Z dz + v i \sigma \frac{\partial^2 Z}{\partial z^2} = 0, \quad \dots \quad (27)$$

which is the same as equation (7).

VIII. On the Frequency Ranges of Non-generating Force exerting Cumulative Influence. By ANDREW STEPHENSON*.

1. **P**ERIODIC non-generating force acting on a system in oscillation about a position of stable equilibrium, exerts a cumulative action in intensifying or diminishing the amplitude if its frequency is contained within any one of a number of ranges lying in the vicinity of 2μ , $2\mu/2$, $2\mu/3$..., where μ is the natural frequency of the system†.

The limits of the leading ranges have been obtained for small non-generating force. It is our object here to give a general method of finding the ranges in magnitude and position when the force is finite, and in particular to obtain the numerical values of the range limits about the double frequency for various intensities.

* Communicated by the Author.

† "On a Class of Forced Oscillations," Quart. Journ. of Mathematics, No. 168, 1906.

2. The complete solution of the equation of motion

$$\ddot{x} + \mu^2(1 + 2 \sum_1^{\infty} \alpha_r \cos rnt)x = 0 \quad . \quad . \quad . \quad (i.)$$

is given by

$$x = \sum_{-\infty}^{\infty} A_r \sin \{(c - rn)t + \epsilon\}, \quad . \quad . \quad . \quad (ii.)$$

where ϵ is arbitrary and

$$A_r \{\mu^2 - (c - rn)^2\} + \mu^2 \{\alpha_1(A_{r-1} + A_{r+1}) + \alpha_2(A_{r-2} + A_{r+2}) + \dots\} = 0.$$

On eliminating the A's we obtain the infinite determinant

$$\begin{vmatrix} . & . & . & . & . & . & . \\ . & \alpha_2 & \alpha_1 & [-1] & \alpha_1 & \alpha_2 & . \\ . & . & \alpha_2 & \alpha_1 & [0] & \alpha_1 & \alpha_2 \\ . & . & . & \alpha_2 & \alpha_1 & [1] & \alpha_1 \alpha_2 \\ . & . & . & . & . & . & . \end{vmatrix} = 0; \quad . \quad (1)$$

where $[r]$ denotes $\{\mu^2 - (c - rn)^2\}/\mu^2$. This equation determines c , and the roots are all included in the form $\pm c_0 - rn$, where r has the zero or any positive or negative integral value*.

Considering n as a variable parameter we observe that the roots can become equal in pairs only through c_0 becoming equal to zero or half of some multiple of n . But the values of n leading to equal roots separate the ranges of n giving a real ' c ' from those giving a complex ' c .' Hence the determination of the ranges of cumulative action is equivalent to the problem of obtaining the values of n which make the roots of the above determinant equal in pairs; i. e. to the problem of finding the roots of each of the two equations in n obtained from (1) by putting $c=0$ and $\frac{1}{2}n$ respectively.

3. When $c=0$, $[-r]=[r]$, and when $c=\frac{1}{2}n$, $[-(r-1)]=[r]$: on account of the symmetry thus introduced the idea suggests itself that for these values of c equation (1) admits of reduction. We shall investigate this question by direct examination of the motion in the limiting cases.

The general solution (ii.) may be written

$$x = \cos \epsilon \left(\sin ct \sum_{-\infty}^{\infty} A_r \cos rnt - \cos ct \sum_{-\infty}^{\infty} A_r \sin rnt \right) \\ + \sin \epsilon \left(\cos ct \sum_{-\infty}^{\infty} A_r \cos rnt + \sin ct \sum_{-\infty}^{\infty} A_r \sin rnt \right).$$

Now in the limit when $c=0$, either $A_{-r}=A_r$ or $A_{-r}=-A_r$,

* Thus far we follow the analysis introduced by G. W. Hill in his lunar theory.

according to which end of the range n approaches. On the former alternative one particular solution is given by

$$\begin{aligned} x &= \sum_{-\infty}^{\infty} A_r \cos rnt \\ &= A_0 + 2 \sum_1^{\infty} A_r \cos rnt. \end{aligned}$$

and the other by

$$\begin{aligned} x &= \text{Lt}_{c=0} \left\{ t \sum_{-\infty}^{\infty} A_r \cos rnt - \frac{1}{c} \sum_{-\infty}^{\infty} A_r \sin rnt \right\} \\ &= \text{Lt}_{c=0} \left\{ t(A_0 + 2 \sum_1^{\infty} A_r \cos rnt) - 2 \sum_1^{\infty} \frac{A_r - A_{-r}}{c} \sin rnt \right\}. \end{aligned}$$

Thus the solutions can be expressed by convergent series of the form

$$x = \sum_0^{\infty} B_r \cos rnt$$

$$\text{and} \quad x = t \sum_0^{\infty} B_r \cos rnt + \sum_1^{\infty} C_r \sin rnt.$$

By direct substitution

$$\begin{aligned} B_0 + \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \dots &= 0, \\ 2\alpha_1 B_0 + \{1 - (n/\mu)^2 + \alpha_2\} B_1 + (\alpha_1 + \alpha_3) B_2 + (\alpha_2 + \alpha_4) B_3 + \dots &= 0, \\ 2\alpha_2 B_0 + (\alpha_1 + \alpha_3) B_1 + \{1 - (2n/\mu)^2 + \alpha_4\} B_2 + (\alpha_1 + \alpha_5) B_3 + \dots &= 0, \\ 2\alpha_3 B_0 + (\alpha_2 + \alpha_4) B_1 + (\alpha_1 + \alpha_5) B_2 + \{1 - (3n/\mu)^2 + \alpha_6\} B_3 + \dots &= 0; \end{aligned}$$

also

$$\begin{aligned} \{1 - (n/\mu)^2 - \alpha_2\} C_1 + (\alpha_1 - \alpha_3) C_2 + (\alpha_2 - \alpha_4) C_3 + \dots &= 2n \cdot B_1, \\ (\alpha_1 - \alpha_3) C_1 + \{1 - (2n/\mu)^2 - \alpha_4\} C_2 + (\alpha_1 - \alpha_5) C_3 \dots &= 2n \cdot 2B_2, \\ (\alpha_2 - \alpha_4) C_1 + (\alpha_1 - \alpha_5) C_2 + \{1 - (3n/\mu)^2 - \alpha_6\} C_3 + \dots &= 2n \cdot 3B_3. \end{aligned}$$

The eliminant of the former set gives

$$\begin{vmatrix} 2 & 2\alpha_1 & 2\alpha_2 & 2\alpha_3 & \dots \\ 2\alpha_1 & 1 - \left(\frac{n}{\mu}\right)^2 + \alpha_2 & \alpha_1 + \alpha_3 & \alpha_2 + \alpha_4 & \dots \\ 2\alpha_2 & \alpha_1 + \alpha_3 & 1 - 4\left(\frac{n}{\mu}\right)^2 + \alpha_4 & \alpha_1 + \alpha_5 & \dots \\ 2\alpha_3 & \alpha_2 + \alpha_4 & \alpha_1 + \alpha_5 & 1 - 9\left(\frac{n}{\mu}\right)^2 + \alpha_6 & \dots \end{vmatrix} = 0, \quad (2)$$

where the term in the $(r+1)$ th row and $(s+1)$ th column is $\alpha_{r+s} + \alpha_{|r-s|}$, unless $r=s$, in which case the term is $1 - s^2(n/\mu)^2 + \alpha_{2s}$.

This equation determines one limit of each of the ranges of n associated with $\mu, \mu/2, \dots \mu/r \dots$. For the other limit $A_{-r} = -A_r$ in (ii.), and the particular solutions are

$$x = \sum_1^{\infty} B_r \sin rnt,$$

$$x = t \sum_1^{\infty} B_r \sin rnt - \sum_0^{\infty} C_r \cos rnt,$$

where

$$\begin{aligned} \{1 - (n/\mu)^2 - \alpha_2\} B_1 + (\alpha_1 - \alpha_3) B_2 + (\alpha_2 - \alpha_4) B_3 + (\alpha_3 - \alpha_5) B_4 + \dots &= 0, \\ (\alpha_1 - \alpha_3) B_1 + \{1 - (2n/\mu)^2 - \alpha_4\} B_2 + (\alpha_1 - \alpha_5) B_3 + (\alpha_2 - \alpha_6) B_4 + \dots &= 0, \\ (\alpha_2 - \alpha_4) B_1 + (\alpha_1 - \alpha_5) B_2 + \{1 - (3n/\mu)^2 - \alpha_6\} B_3 + (\alpha_1 - \alpha_7) B_4 + \dots &= 0, \\ (\alpha_3 - \alpha_5) B_1 + (\alpha_2 - \alpha_6) B_2 + (\alpha_1 - \alpha_7) B_3 + \{1 - (4n/\mu)^2 - \alpha_8\} B_4 + \dots &= 0, \\ &\dots \end{aligned}$$

$$\begin{aligned} C_0 + \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3 + \dots &= 0, \\ 2\alpha_1 C_0 + \{1 - (n/\mu)^2 + \alpha_2\} C_1 + (\alpha_1 + \alpha_3) C_2 + (\alpha_2 + \alpha_4) C_3 + \dots &= 2n \cdot B_1, \\ 2\alpha_2 C_0 + (\alpha_1 + \alpha_3) C_1 + \{1 - (2n/\mu)^2 + \alpha_4\} C_2 + (\alpha_1 + \alpha_5) C_3 + \dots &= 2n \cdot 2B_2 \\ &\dots \end{aligned}$$

Hence,

$$\begin{vmatrix} 1 - \left(\frac{n}{\mu}\right)^2 - \alpha_2 & \alpha_1 - \alpha_3 & \alpha_2 - \alpha_4 & \alpha_3 - \alpha_5 & \dots \\ \alpha_1 - \alpha_3 & 1 - 4\left(\frac{n}{\mu}\right)^2 - \alpha_4 & \alpha_1 - \alpha_5 & \alpha_2 - \alpha_6 & \dots \\ \alpha_2 - \alpha_4 & \alpha_1 - \alpha_5 & 1 - 9\left(\frac{n}{\mu}\right)^2 - \alpha_6 & \alpha_1 - \alpha_7 & \dots \\ \alpha_3 - \alpha_5 & \alpha_2 - \alpha_6 & \alpha_1 - \alpha_7 & 1 - 16\left(\frac{n}{\mu}\right)^2 - \alpha_8 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0, (3)$$

where the term in the r th row and s th column is $\alpha_{|r-s|} - \alpha_{r+s}$, if $r \neq s$; if $r=s$ the term is $1 - s^2(n/\mu)^2 - \alpha_{2s}$. This equation determines the other limits.

The case $c = \frac{1}{2}n$ admits of similar treatment. When c has this value we find that either

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 + \alpha_1 & \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & \dots \\ \alpha_1 + \alpha_2 & 1 - 9\left(\frac{n}{2\mu}\right)^2 + \alpha_3 & \alpha_1 + \alpha_4 & \dots \\ \alpha_2 + \alpha_3 & \alpha_1 + \alpha_4 & 1 - 25\left(\frac{n}{2\mu}\right)^2 + \alpha_5 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0, \quad (4)$$

or

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 - \alpha_1 & \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \dots \\ \alpha_1 - \alpha_2 & 1 - 9\left(\frac{n}{2\mu}\right)^2 - \alpha_3 & \alpha_1 - \alpha_4 & \dots \\ \alpha_2 - \alpha_3 & \alpha_1 - \alpha_4 & 1 - 25\left(\frac{n}{2\mu}\right)^2 - \alpha_5 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0, \quad (5)$$

where the term in the r th row and s th column is

$$\alpha_{|r-s|} \pm \alpha_{r+s-1},$$

except in the case $r=s$, for which it has the form

$$1 - (2s-1)^2(n/2\mu)^2 \pm \alpha_{2s-1}.$$

Hence the ranges of n associated with $2\mu, \dots, 2\mu/r$ are obtained.

Thus when $c=0$ equation (1) splits up into (2) and (3); and when $c = \frac{1}{2}n$, into (4) and (5).

4. When the non-generating force is simply periodic

$$\ddot{x} + \mu^2(1 + 2\alpha \cos nt)x = 0.$$

If α is so small that its square may be neglected, it is evident from (4) and (5) that the range of cumulative effect about the double frequency, 2μ , is $2\mu(1 \pm \frac{1}{2}\alpha)$. To a second approximation the limits are given by

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 \pm \alpha & \alpha \\ \alpha & 1 - 9\left(\frac{n}{2\mu}\right)^2 \end{vmatrix} = 0,$$

whence

$$n = 2\mu(1 \pm \frac{1}{2}\alpha - \frac{1}{16}\alpha^2). \quad \dots \quad (6)$$

For the limits of the range about μ , from (2) and (3)

$$\begin{vmatrix} 1 & \alpha & \\ 2\alpha & 1 - \left(\frac{n}{\mu}\right)^2 & \alpha \\ & \alpha & 1 - 4\left(\frac{n}{\mu}\right)^2 \end{vmatrix} = 0,$$

$$\text{and} \quad \begin{vmatrix} 1 - \left(\frac{n}{\mu}\right)^2 & \alpha & \\ \alpha & 1 - 4\left(\frac{n}{\mu}\right)^2 & \end{vmatrix} = 0,$$

$$\begin{aligned} i. e. & \quad n = \mu(1 - \frac{5}{6} \alpha^2) \\ \text{and} & \quad n = \mu(1 + \frac{1}{6} \alpha^2). \end{aligned} \quad \dots \dots \dots (7)$$

Again, for the range associated with $2\mu/3$,

$$\begin{vmatrix} 1 - \left(\frac{n}{2\mu}\right)^2 \pm \alpha & \alpha & \\ \alpha & 1 - 9\left(\frac{n}{2\mu}\right)^2 & \alpha \\ & \alpha & 1 - 25\left(\frac{n}{2\mu}\right)^2 \end{vmatrix} = 0,$$

$$\text{whence} \quad n = \frac{2\mu}{3} (1 - \frac{9}{32} \alpha^2 \pm \frac{81}{128} \alpha^3). \quad \dots \dots \dots (8)$$

These results are in agreement with those obtained previously by another method*.

5. From the physical point of view interest attaches to the change in the breadth and position of the range about the double frequency when the intensity of the force is increased. For this range the roots approach rapidly towards their limiting values if α is not too large, and we readily find:—

α .	Range of $\left(\frac{n}{2\mu}\right)^2$.	Range of $\frac{n}{2\mu}$.
0.25	0.7607, 1.2560	0.872, 1.121
0.50	0.5619, 1.5197	0.749, 1.232
0.75	0.4423, 1.7873	0.665, 1.337
1.00	0.4007, 2.0571	0.633, 1.434
1.25	0.3983, 2.3284	0.631, 1.526

and so on.

* *Loc. cit.* § 3.

If Z_n denotes the determinant obtained by taking the terms common to the first n rows and columns of either (4) or (5), when all the α 's except α_1 are zero,

$$Z_n = \{1 - (2n-1)^2 z\} Z_{n-1} - \alpha^2 Z_{n-2},$$

where $z = \left(\frac{n}{2\mu}\right)^2$. Hence if z_0 is a root of $Z_{n-1} = 0$, and if the corresponding root of $Z_n = 0$ differs from z_0 by a small quantity s , then approximately

$$s = \frac{\alpha^2 Z_{n-2}}{\{1 - (2n-1)^2 z\} \frac{dZ_{n-1}}{dz}}, \text{ when } z = z_0.$$

This affords a ready method of testing the approximation given by z_0 .

To the degree of accuracy shown in the table the last three lower limits follow from $Z_3 = 0$; the rest of the roots exhibited are given by $Z_2 = 0$, with the small correction s if necessary.

6. The cumulative effect of nongenerating force of long period, being comparatively feeble, is likely to escape notice, except in the case of a system in which the motional resistance is inappreciable and the time of observation extended. As both of these conditions hold for the solar system, it is natural to inquire whether any outstanding discrepancies between astronomical calculations and observations could be traced to the neglect of such slow cumulative action. Records of solar eclipses are available over a long period, and it is found that the calculated paths exhibit a regularly varying error which becomes more marked with the remoteness of the eclipse*. The regularity of variation is held to confirm the accuracy of the record; and the question remains as to how the discrepancy arises.

I would suggest that it may be due to the cumulative action of some lunar disturbance of relatively long period--an action requiring for its detection a higher approximation, perhaps, than has been attained in existing calculations.

* See, for example, "The Moon's Motion," *Nature*, Oct. 1908, p. 599.

IX. *On the Number and the Absorption by Matter of the β Particles emitted by Radium.* By WALTER MAKOWER, Demonstrator and Assistant Lecturer in Physics in the University of Manchester*.

WHEN radium is in radioactive equilibrium with its products of disintegration, it is known to emit α , β , and γ rays. The α rays are emitted by the radium itself, the emanation, radium A, and radium C; whereas the β rays are emitted by radium C and by radium B only†. Now it is known that the β rays consist of negatively charged particles projected with high velocity, whereas the γ rays carry no charge with them; so that a mass of radium so situated in an insulated enclosure that only the β and γ rays can leave it, should lose a negative charge, and therefore become charged with positive electricity. That this is the case has been shown by M. & M^{me} Curie‡, Strutt§, and others. It is a matter of importance to ascertain the magnitude of the negative charge carried away from a known quantity of radium in radioactive equilibrium, and hence to deduce the number of β particles emitted.

The first attempt was made by Wien||, from whose data it may be deduced that, if each β particle carries a charge of 3.4×10^{-10} electrostatic unit, one gram of radium emits 1.13×10^{10} β particles per second. The method of experiment consisted in suspending in a very perfect vacuum a known quantity of radium in an insulated platinum vessel connected to an electrometer, and measuring the charge acquired by the electrometer in a known time. The value obtained is, however, certainly too low, since no account was taken of the absorption of the rays by the walls of the platinum vessel. A subsequent series of experiments was carried out by Rutherford¶, in which this source of error was eliminated. A lead rod, 4 centimetres long and 4 millimetres in diameter, was made active by exposure to radium emanation for a sufficient time to obtain the maximum activity. After testing the quantity of active deposit thus collected on the lead rod by comparison with a standard quantity of radium, the rod was covered with just sufficient aluminium-foil (0.53 millim. thick) to absorb the α rays, and transferred to a vessel in which the charge acquired by

* Communicated by the Author.

† H. W. Schmidt, *Phys. Zeit.* vi. p. 897 (1905).

‡ M. & M^{me} Curie, *Comptes Rendus*, cxxx. p. 647 (1900).

§ Strutt, *Phil. Mag.* Nov. 1903.

|| Wien, *Phys. Zeit.* iv. (1903).

¶ Rutherford, *Phil. Mag.* Aug. 1905.

the rod when insulated in a perfect vacuum could be measured by a quadrant-electrometer. As a result of his experiment, Rutherford concluded that "the total number of β particles expelled by 1 gram of radium per second is 7.3×10^{10} " *.

In the light of recent investigations on secondary radiation, it seemed not impossible that in the above-described experiments errors might have arisen on account of secondary radiation of the β type set up when the rays from the active deposit impinged on the lead rod and on the containing cylinder. As it is difficult to estimate to what extent the value obtained was vitiated by this circumstance, it seemed desirable to repeat the experiments in such a way as to reduce the possibility of errors due to secondary radiation to a minimum. It was with this object, and at the suggestion of Professor Rutherford, that the following measurements were undertaken.

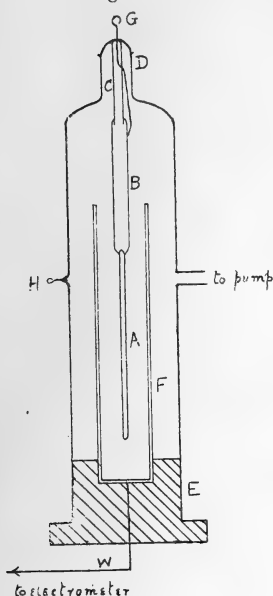
Method of Experiment.

As somewhat large quantities of radium would have been required to carry out the experiments satisfactorily, it was thought advisable to work, not with radium itself but with the emanation. The emanation from a considerable quantity of radium was therefore collected in a fine glass tube, about 1 millim. in diameter, the walls of which were exceedingly thin. When the emanation had been transferred into the tube, it was sealed off so that the emanation could not escape. After sufficient time had elapsed for radioactive equilibrium to be established, the quantity of emanation which had been collected was determined by comparing the γ radiation emitted from the tube with that from a standard quantity of radium. The quantity of radium emanation in the tube at any subsequent time could then be deduced from the well-known rate of decay of the emanation. The tube, which was about 7 centims. long, was then covered with thin aluminium-leaf to render it conducting, and introduced into the measuring apparatus, the arrangement of which can be seen from fig. 1. The tube A containing the emanation was attached by sealing-wax to the brass tube B, which was in turn cemented to the glass tube C fused on to the stopper D of an internally silvered glass vessel of 4.4 centims. diameter. By inserting the stopper D, the tube A could be made to lie along the axis of the glass vessel. The other end of the glass

* Taking the value 4.65×10^{-10} for the charge carried by the β particle, instead of 3.4×10^{-10} , which is probably too low, this value should be 5.33×10^{10} .

vessel was closed by a large ebonite stopper E, into which fitted a brass cylinder F, of diameter 2 centims., as indicated in the diagram, and the thickness of whose walls was 1.3 millim.; the length of this tube was 11 centims. When

Fig. 1.



in position, the tube A lay along the axis of the brass tube F. The cylinder F could be connected to a quadrant electrometer by the wire W passing through the ebonite stopper E. By means of the platinum wire G, fused through the stopper of the glass vessel, the brass tube B, and therefore the aluminium-leaf covering the tube containing the emanation, could be maintained at any desired potential by a battery of accumulators. The platinum wire H, sealed through the glass vessel, served to connect the silvered surface of the glass to earth.

To carry out an experiment, the silvered glass vessel was evacuated as perfectly as possible by means of charcoal and liquid air, and, by a quadrant electrometer, the negative charge reaching the brass cylinder F from the tube containing the emanation on account of the β rays emitted by it, measured. The walls

of the glass tubes containing the emanation, in the two sets of experiments which were made, were .12 millim. and .078 millim. thick respectively. This thickness of glass was sufficient to completely stop the α rays coming from the emanation, but insufficient to stop more than a small fraction of the β rays from radium C. The slowly moving β rays from radium B are nearly, though not completely, absorbed by the above-mentioned thickness of glass. It will thus be seen that the β radiation reaching the brass cylinder F consisted, for the most part, of the β rays from radium C, together with a small amount of radiation from radium B. In order to find the charge which would be acquired by the cylinder on account of the radium C contained within the glass tube A alone, it is necessary to apply corrections for the small quantity of radium reaching it from the radium B in the glass tube, and also for the absorption of the rays from radium C by the walls of the glass tube. The method of applying these corrections will appear later.

In order to eliminate errors due to faulty insulation and to the ionization of the residual gas between the tube containing the emanation and the brass cylinder, observations were made first with the tube containing the emanation connected to the positive pole, and then to the negative pole of twenty storage-cells. By taking the mean of the charges acquired by the brass cylinder in a fixed time in these two cases, the above mentioned errors were eliminated.

Assuming that all the β rays are stopped by the brass cylinder, it is clear that, if the capacity of the glass tube together with that of the electrometer and connexions is known, the total negative charge emitted per second by the known quantity of radium C contained in the glass tube can be calculated. Now it is known that the charge carried by the β particle is 4.65×10^{-10} electrostatic unit. Hence, from the above measurements, the number of β particles emitted by the quantity of radium C in equilibrium with one gram of radium can be calculated. In the experiments here to be described, the current passing from the emanation to the brass cylinder was so large that it was necessary to insert, in parallel with the electrometer, a large capacity. On this account it was necessary to determine the capacity of the electrometer and its connexions only roughly. This was done by screening off part of the radiation from the tube containing the emanation by glass tubes placed over it, and, when the radiation had been sufficiently reduced to get readable deflexions of the electrometer, comparing the deflexion produced in a given time with and without a condenser of suitable capacity in parallel with the electrometer.

It has already been mentioned that these measurements were undertaken with a view to avoiding errors due to secondary radiation of the β type set up when β rays impinge on an obstacle. The following considerations will show that the method described above is calculated to reduce such errors to a minimum. The radiation proceeding from the tube containing the emanation fell upon the brass cylinder, where it was absorbed. During this process secondary rays were emitted from the inside of the brass cylinder, and so would have escaped detection by the electrometer had not steps been taken to reabsorb them by the brass cylinder. Now any radiation proceeding from the inside of the walls of the brass cylinder will impinge on some other portion of the cylinder, since this forms an almost perfect enclosure, and will therefore ultimately all be absorbed and give up its charge, except for the small amount of radiation

which escapes at the open end of the cylinder, and the secondary radiation which, impinging on the walls of the glass tube containing the emanation, is there reabsorbed. The radiation escaping at the top of the cylinder was certainly small, and did not amount to one per cent. of the total radiation; the secondary radiation absorbed by the glass tube must also have been inconsiderable, since the thickness of the glass was too small to absorb much of the radiation falling on it. If, further, we take into consideration the small surface of the glass tube, which amounted to only one-twentieth of the area of the brass cylinder, it is evident that errors due to secondary radiation were without effect within the limits of experimental errors.

The following numbers illustrate the method of measurement:—With the emanation from 13·17 milligrams of radium-bromide contained in a glass tube ·078 millimetre thick, the electrometer charged up ·238 volts per minute when the tube containing the emanation was connected to the positive pole of a battery of 20 storage-cells, and ·281 volts when connected to the negative pole. The capacity of the whole system connected to the electrometer was 9150 centimetres. Assuming the charge carried by the β particle to be $4\cdot65 \times 10^{-10}$ electrostatic unit, this gives as the number of β particles emitted per second by the radium C in equilibrium with 1 gram of radium, the value $3\cdot68 \times 10^{10}$. A second experiment with the same glass tube gave $3\cdot41 \times 10^{10}$. Having regard to the easy absorption of the β rays from the active deposit from radium for small layers of absorbing material, these numbers are in fair agreement with the number $5\cdot33 \times 10^{10}$ found by Rutherford when the radiation had to traverse only ·053 millimetre of aluminium.

The Absorption of the β Rays by Glass.

It has already been pointed out that it is necessary to apply a correction to the number of β particles emitted by radium given above, on account of the absorption of the rays by the walls of the thin glass tube containing the emanation. In order to determine the magnitude of the correction to be applied, it was necessary to study the law of variation of the charge emitted by the emanation sealed in the glass tube after traversing different thicknesses of glass. It was also thought that experiments on this point might throw light on the interesting question of the nature of the process by which the β rays are absorbed by matter. If, as is sometimes supposed, the absorption of the β rays consists in a scattering

of the primary rays, then, with the apparatus described above, the quantity of electricity reaching the outer brass cylinder should decrease little, if at all, when screens of glass are interposed between it and the tube containing the emanation. For after scattering, the quantity of radiation reaching the brass cylinder will be the same as before. On the other hand, if the rays are actually stopped on passing into the glass, leaving their charge on the glass by which they are absorbed, then the charge reaching the brass cylinder will be diminished by interposing screens of glass. This was found to be the case; so that the absorption of the β rays by glass cannot, at any rate, be entirely accounted for by considering the rays to be scattered. If, further, the law of absorption as found by measuring the charge transmitted through different thicknesses of glass is found to be identical with that as measured by the ionization produced by the β rays after absorption by different thicknesses of glass, then the falling off of the ionization due to the rays must be entirely attributed to the stopping of the negatively charged particles of which the rays are known to consist. It will be seen from the experiments now to be described that such is the case*.

The method of experiment was the same as that already described, the thickness of glass through which the radiation passed before reaching the brass cylinder being increased by covering the glass tube containing the emanation successively by a series of thin glass tubes of gradually increasing diameter. The glass tubes so interposed were always coated with thin aluminium-leaf to render them conducting.

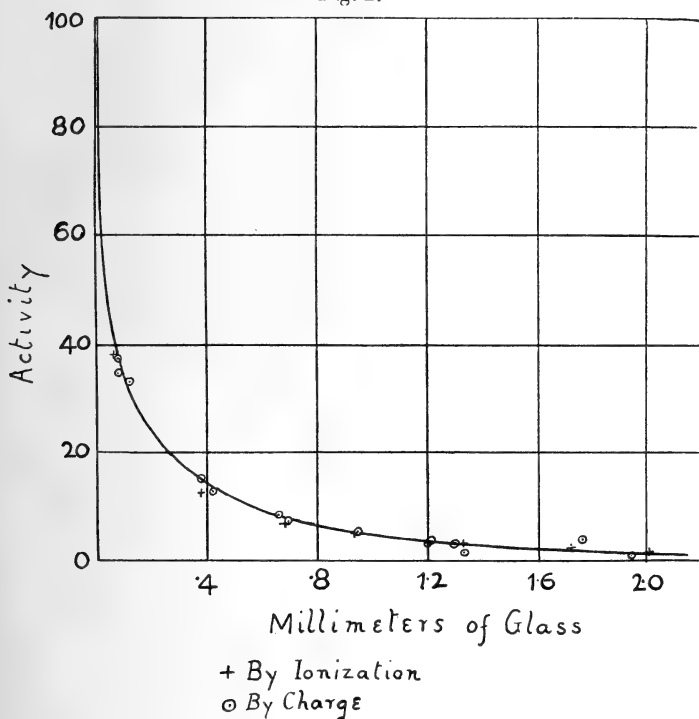
The law of absorption of the β rays from radium B and C by aluminium, as measured by the ionization produced by the rays after transmission through different thicknesses of that metal, have been very carefully investigated by Schmidt†. As the densities of glass and aluminium are nearly the same, it might be expected that the absorption of the β rays by these two substances would be not very different. It was, however, necessary to find out whether this supposition was correct or not, and a series of experiments was carried out to test the question.

* It has been shown by Crowther (Proc. Roy. Soc. A. lxxx. 1908) that the β rays are scattered after traversing very small thicknesses of matter. The present experiments are, however, quite consistent with this phenomenon, since the rays from the emanation had always to traverse the walls of the containing tube. But it seems clear that the diminution of intensity of the β rays on traversing matter is ultimately effected by an actual stoppage of the particles of which the rays consist.

† H. W. Schmidt, *Ann. der Phys.* xxi. p. 609 (1906) and *Phys. Zeit.* vii. p. 764 (1906).

For this purpose, an aperture was cut in an ordinary β -ray electroscope and closed by a thin aluminium filter just thick enough to absorb the α rays. The thin-walled glass tube was then placed just outside the electroscope, as close up to the aluminium as possible, and various thicknesses of glass tubing interposed between the emanation and the electroscope by passing glass tubes over the tube containing the emanation. The results obtained were then compared with those given by Schmidt for aluminium. The results are shown graphically in fig. 2, from which it will be seen that the absorption-curves for glass and aluminium are substantially the same. The curve

Fig. 2.



is drawn from the values of the ionization obtained by Schmidt when different thicknesses of *aluminium* were interposed between the radium B and C and the electroscope. The points marked (+) indicate the value of the ionization obtained in the experiment with *glass* when the thickness of glass traversed by the β rays was increased by superposing successive glass tubes over the tube containing the emanation.

It will be seen that the absorption-curves for aluminium and glass do not differ appreciably from each other. The points marked (⊙) indicate the value of the charge in arbitrary units received by the brass cylinder surrounding the tube containing the emanation in the experiments described above. It will be seen that these points, too, lie fairly closely on the curve obtained for aluminium by measuring the ionization caused by the rays after passing through different thicknesses. This shows that when the β rays are absorbed by glass, the charge carried by the particles which are absorbed remains in the glass, indicating that the rays are actually stopped by the glass and not merely scattered.

Corrected Value of the number of β particles emitted by Radium C.

The value for the number of β particles emitted by radium C given above is incorrect, for two reasons. In the first place, some of the radiation from radium C is absorbed even in passing through the very thin glass tube used to contain the emanation; and in the second place, a small amount of radiation passing through the walls of the glass tube is due to the β rays from radium B. It remains to apply a correction for these two errors. Now the absorption-curves for the β rays from radium B and C have been calculated *separately* by Schmidt*. On the assumption that radium B and C, when in radioactive equilibrium, emit the same number of β particles, it is possible to deduce the number of particles emitted by either separately, *i. e.* the number which would impinge on the outer brass cylinder (fig. 1) when no glass is interposed between the emanation and the cylinder. Table I. shows the experimental numbers obtained, and fig. 3 shows the method by which the above-mentioned correction was applied. The curve obtained from the experiments described above is shown by the continuous line. The dotted curve has been obtained by subtracting from the ordinates at each point the charge due to the β rays from radium B, and then extrapolating back to thickness zero from the values given by Schmidt in his paper. The value so obtained for the number of β particles emitted by radium C per gram of radium per second is 5.0×10^{10} .

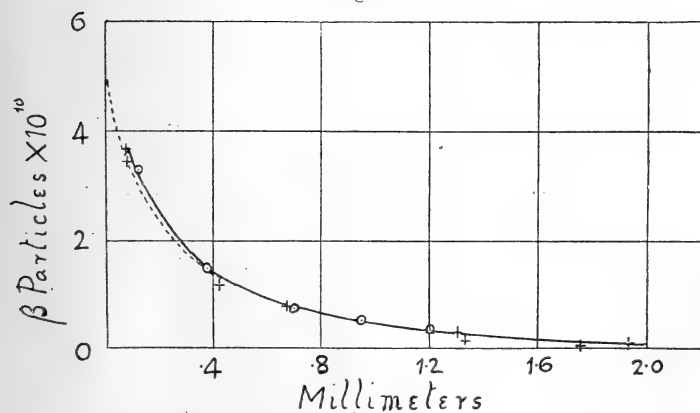
In a recent determination of the number of α particles emitted by radium C Rutherford and Geiger give the number 3.4×10^{10} . Considering the uncertainty in applying the corrections just given, the values for the numbers of α and β particles emitted

* Schmidt, *loc. cit.*

TABLE I.

Thickness of glass, in millimetres.	Number of β particles per second from radium C in one gram of radium.
First Series of Experiments.	
·120	3.27×10^{10}
·376	1.51 „
·695	·722 „
·945	·533 „
1.205	·342 „
Second Series of Experiments.	
·078	3.68×10^{10}
·078	3.41 „
·409	1.19 „
·667	·79 „
1.199	·311 „
1.299	·297 „
1.329	·158 „
1.76	·0394 „
1.936	·0834 „

Fig. 3.



by radium C are in fair agreement. It would therefore seem that for every α particle emitted by radium C, one, or possibly two, β particles are simultaneously emitted. It should, however, be noticed that this conclusion has been reached on the assumption that radium B and radium C, when in radioactive equilibrium, emit the same number of β particles.

Conclusions.

(1) The β rays are stopped and not merely scattered when they pass through matter.

(2) The number of β particles emitted per second by the radium C in radioactive equilibrium with one gram of radium is 5.0×10^{10} .

I am indebted to Professor Rutherford for the interest which he has taken in this investigation and for many valuable suggestions. My thanks are also due to Dr. Beattie for comparing the condensers used in the above experiments with his standards.

X. *The Absorption of β Rays by Liquids.* By NORMAN CAMPBELL, M.A., *Fellow of Trinity College, Cambridge* *.

1. IF I_0 is the intensity of a pencil of β rays, I the intensity of the same pencil when it has passed through a uniform layer of absorbing material of thickness x , then the coefficient λ of absorption of the material for the rays is defined by the equation :

$$I = I_0 e^{-\lambda x}.$$

If ρ is the density of the absorbing material, Crowther (Phil. Mag. Oct. 1906, p. 388) has shown that the value of λ/ρ for a compound containing the proportions x, y, z, \dots of the elements X, Y, Z, can be calculated from the known values of the same quantity for the elements by means of the formula :

$$(\lambda/\rho)_{xyz\dots} = x \cdot (\lambda/\rho)_x + y \cdot (\lambda/\rho)_y + z \cdot (\lambda/\rho)_z + \dots$$

Crowther's work extended only to solid compounds. Some experiments on the β rays from potassium described in a recent paper (Proc. Camb. Phil. Soc. xiv. 6, 557) indicated that the law was not true for such rays absorbed in solutions of potassium salts : it appeared that the value of λ/ρ for the solution might be greater than either of those for the solute

* Communicated by the Author.

or the solvent. Accordingly it seemed desirable to discover whether the abnormality arose from the rays or from the solutions. The following pages contain a brief description of measurements of the value of λ/ρ for the β rays of uranium passing through various solutions and other liquids.

2. The apparatus used was identical in principle with that of Crowther and needs no lengthy description, except in regard to one detail. It was thought that some difficulty might be found in obtaining layers of liquids which should be thin enough to be penetrated by a measurable proportion of the rays and yet be uniform in thickness. Such layers must not be much more than 0.2 cm. thick and, if the liquid were merely poured into a tray covering the active material, a slight deviation of the tray from the horizontal and a distortion of the surface by capillary action might cause variations in the thickness sufficient to produce notable errors in the measurements. Accordingly, the device was adopted of absorbing the liquid in a sheet of thick filter-paper placed in a carefully levelled tray.

In the final form of the apparatus this tray was formed out of two zinc rings, 16.5 cm. in external diameter, 8.5 cm. in internal diameter, and 0.35 cm. thick, screwed tightly together and enclosing between them a sheet of mica about 0.02 cm. thick. The upper surface of the mica and the internal edge of the upper zinc ring formed the tray in which the filter-paper was placed. In the tray formed by the lower surface of the mica and the internal edge of the lower zinc ring, was placed about 50 grammes of uranium oxide held in place by paraffin-wax melted over its surface. The rings rested upon 3 V-ways, along which they could be pushed into a definite position under the ionization-vessel: the ways were levelled so that the upper surface of the mica was accurately horizontal when the rings were in the fixed position. A thin sheet of mica rested on the upper ring and formed a cover to the tray, preventing evaporation of the liquid.

The ionization-vessel was a circular cylinder, 16.5 cm. in diameter and 8.0 cm. high. The top and the sides were of zinc sheet, the bottom, through which the rays passed, was of tinfoil. An electrode, passing through ebonite insulation and the usual guard-ring, led to a Wilson tilted electroscope. The natural ionization of the vessel was measured every hour and, of course, subtracted from the observed ionization when the uranium rays were acting.

3. In measuring the absorption coefficient of the solid solutes, measurements were made of the value of I for about

ten different values of x . (The latter quantity was always determined from the mass of the substance placed in the tray and the known area (σ) of the tray. Since the value of λ/ρ and not of λ is required, a knowledge of the density is not necessary.) It was found that the most satisfactorily uniform layer of such solids as were used could be obtained by reducing the solid to fine powder and sprinkling it over the surface of the tray out of a vessel of the nature of a pepper-pot. The usual graphical method of determining λ/ρ from such observations was not sufficiently delicate to take full advantage of the accuracy of the observations: this quantity was found by solving by the least square method the equations of form:

$$\log I_0 - \log I = \lambda/\rho \cdot \frac{1}{\sigma} \cdot m$$

resulting directly from the observations. I_0 and λ/ρ are the unknowns. The calculated probable error was never more than 1 per cent.

However, since only comparative measurements on the solutions of different concentrations were required, it was thought in the first instance sufficient to make observations on the value of I for only one value of x , the same for all the solutions. Accordingly in each case the contents of a pipette (6.691 c.c.) were poured upon the filter-paper. Ten measurements of this nature were made on each solution: the agreement between the calculated values of λ/ρ was such that the probable error of the mean was always less than 0.8 per cent.

The composition of the solutions was estimated by preparing them by dilution of a saturated solution, the content of which was known by analysis. The values thus obtained were confirmed from density measurements with the aid of Landolt and Bornstein's tables.

4. Three solutions were investigated in this manner, namely, aqueous solutions of potassium carbonate, potassium acetate, and calcium chloride. The following diagrams (1, 2, 3) indicate the results obtained.

The straight line on the diagram joins the points representing the pure solvent and the pure solute. If the law enounced by Crowther for solid compounds were true for liquids, all the points would lie on this line. The points do not lie on the line, and it will be noted that in the case of each of the three solutions, a curve drawn through the observed points seems to possess a common feature, namely, a point of inflexion nearly midway between the solvent and

the saturated solution. However, since the largest deviation of any one point from the straight line is not four times the probable error of that point, it is scarcely legitimate to

Fig. 1.

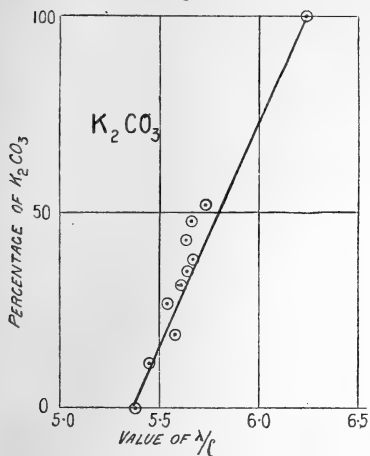


Fig. 2.

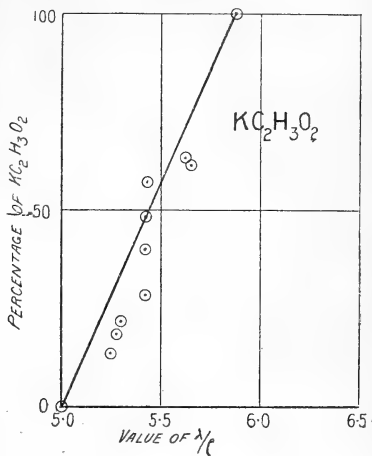
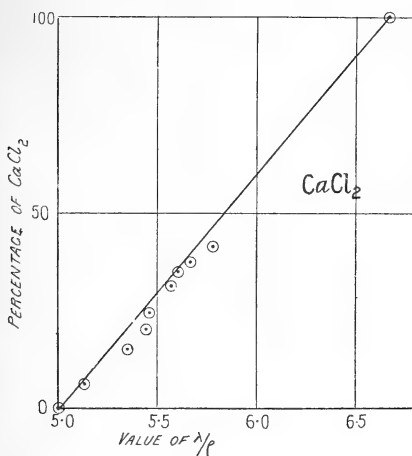


Fig. 3.



draw any definite inference from the results. The accuracy attained is not sufficiently great to determine the issue investigated.

5. In endeavouring to increase the accuracy of the observations it was found that, if a larger volume of solution was used, a smaller value was found for λ/ρ . It appeared

that the "absorption curve" for the solutions was not accurately exponential and detailed investigations on this point were made.

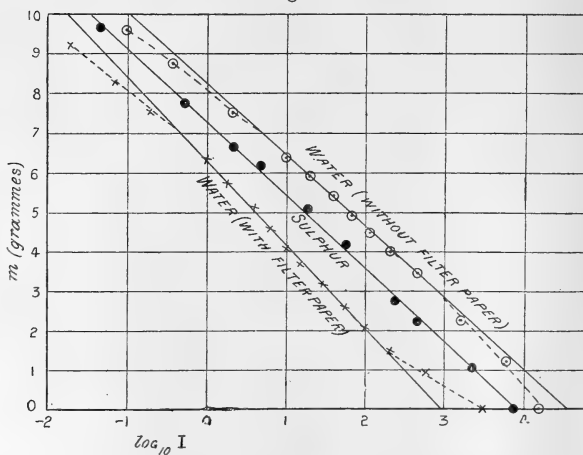
In the first place, it was investigated whether, through any defect in the apparatus, the absorption curve for a solid deviated from the exponential form. Table I. gives the results for measurements on sulphur, a solid which was convenient for the purpose. The results are plotted in diagram 4, and it will be seen that there is no evidence of a departure from the theoretical straight line. (m is the mass of the substances in the tray.)

TABLE I.—Sulphur.

$m.$	$\log_{10} I.$
(1) 0	$+ \cdot 3874 = \log_{10} I_0$
(2) 1·025	·3304
(3) 2·244	·2648
(4) 2·757	·2387
(5) 4·109	·1741
(6) 5·004	·1255
(7) 6·120	·0664
(8) 6·680	·0327
(9) 7·755	—·0285
(10) 9·667	·1324

Calculated value of λ/ρ $6\cdot967 \pm 0\cdot038$.
(Crowther gives $6\cdot6$.)

Fig. 4.



Other solids investigated were filter-paper, aluminium, and the solutes of the solutions used. Observations of these salts,

which are hygroscopic, were perforce less accurate than the others, but in no case was any deviation of the absorption curve from the exponential form indicated.

Table II. gives measurements made on various masses of water poured on filter-paper in the tray. The results are plotted also in diagram 4 and the deviation from the straight line is obvious. The dotted curve through the observed points is concave upwards for small masses and concave downwards for large. The intermediate part of the curve is approximately straight and from it a value of λ/ρ can be deduced. Taking the observations numbered 3 to 12 inclusive the value for λ/ρ is obtained,

$$6.243 \pm 0.064.$$

Observations were then taken on the absorption of water poured directly into the tray without the use of filter-paper to contain it. Table III. gives these observations which are

TABLE II.

Water with filter-paper.

$m.$	$\log_{10} I.$
(1) 0	+·3475
(2) 0.897	·2750
(3) 1.492	·2290
(4) 2.064	·1993
(5) 2.585	·1738
(6) 3.158	·1443
(7) 3.673	·1162
(8) 4.087	·0997
(9) 4.560	·0797
(10) 5.101	·0599
(11) 5.713	·0253
(12) 6.317	-·0066
(13) 7.555	·0722
(14) 8.281	·1160
(15) 9.201	·1702

$$\lambda/\rho = 6.243 \pm 0.064.$$

TABLE III.

Water without filter-paper.

$m.$	$\log_{10} I.$
(1) 0	+·4160 = $\log_{10} I_0$
(2) 1.203	·3761
(3) 2.262	·3198
(4) 3.489	·2615
(5) 3.998	·2297
(6) 4.442	·2031
(7) 4.893	·1818
(8) 5.381	·1584
(9) 5.877	·1290
(10) 6.355	·0976
(11) 7.524	·0302
(12) 8.736	-·0425
(13) 9.615	·1013

$$\lambda/\rho = 7.176 \pm 0.082.$$

also plotted in diagram 4. It appears again that the absorption is not exponential: the dotted curve is concave downwards both for large and small masses. Taking the figures numbered 4 to 10 inclusive, which lie fairly well on a straight line, the value of λ/ρ is obtained

$$7.176 \pm 0.082.$$

Various modifications of the apparatus were tried to reduce the curve to its normal form. Changes in the shape and

position of the ionization-vessel may change the absolute values of the absorption coefficient of solids as well as liquids by as much as 10 per cent., but they did not seem to have any effect upon their relative values or upon the shape of the absorption-curve for the liquids. Other liquids were tried—paraffin, aniline, alcohol, as well as several solutions, but the curves with and without filter-paper all showed the same peculiarities. In all cases the absorption coefficient (deduced from the straight part of the curve) was less when filter-paper was used than when it was not used. Different thicknesses of filter-paper produced little, if any, effect upon the observations. Since the curvature of the curves with and without filter-paper are in opposite directions in their earlier parts, it might be possible to find paper so thin that the curve should be straight; but the thinnest paper obtainable gave results differing very little from the thickest.

The shape of the early part of the curve without filter-paper can be explained easily on the assumption that the thin layers do not spread themselves evenly over the tray; for a non-uniform layer would absorb less rays than a uniform layer of the same mass. But I cannot offer any explanation of the other deviations from a straight line, or (the most important point of all) of the difference between the absorption coefficients calculated with and without the filter-paper.

It should be noted that the calculated probable error of λ/ρ for a liquid is always greater than that for a solid, and greater than that estimated from the known errors of the constituent observations. This fact would seem to indicate that that part of the curve for solutions which appears to be a straight line is not so in reality—as might be expected from the form of the remainder of the curve.

6. It appears doubtful, then, whether liquids possess a true absorption coefficient as defined in the first paragraph; at any rate it is obvious that that quantity differs according to the method of measurement. Nevertheless, it was thought worth while to continue the comparison of the absorption of solutions of different concentrations by adopting throughout the same method of calculating the absorption coefficient. Since rather more consistent results were obtained with the use of filter-paper, that method was adopted. The absorption is approximately exponential for masses of the liquid between 2 and 7 grammes; some ten measurements were made on each solution between these limits, and the value of λ/ρ obtained from them by a least square calculation.

Diagrams 5–8 give the results obtained. Four solutions

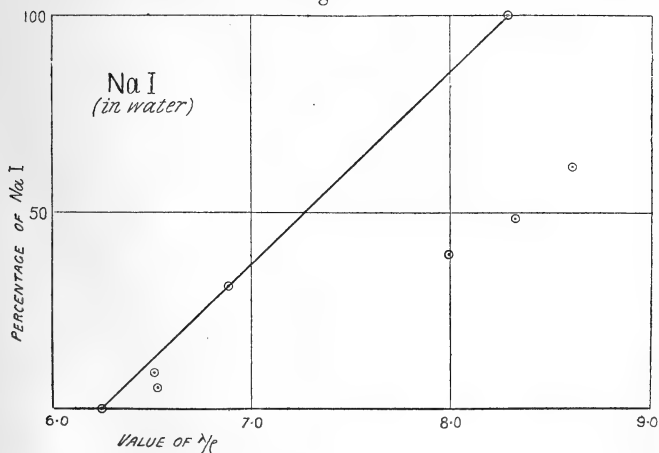
were used—aqueous solutions of sodium iodide, potassium carbonate, potassium nitrate, and an alcoholic solution of sodium iodide. The actual figures are given in Table IV. for the first of these solutions in order to show the order of the errors. (The error for the solid iodide is exceptionally large because that salt is very hygroscopic.)

TABLE IV.
NaI in Water.

Water	$\lambda/\rho = 6.154 \pm .063$
NaI 5.4 p. cent.	$6.528 \pm .059$
NaI 9.1 „	$6.515 \pm .065$
NaI 31.4 „	$6.881 \pm .056$
NaI 39.3 „	$7.981 \pm .073$
NaI 48.24 „	$8.324 \pm .069$
NaI 61.76 „	$8.611 \pm .051$
NaI (solid)	$8.281 \pm .094$

The results are so different for different substances that it is impossible to draw any general conclusions. Except possibly in the case of potassium carbonate, it is clear at

Fig. 5.



once that the linear relation given by Crowther for solids does not hold for liquids, if their absorption is measured by

Fig. 6.

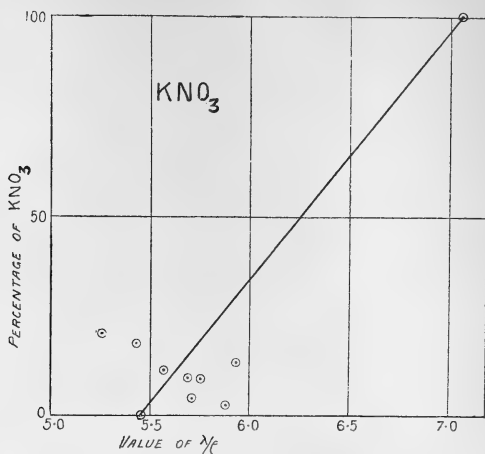
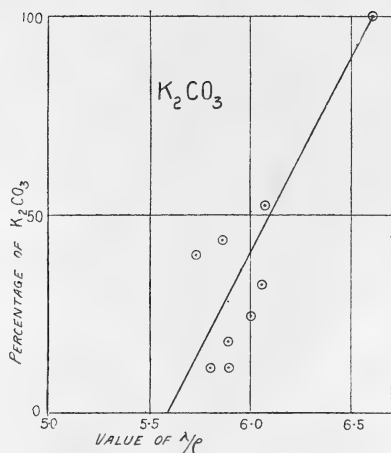
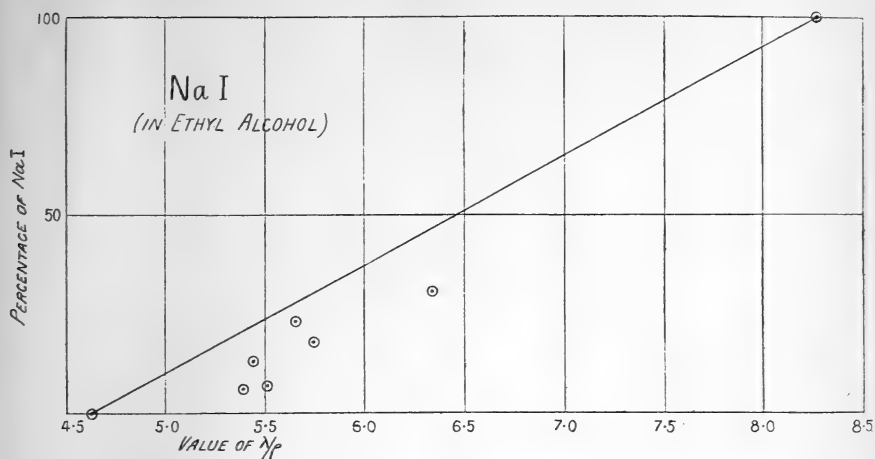


Fig. 7.



the method selected. It is interesting to note that in the case of an aqueous solution of potassium nitrate, the value of λ/ρ seems to decrease with an increase in the concentration, although the value of the same quantity for the solute is notably greater than that for the solvent.

Fig. 8.



7. One more experiment must be described. An attempt was made to discover whether the same mass of substance possessed the same absorption in the solid and liquid states. A certain volume of water was poured into the tray (without filter-paper) and frozen by pouring liquid air into a metal vessel covering the tray. The tray was then placed under the ionization-vessel and the readings of the electroscope taken while the ice melted. Similar observations were taken by pouring melted paraffin-wax into the tray and taking readings while it solidified. The measurements were not very reliable because the temperature changes of the walls and the air of the ionization-vessel undoubtedly produce small changes in the current through it; but no certain change in the current could be detected while the substance changed from the solid to the liquid state or *vice versa*. Such a conclusion is to be expected. It is very improbable that the complications of the absorption of the rays by liquids are due to some peculiarity in the condition of the molecules of the substances, for if absorption were a property dependent on the state of the molecules, it would not follow the atomic law stated by Crowther for solids. It is much more probable that they are due to some peculiarity of the distribution of the liquid in the tray, and this distribution would not be altered greatly by the solidification of the substance *in situ*. The detailed investigation of the absorption of any substance in two different states involves the measurement of ionization currents at different temperatures. The accuracy obtainable in such measurements at any temperature other than atmo-

spheric is so much less than that at atmospheric temperature, that it seems doubtful whether the results would be sufficiently definite to lead to any certain conclusion.

8. It is not pretended that the experiments which I have described are in any way final. My only object in undertaking them was to compare the potassium rays with the β rays of uranium. The absorption of the former in solutions is abnormal and so, it appears, is the absorption of the latter. Moreover, the abnormalities are, in general, of the same kind. The object of the investigation is attained, and it is not proposed to pursue it further in the immediate future. Light is far more likely to be thrown upon the difficult question of the absorption of β rays by experiments, such as those of McClelland, Schmidt, and Crowther, on the secondary radiation excited, than by any direct measurements of the absorption coefficient.

I am much indebted to Mr. Crowther for valuable suggestions and for the benefit of his great experience in work of this nature.

Summary.

The absorption of the β rays of uranium in liquids (especially solutions) has been investigated. It is doubtful whether liquids possess a true absorption coefficient defined as a quantity occurring in an exponential equation.

One of several possible, but inconsistent, methods of estimating a quasi-absorption coefficient has been selected. It appears that the relation between the values of λ/ρ (where λ is estimated by this method) for the solution, the solvent and the solute is not linear. It is possible that the value for a solution should be greater than either of the values for the solvent or the solute, or should be less than either of those values.

Sept. 3, 1908.

XI. *Note on the Amount of Water in a Cloud formed by Expansion of Moist Air.* By W. B. MORTON, Professor of Natural Philosophy, Queen's College, Belfast*.

THE calculation of the total mass of water condensed, when a volume of air saturated with vapour receives a given sudden expansion, is an important step in Prof. J. J. Thomson's determination of the charge on a gaseous ion†.

* Communicated by the Physical Society: read June 12, 1908.

† 'Conduction of Electricity through Gases,' p. 122.

In conjunction with the size of a single drop, as found from the rate of fall of the cloud, it gives the number of drops in unit volume and so the number of the ions which form the nuclei of the drops. In making the calculation Prof. Thomson, following C. T. R. Wilson, makes the assumption that the air is cooled down to the full extent by the adiabatic expansion before the drops begin to form. Then condensation proceeds and the latent heat liberated warms the air until the density of the remaining vapour is such as to saturate the air at the increased temperature. This process is, of course, irreversible. There is no reason to doubt that it represents closely the actual experimental conditions; but with a view to estimate the amount of error which would be introduced by a small departure from the assumed process, it seems of some interest to calculate, for comparison, the result of the other extreme assumption, viz., that the expansion and condensation exactly keep pace with each other, the density of vapour at each stage being the saturation-density corresponding to the instantaneous value of the temperature. In this case the process would be reversible, and we can equate the initial and final values of the entropy of the combined mass of air and water-substance. Since an irreversible change is accompanied by an increase of entropy, it is easy to see that on the new assumption we shall arrive at a lower value for the density of the remaining vapour and higher value for the mass of water in the cloud.

In both cases the solution is most easily obtained by a graphical method. We plot a curve connecting the vapour density ρ and temperature t at any stage of the condensation, and find where it intersects the curve representing the maximum density of saturated vapour at different temperatures. In the irreversible case the former curve is the straight line.

$$\rho = \frac{\rho'}{x} - \frac{CM}{L}(t - t_2)^*;$$

where (using the notation of the authors quoted) ρ' is the initial saturation density, x the expansion ratio, t_2 the calculated temperature after adiabatic expansion, M the mass of unit volume of air after expansion, C specific heat of air at constant volume, L latent heat of the vapour. The heat-capacity of the drops is neglected in comparison with that of the air. For the reversible case the equation of constant

* Equation (2), *loc. cit.*

entropy comes out

$$(xMC + sp') \log \frac{273 + t}{273 + T} + L \left\{ \frac{x\rho}{273 + t} - \frac{\rho'}{273 + T} \right\} + xMR \log x = 0,$$

where, in addition to the symbols already define, s is the specific heat of water, T the temperature before expansion, and R the gas constant for one gram of air.

I have plotted this curve for the particular case discussed in detail by Prof. Thomson, viz., starting from an initial temperature of $16^{\circ}\text{C}.$, and giving an expansion of 1.36. The graph comes out practically a straight line running parallel to the irreversible line, both lines being nearly perpendicular to the saturation-density curve with which their intersections are sought. The numerical values come out on the reversible hypothesis,

$$\rho = 4.87 \times 10^{-6}, \quad t = 0.5, \text{ mass of cloud per cm.}^3 \quad 5.06 \times 10^{-6} \text{ gm.}$$

as compared with the values obtained on the accepted assumption by Prof. Thomson,

$$\rho = 5.15 \times 10^{-6}, \quad t = 1.2, \text{ mass of cloud per cm.}^3 \quad 4.77 \times 10^{-6} \text{ gm.}$$

So that even on the extreme assumption here made we arrive at a value differing by only 6 per cent. from the other. We can infer that no error, comparable with the other inaccuracies attached to the methods of measurement, will be involved if the assumption of complete adiabatic cooling of the air does not exactly correspond with the facts.

27th May, 1908.

XII. *The Coefficient of Viscous Traction of Lead and Tin Alloys.* By ALBERT ERNEST DUNSTAN, B.Sc.*

FROM results obtained in experiments where pitch, shoemakers' wax, and the like materials were subjected to torsional stresses, Trouton (Phil. Mag. 1904, xix. p. 34) found that the rate of twisting was approximately proportional to the driving couple, and the proportionality between the shearing stress and the induced flow was sufficiently exact to render useful the introduction of a coefficient of viscous flow for such materials. The viscous flow was shown also by the same author (Proc. Roy. Soc. 1906, A. lxxvii. p. 426) in the case of drawn-out rods of the same materials

* Communicated by the Author.

to be proportional approximately to the force of traction per sq. cm. of cross-section, *i. e.*

$$\frac{F}{A} = \lambda \frac{dv}{dx} \dots \dots \dots (1)$$

F being the shearing force in dynes ;

A being the area of cross-section in sq. cms. ;

v being the velocity in cm.-secs. of any point distant *x* cms. from the point of support ; and

λ being the constant for the particular material,

this constant being termed by Trouton the coefficient of viscous traction.

Trouton and Rankine (*Phil. Mag.* 1904, viii. p. 538), in the case of a lead wire stretched beyond its elastic limit (and for this material the elastic limit is practically obtained with so small a load as 10 grams, though not by the weight of the wire itself), pointed out that :

(1) The viscous flow has a larger initial rate than that which obtains in the subsequent steady period.

(2) The tractive force is hence not strictly proportional to the rate of flow, so that

$$T - T_0 = \lambda \frac{dv}{dx}, \dots \dots \dots (2)$$

a correction T_0 being made for the initial effect. A close connexion exists between the ordinary coefficient of viscosity as usually determined and this new coefficient of viscous traction, *viz.*

$$\eta = 3\lambda.$$

This equation (Trouton, *Proc. Roy. Soc. A.* lxxvii. p. 438), obtained, analytically, from the flow of a viscous cylinder of incompressible material, was confirmed by direct comparison of results from tractive experiments, sagging of a beam and Poiseuille's capillary tube method.

Thus from a mixture of pitch and tar,

	λ .	η .	λ/η .
i.	4.3×10^7	1.4×10^7	3.07
ii.	3.3×10^7	1.0×10^7	3.30
iii.	12.9×10^7	4.2×10^7	3.07

In the present communication, this valuable method for determining viscosity coefficients of solid bodies at the ordinary temperatures has been applied to lead, tin, and alloys of these metals.

The following conclusions, which have been deduced in a *Phil. Mag.* S. 6. Vol. 17. No. 97. *Jan.* 1909. O

series of papers on the viscosity of liquid mixtures, may be briefly summarized in connexion with this work on alloys (Dunstan, Chem. Soc. Journal, 1904, lxxxv. p. 817; 1905, lxxxvii. p. 11; 1907, xci. p. 83).

1. There is an intimate relationship for all substances between the viscosity coefficient and the molecular condition.

2. In the case of liquid mixtures of completely miscible components, the viscosity concentration curves afford evidence of the mutual effect of the one component on the other; and from observations of the tangential angles at each end of the viscosity concentration curve, it is possible to obtain, firstly, the relative molecular masses of the component liquids; and, secondly, evidence as to their dissociation on the one hand, and as to their homogeneous or heterogeneous association on the other.

3. Maxima in the viscosity concentration curves are found at points of simple molecular composition (*e. g.* 1 alcohol 3 water), and are due to the associations of molecules of each compound in simple ratios.

4. Minima are also found at points of simple molecular composition, and are due to the partial or ultimate fission of one complex component by the other.

5. For any chemical compound

$$\eta = A + B \log x,$$

where η is the viscosity coefficient of the compound, A is a general constant for all substances, B is a specific constant belonging to each family of closely related substances, and x is the molecular weight of the compound.

It is obvious, then, that an extension of this previous work was desirable; and its application to the question of the molecular condition of the components of alloys was rendered possible by Trouton's discovery of the relation described above between viscosity as ordinarily defined and viscous traction.

In this preliminary paper, one of the simplest ranges of alloys was chosen for testing the validity of the above conclusions on the variation of the coefficient of viscosity with concentration. The lead and tin system was selected because the metals are readily obtained in a pure condition, the alloys can be conveniently made without any volatilization, and can be easily drawn into uniform wires. The two metals, moreover, are closely allied chemically, their melting-points lie nearly together, and consequently there is little chance of chemical union; hence, we should expect that each

component should behave independently in the mixture, contributing its own share to the total observed effect.

A considerable amount of work has been done on this range of alloys, from the cryoscopic point of view. Roberts-Austen ('Engineering,' 1897, lxiii. p. 233), in determination of the freezing-points of lead-tin solutions, obtained an eutectic point at 180° , corresponding with 32 per cent. lead. This eutectic point would fall at a concentration of 4 Sn 1 Pb, or Sn_4Pb . In various communications Heycock and Neville (Trans. Chem. Soc. 1892, lxi. p. 904; Phil. Trans. A. xcvii. p. 25, lxxi. pp. 384 & 892) showed that tin may attain varying degrees of molecular complexity in different solutions, thus Sn_4 in lead, Sn in zinc and bismuth, Sn_2 in silver, whereas lead is monatomic in tin, cadmium, and copper, and very slightly associated in silver and zinc solutions.

The alloys used in this work were made up of approximately definite percentage composition, and drawn by the London Electric Wire Company. They were of 1 mm. diameter, and usually a length of from 70–80 cms. was used. The wire, enclosed in folds of paper, was suspended from a pair of wooden jaws, screwed tightly together. This method almost entirely obviated slipping at the suspension. To check any error from this cause an index affixed to the wire quite close to the jaws was observed through a vernier microscope reading to .001 cm. Any slip would thus be detected, determined, and corrected for in the calculations. A similar pair of jaws was screwed on to the lower end of the wire, and served to carry the stressing load. Near the bottom end of the wire, at an accurately determined initial position, was affixed an index mark on mica, and this also was followed by a vernier microscope during the time of flow. The actual flow in cms., corrected for slip, was plotted against time in minutes, and the data for calculation were taken off the curves when a steady rate had set in. As a rule the same loads (about 500 grams) were used throughout the experiments, for it was found that the rate of flow was affected very considerably by increase in the tractive force, so much so that the viscosity coefficient cannot be taken as constant, but must be accepted as a function of that force. The experiments have been carried out at the laboratory temperature (varying at different times between 10° and 22°), and there is undoubtedly a marked temperature coefficient to be introduced. Further experiments are now in progress, with a view of following quantitatively these two factors.

Discussion of Results.

Generally speaking very regular straight-line curves obtain for each wire. The extension increases quite steadily with time over a long range, so long as the load is not excessive. It is very clearly shown that increase in concentration of tin lowers the viscosity, increasing the stretch per minute. Pure tin is markedly mobile under a load of 500 grms., pure lead is proportionally scarcely affected by such a load.

For a constant load the curves, after a long period of stress, tend to fall towards the horizontal, a state of fatigue sets in; but after an extra load is suddenly put on the mobility markedly increases, a sudden loosening of molecular attraction occurs, which soon passes away however, when the necessary rearrangement of groupings comes into play. From this point a normal state of affairs obtains, but with a noticeable increase on the previous rate of extension.

Whether similar curves would be obtained with virgin wires remains to be settled. In all probability the gradual change of curvature would disappear and a discontinuous series of straight lines would be obtained. Undoubtedly, much depends on the previous history of the wires. Too much stress should not be laid, however, on the curves after a certain limiting line, because the progressive thinning out of the wires causes an apparent diminution of viscosity, and ultimately the rupture of the material. In fig. 1 values for the coefficient of viscous traction are plotted against percentage composition. The inner curve is for an approximately uniform load of 500 grams, and the nature of the curve is undoubtedly hyperbolic, asymptotic to the axes. The effect of adding tin to lead is very great, 10 per cent. tin lowers the arbitrary viscosity coefficient from 160 to 37; beyond this further addition of tin is almost without effect; similarly, addition of lead to pure tin produces no noticeable change, till 80 per cent. has been alloyed with it.

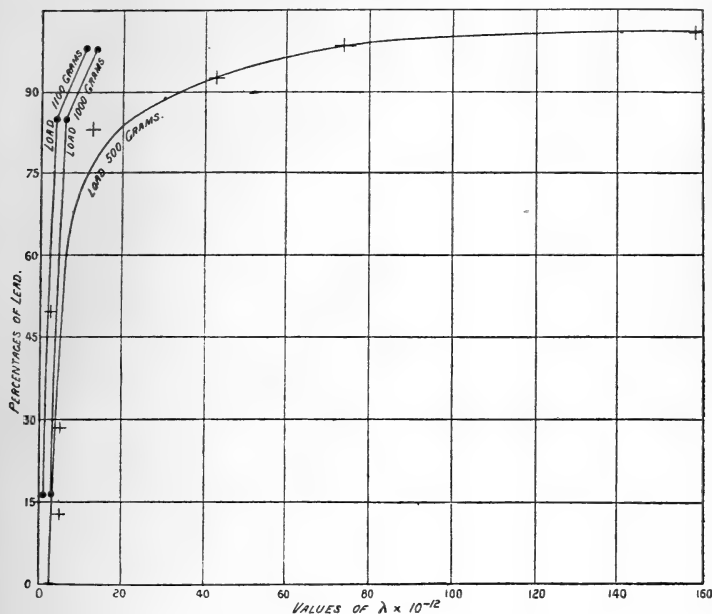
The greatest divergence of this curve from the straight line joining the extreme points of the curve is at 70-75 per cent. lead.

Now it has been found in previous work that such a point of maximum divergence always lies at some simple molecular concentration. The nearest molecular concentration in this case is 2 Pb 1 Sn.

The addition of lead to tin does not markedly affect the viscosity of the lead; it does, however, throw the curve well out of the normal. One must assume, therefore, that there

is a certain amount of association existing between the two metals in the solid state, which is not made apparent when they are fused, since the freezing-point curve is of the ordinary type, indicative of no such loose combination.

Fig. 1.

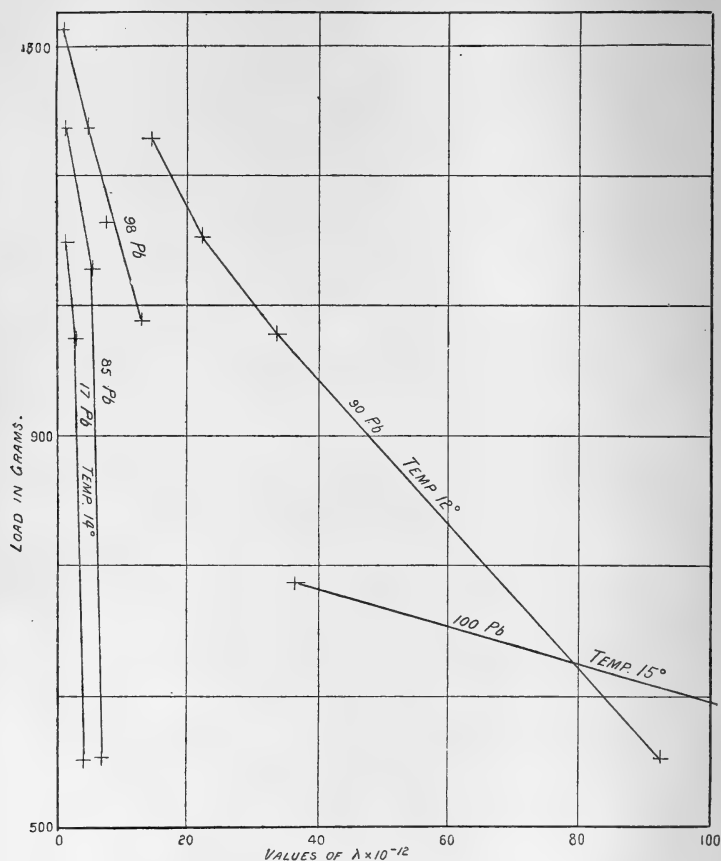


It is only natural to expect a greater degree of association in the solid state. The enormous effect of the addition of tin to lead may be explicable on the assumption of the complexity of the tin molecule, which has been seen to approximate Sn_4 . The dissociation of this and the consequent building up of lead-tin complexes would account for the departure of the curve from the normal. The two outer curves in fig. 2 (p. 198) represent the viscosity concentration curves for greater loads than 500 grams, viz., 1000–1100 grams. It is at once evident that the already low viscosity coefficient of tin is not materially affected by increased stressing force, but a very considerable effect is to be observed in the case of lead and its higher alloys.

It seems very possible that at a given limiting load these two metals and their alloys would attain an equality of mobility; at any rate, the difference between their viscosity

coefficients would not be observable by this method. Undoubtedly in this limiting case we should be dealing with a

Fig. 2.



pseudo liquid; in fact, from the point of view indicated by this work, lead, and particularly tin, are quite comparable with such viscous liquids as the pitch-tar mixtures with which Trouton experimented.

In fig. 2 some attempt has been made in a preliminary way to investigate the character of the effect of load on the coefficient of viscous traction. The effect is most pronounced with large percentages of lead, so much so that the question

arises whether the introduction of the idea of a coefficient can be really helpful in the case of these alloys and such like substances. It must, however, be considered that, as in the case of permeability in magnetism, a coefficient, though a function of the force applied, may yet render assistance in unravelling the intricacies of complicated phenomena.

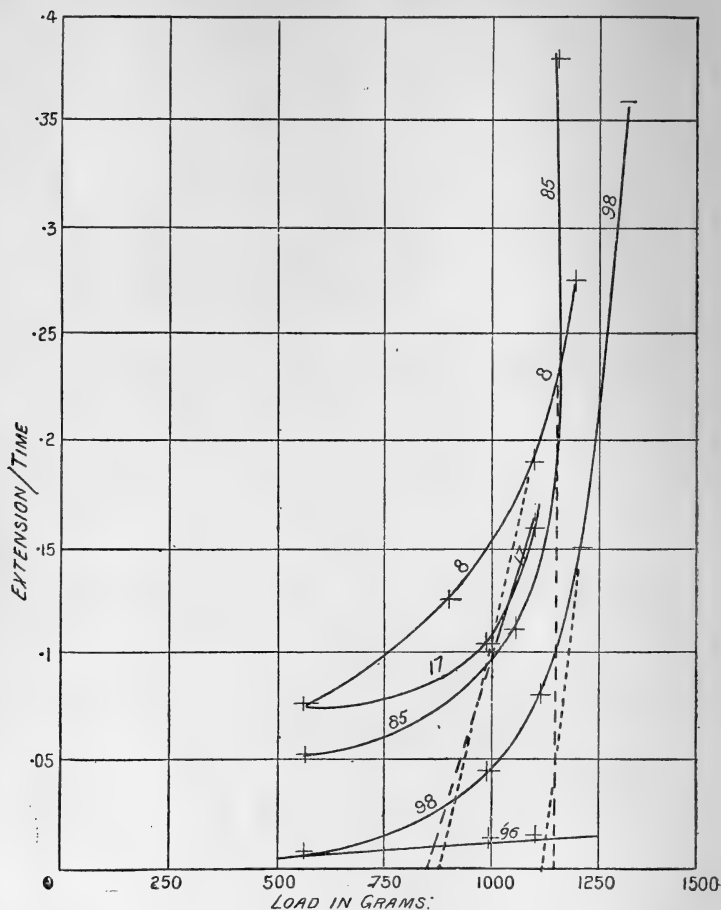
Particular mention must be made of the behaviour of those alloys containing more than 10 per cent. of tin. It may be at once pointed out that the curves for the alloys 98 Pb and 90 Pb and for the pure metal Pb are quite comparable in general contour. With increasing load there is the same approximation to the quasi-liquid state, the viscosity falls to the same limiting value, but for the alloys 85 Pb and 17 Pb much the same mobility is observable, no matter what load is operating.

As mentioned already, it must be remembered that the application of the term "coefficient of viscosity" to these cases may be somewhat misleading. It is a moot point whether such a term can be applied fairly with its usual understood connotation, viz. that it is a constant. The reciprocal expression "coefficient of mobility" might perhaps be better as not presenting an idea already stereotyped by usage. The chief difficulty is the determination of the limiting stress, before which no displacement other than that within the elastic limit occurs. That there is such a limit even for liquids appears certain. Consequently the method used in this paper, of assuming coefficients of viscosity to be comparable when determined with the same stressing force, is to be regarded only as an approximation. This will be quite evident when fig. 3 (p. 200) is considered. Here are plotted rates of extension against loads, and it is clear that comparable rates of flow can be determined only when the initial rate is neglected; in other words, when the intercept of the tangent on the load axis should be subtracted from the total load. Up to the present the data at hand are not sufficient for this correction to be made in every case.

General Conclusions.

1. When a wire composed of pure lead or tin or alloys of these metals is stretched beyond its elastic limit (in these cases this limit is exceeded by a very small load) the rate of extension proceeds quite steadily, and the material undergoes a viscous flow. From this rate of flow a coefficient of viscous traction may be deduced for any given load.

Fig. 3.



2. On plotting the corresponding coefficients of viscous traction against the percentage composition of the alloys a hyperbolic curve is obtained, which indicates that the relative effect of tin on the mobility of lead is very great, whilst that of lead on the mobility of tin is very small. The maximum divergence from the normal did these alloys follow the mixture law, is at 2 Pb 1 Sn.

3. The coefficient of viscous traction is a function of the load, and in some cases is very considerably affected by the load used. This statement is, however, strictly applicable only to those alloys containing less than 10 per cent. of tin.

Tabulated Results.

Percentage of Lead in wire.	Load in grams.	Time of expt. in minutes.	Total extension in cms.	Coefficient of Viscous Traction $\times 10^{12}$.	Temperature.
100	501	30	·082	160	15·5
	754	12	1·112	35·4	15·5
98	501	60	·218	73	17
	1016	9	·391	13·5	19
	1116	5	·410	7·99	19
	1216	4	·601	4·77	19
	1316	5	1·798	2·12	19
96	567	19	·067	92·2	12
	1007	42	·522	32·1	12
	1107	48	·724	21·1	12
	1207	51	·923	15·4	12
92	501	55	·303	41·4	10
85	569	8	·430	6·91	15
	1069	8	·880	5·74	15
	1169	5	1·895	1·88	15
84	500	42	·941	12·72	20
80	500	12	1·836	2·28	22·5
50	500	7	1·150	2·65	15
29	500	14	1·155	4·71	17
	500	46	2·892	3·57	19
20	509	5	·686	3·15	12
17	567	10	·779	4·15	14
	1000	21	2·690	2·99	14
	1100	27	4·305	2·34	14
14	500	42	3·005	4·14	15
4	500	4·03	10
0	498	16	3·135	2·17	22·5

The author desires to thank Professor Trouton, in whose laboratory this work was carried out, for much kind advice and assistance.

London, University College, W.C.

XIII. *The Grating Spectrum of Radium Emanation.* By T. ROYDS, M.Sc., 1851 Exhibition Scholar, Manchester University*.

IN order to measure with greater accuracy the wave-lengths of the lines in the spectrum of radium emanation, it is desirable to photograph the spectrum with an instrument of the highest power practicable. The fact that when a discharge is passed through a vacuum-tube containing pure radium emanation, the emanation rapidly disappears into the glass walls of the tube, sets a limit to the dispersion which can be employed. Using, however, a concave grating suitable for faint spectra, having a radius of only one metre, it has been possible to photograph the more intense lines in the spectrum of radium emanation, and also, incidentally, to extend the spectrum farther into the ultraviolet than has previously been photographed.

This concave grating, which has been fully described previously†, has a width of 3.5 inches, and is ruled with 15,000 lines to the inch. Its radius of curvature is 1 metre, and the dispersion in the first order spectrum amounts to 16.8 Å.U. per mm. The grating, the slit, and the camera are mounted in a light-tight wooden case and the photographs are taken on a photographic film which is bent to the requisite curvature by the shape of the film-carrier.

A small vacuum-tube of about 50 c.mms. capacity made out of capillary tubing and fitted with platinum electrodes, was filled to a pressure of about 0.1 mm. with radium emanation purified by the method and apparatus of Prof. Rutherford‡. A quartz condensing lens was employed and the iron arc used to give a comparison spectrum. The comparison spectrum was obtained alongside on the same film by moving a suitably perforated screen in front of the slit between the two exposures§.

* Communicated by Prof. E. Rutherford, F.R.S.

† de Wetteville, Phil. Trans. A 204. p. 139.

‡ Rutherford, "Experiments with Radium Emanation," Phil. Mag. Aug. 1908.

§ This method was employed for two reasons: (1) to avoid the risk of displacing the film-carrier into which the comparison shutters commonly used with the grating were slid in front of the film; and (2) in the case of the end-on discharge when the image of the tube was a faint dot on the slit, to avoid the possibility of the height of the spectrum being such that the spectrum was entirely cut out by the shutter—an adjustment which it is difficult to carry out in the dark, and, moreover, every moment of exposure is valuable.

This method of using different portions of the slit for the two exposures is open to objection if the slit is not exactly parallel to the rulings of the grating owing to the astigmatism; the adjustment was, however, carefully made by Kayser's method, and, in any case within reason, the error would be small.

Three photographs were obtained, two being of the end-on discharge seen through a quartz window in the end of the tube; in the latter cases it was not possible to remove the hydrogen completely by a preliminary heating of the tube for fear of cracking the quartz-glass joint. The photographs were measured up by means of a Kayser's measuring machine with a pitch of $\frac{1}{3}$ mm.

About 35 of the more intense lines previously obtained by Rutherford and Royds* with a glass prism can be seen on the photographs together with four additional lines farther in the ultraviolet. The new lines $\lambda\lambda$ 4631·89, 4628·08 near the intense line λ 4625·58 are only seen in one photograph (which had the longest exposure), but they may have been hidden in the prism photographs by the broadening of the intense line due to overexposure; the lines $\lambda\lambda$ 4572·66, 4568·24 are only seen in the same photograph. The spectrum presents the same characteristics as those previously obtained, the relative intensities being those of photograph 3 of the former paper rather than those of photograph 2.

The wave-lengths of the spectrum obtained by means of the grating are given in the accompanying list. The error

Wave-lengths of the Emanation Lines.

λ (Grating).	Intensity.	λ (Prism).	λ (Grating).	Intensity.	λ (Prism).
5084·45	1	5084·5	4568·24	2	not observed
4979·02	3	4979·0	4508·68	7	4509·0
4861·55	10	(4861·49 H β)	4503·89	2	4504·0
4828·46	1	4827·8	4439·88	1	4440·6
4817·33	5	4817·2	4435·25	3	4435·7
4796·73	1	4796·7	4371·70	1	4372·1
4768·16	5	4767·9	4349·81	15	4350·3
4721·70	2	4721·5	4340·69	4	(4340·66 H γ)
4701·88	3	4701·7	4203·39	10	4203·7
4680·92	10	4681·1	4166·59	20	4166·6
4671·58	1	4671·8	4114·71	6	4114·9
4644·29	15	4644·7	4017·90	6	4018·0
4634·13	3	(4634·15 H)	3981·83	15	3982·0
4631·89	2	not observed	3971·71	8	3971·9
4628·08	1	not observed	3957·30	8	3957·5
4625·58	15	4625·9	3664·96	5	3664·6
4609·40	10	4609·9	3612·76	3	3612·2
4604·46	6	4604·7	3122·02	0	
4580·03	3	(4580·1 H)	3100·31	1	
4577·77	8	4578·7	3089·15	0	
4572·66	0	not observed	3005·84	1	

* Rutherford and Royds, Phil. Mag. Aug. 1908.

of the measurements is about 0.1 \AA.U. ; it will be seen that the measurements of Rutherford and Royds are in good agreement.

Brief glances at the spectrum when the jar and spark-gap discharge is sent through the vacuum-tube indicate that the spectrum remains unaltered.

These experiments were made with the aid of the radium loaned to Prof. Rutherford by the Austrian Academy of Sciences. To Prof. Rutherford I am also indebted for the use of the apparatus and methods for purifying the emanation.

Physical Laboratories,
Manchester University.

XIV. *The Experimental Study of Huygens's Secondary Waves.*

By C. V. RAMAN, M.A.*

[Plate III.]

IN the Phil. Mag. for Nov. 1906 (pp. 495-498), I published a note on the diffraction-bands formed when a rectangular aperture is held very obliquely in a parallel beam of light. I showed that the bands cease to be of the same symmetrical type as the fringes formed when a rectangular aperture is held normally. They are not equidistant, the band-width increasing progressively from one side of the pattern to the other. Further, the number of bands visible on one side of the pattern is limited. The photographs of the effect published with this paper (Plate III.) exhibit these features.

Further observation of the diffraction-bands on the spectrometer, made by the methods I described in the paper referred to (*i. e.* of observing through the telescope the image of the slit of the instrument formed by light reflected very obliquely at the face of a prism, or by light passing through a rectangular aperture cut in a thin sheet of metal and held very obliquely on the table), elicited the following: it was found that the bands on one side of the pattern were fainter than those on the other, the difference becoming very large as grazing incidence was approached. This feature is visible on all the three photographs in the Plate. The effect is inexplicable on the ordinary (non-analytical) theory of diffraction.

* Communicated by the Author.

The illumination at any point in the pattern is, as deduced by the ordinary method, proportional to

$$\sin^2 \frac{\pi a}{\lambda} (\sin i - \sin \theta) \bigg/ \frac{\pi^2 a^2}{\lambda^2} (\sin i - \sin \theta)^2,$$

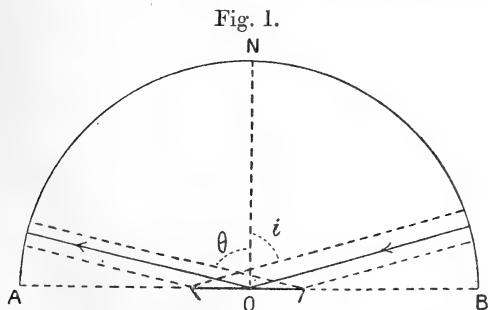
a being the aperture, λ the wave-length, and i , θ being the angles of incidence and diffraction respectively. Plotting this expression against θ , it is seen to be the ordinary symmetrical curve $\sin^2 x/x^2$ with its abscissæ distorted but its ordinates the same: (1) the maxima of illumination in corresponding bands on either side of the central one ($i = \theta$) are equal: (2) the illuminations at corresponding points on either side of the diffraction-pattern

$$(\sin i - \sin \theta_1 = \sin \theta_2 - \sin i),$$

are equal: (3) as the largest value of θ admissible is $\frac{\pi}{2}$, it follows that the curve of illumination at this point drops suddenly to zero; in other words, there is a discontinuity in the illumination-curve at this point. *All three results are contradicted by observation.* As has been stated above, the bands on the side nearer to the limiting plane $\theta = \frac{\pi}{2}$ were found to be fainter than those on the other side and the illumination at points in the diffraction-pattern decreased to zero as the limiting plane was approached.

The diffraction-fringes were observed through a nicol; there was no relative change in the illumination at different points in the pattern as the nicol was rotated, and at very oblique incidences no change at all.

An explanation of the effect was sought for on the following lines: each element of the reflecting surface may



be supposed to send out hemispherical secondary wavelets (fig. 1) and the illumination in the diffraction-pattern may be

determined by integrating the effects of these secondary waves. If the amplitude of the disturbance in a secondary wave in the direction in which the diffraction-pattern is formed varies rapidly with the obliquity, such variation would have to be taken into account and we should have an explanation of the difference between the observed result and that predicted by the ordinary theory. In no diffraction-experiment so far known, with apertures of ordinary size, has the variation of the amplitude with the obliquity, in a secondary wave, manifested itself. In the well-known Fresnel-Arago Circular-disk experiment, the fact that as the disk is approached the illumination along the axis of the disk decreases, cannot be taken to be an obliquity effect, it being more or less entirely due to the large increase in the effect of minute irregularities in the rim of the disk or of minute inaccuracies in its setting, as the latter is approached. Some authorities have gone so far as to deny that an obliquity-effect is possible at all. The present paper will show that this last view is erroneous. In the case considered in this paper, if we assume that the effect of an element at O (fig. 1) of the reflecting surface is zero at points on the lines OA, OB and a maximum in the direction ON, the rate of variation with respect to θ of the amplitude in the secondary wave ANB would be a maximum in either of the directions OA, OB, and zero in the direction ON. If plane waves of light are incident on the reflecting surface at a very oblique angle, the diffraction-pattern (as observed in a telescope focussed for infinity) is formed in the neighbourhood of the direction OA, and the variation of the effect of an element with θ , the angle of diffraction, would have large effects. The intensity in the diffraction-pattern would be zero in the direction OA, and at a point at which $\theta > i$, would be less than at the corresponding point at which $\theta < i$.

The point will now be investigated mathematically. Take the case of an aperture of any shape and of dimensions large compared with λ , cut in a thin perfectly reflecting sheet of infinite extent, and let parallel waves of light be incident on the aperture at an angle i . The light passing through the aperture falls upon the object-glass (focal length f) of a telescope focussed for infinity. Let x, y be coordinates in the plane of the aperture and ξ, η in the focal plane of the telescope, y and η being parallel, and assume that the effect of an element dx, dy of the aperture is, in the focal plane, equal to

$$-\frac{A}{\lambda f} dx dy F(i, \theta) \sin \frac{2\pi}{\lambda} (Vt - c - P),$$

where A is the amplitude in the incident wave, c is a constant, θ the angle of diffraction, and P the path-difference of the effects at ξ , η , of the element $dx dy$ and of an element at the origin in the plane of the aperture. The rigorous expression for P is a little complicated, but if the angle of incidence is not so large that $(\sin i - \sin \theta)$ cannot be put equal to $\sin(i - \theta) \cos i$, P can be shown to be equal to

$$- \frac{x\xi \cos i + y\eta}{f}.$$

The amplitude of the effect at any point in the diffraction-pattern is therefore

$$- \iint \frac{A}{\lambda f} dx dy F(i, \theta) \sin \frac{2\pi}{\lambda} \left(Vt + \frac{x\xi \cos i + y\eta}{f} - c \right),$$

the double integral being taken over the whole of the aperture. Putting $x \cos i = p$ and $y = q$, p and q being therefore the projections of x and y on the wave-front, the above given expression reduces to

$$- \iint \frac{A}{\lambda f} dp dq \cdot \frac{F(i, \theta)}{\cos i} \sin \frac{2\pi}{\lambda} \left(Vt + \frac{P\xi + q\eta}{f} - c \right),$$

the integral being taken over the whole of the projection of the aperture on the wave-front. Now, the quantity of energy passing, per unit of time, through the obliquely held aperture, must, at any rate approximately, be equal to the quantity that would pass through an aperture identical with the projection of the first on the wave-front, cut in a screen held parallel to the waves. From this it follows that, *provided we assume* $F(i, \theta)$ *does not vary sensibly* throughout the diffraction-pattern, it is equal to $\cos i^*$. This, on the assumptions made, agrees with the expression deduced by Kirchhoff in his 'rigorous' formulation of Huygens's principle

$$- \frac{A}{2R\lambda} (\cos i + \cos \theta) \sin \frac{2\pi}{\lambda} (Vt - R) dx dy,$$

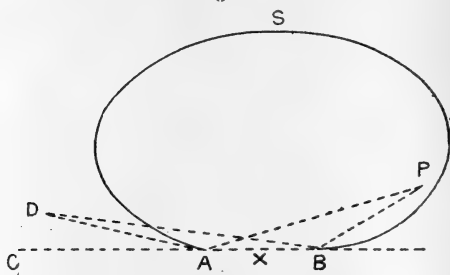
* From this and the integral given above, it appears that the diffraction-bands due to an aperture held obliquely are, at moderate incidences, identical with those due to its projection held normally: a proposition that might seem otherwise obvious, were it not for the fact that it is not true for very oblique incidences. As an instance, it can be shown experimentally that the diffraction-pattern observed when a circular aperture is held obliquely in front of a telescope directed at a point-source consists of a system of ellipses.

for in the direction of the wave-normal, $\cos i = \cos \theta$ and the expression reduces to

$$-\frac{A}{R\lambda} \cos i \sin \frac{2\pi}{\lambda} (Vt - R) dx dy. \quad . \quad . \quad (1)$$

The question now to be discussed is, whether we can always assume $F(i, \theta)$ to be appreciably the same in all the directions with which we are concerned, *i. e.* throughout the diffraction-pattern. Taking Kirchhoff's formula, the approximation clearly becomes inadmissible as i approaches $\frac{\pi}{2}$. In this case $\cos i$ and $\cos \theta$ are both small and a variation in θ affects the value of the expression very largely. The value of i at which the approximation ceases to represent matters fairly well depends upon the size of the aperture. θ will, in the diffraction-pattern, range from $\frac{\pi}{2}$ to i and less. The value of the factor $(\cos i + \cos \theta)$ will vary from $\cos i$ to $2 \cos i$ and more. Kirchhoff's formulation of Huygens's principle thus leads us to expect that at oblique incidences we should observe some phenomena due to the variation of the obliquity, which are inappreciable in the case of normal incidence. But though Kirchhoff's formula is able to indicate this, it itself does not hold at such incidences. It will be remembered that his formula is a purely mathematical deduction holding rigorously only in the case in which the wave-surfaces are not limited by screens of any kind. When they are limited by screens and mirrors, Kirchhoff's formula does not entirely meet the physical circumstances of the case and at oblique incidences leads to results widely differing from the truth.

Fig. 2.



For example, let P be a point-source inside a surface S , such as that in the diagram, which is closed everywhere except over the opening AB . Take a point D inside the plane ABC at this point, the intensity ought obviously to be zero, D being

inside the plane of the aperture. But Kirchhoff's formula leads to a different result. An element of the surface AB at \mathbf{X} would have an effect proportional to

$$(\sin |\mathbf{P} \times \mathbf{B}| - \sin |\mathbf{D} \times \mathbf{A}|),$$

and if the angle PAB be not very large, the path-difference $\text{PA} + \text{AD} - \text{PB} - \text{BD}$ would not be very large compared with λ and therefore there ought to be a finite effect at D. This absurd result discredits the applicability of Kirchhoff's formula to experiment, and further shows that the investigation of the correct obliquity factor has an actual tangible relation to experiment.

We now proceed to obtain a solution of the general differential equation of wave-motion in the form of a surface-integral which satisfies the requisite boundary conditions. Let S be the surface over which the integration is to be effected and r be the distance of a point in the space around from an element dS of the surface.

$$\frac{e^{-ikr}}{r}$$

is a solution of the symbolical equation

$$(\nabla^2 + k^2) = 0.$$

If dn be an element of the normal to the element dS , then

$$\frac{d}{dn} \left(\frac{e^{-ikr}}{r} \right)$$

is also a solution of the differential equation. If ϕ is an expression which is a function of the position of dS on the surface but does not contain r , then

$$\frac{1}{2\pi} \iint \frac{d}{dn} \left(\frac{e^{-ikr}}{r} \right) \phi dS \quad . \quad . \quad . \quad . \quad (a)$$

is a solution of the equation. This integral is the well-known expression for the potential of a sheet of double sources of sound, provided one of the directions of the normal n be regarded as positive and the other negative. The value of the integral at the surface itself is, on one side of it $+\phi$, on the other $-\phi$: for, regarding one of the directions of the normal as positive, the value of the integral at a point

indefinitely near the positive side of the surface

$$\begin{aligned}
 &= - \int \frac{d}{dr} \left(\frac{e^{-ikr}}{r} \right) \phi r dr \cos \theta \\
 &= - \phi N \int_N^\infty \frac{d}{dr} \left(\frac{e^{-ikr}}{r} \right) dr, \quad \dots \quad (b)
 \end{aligned}$$

where N is the indefinitely small distance of the point from the nearest element of the surface and θ is the angle between n and r .

$$\begin{aligned}
 &= \lim_{N \rightarrow 0} \phi N \frac{e^{-ikN}}{N} \\
 &= \phi.
 \end{aligned}$$

In the particular case in which S is an infinite plane and ϕ is constant over the whole of it, the expression (b), instead of being an approximation true in the limit ($N=0$), is perfectly rigorous for all values of N and expression (a) reduces to

$$\pm \phi N \frac{e^{-ikN}}{N} = \pm \phi e^{-ikN},$$

which is the velocity potential of two sets of aerial waves proceeding from the infinite plane, on opposite sides of it, to an infinite distance.

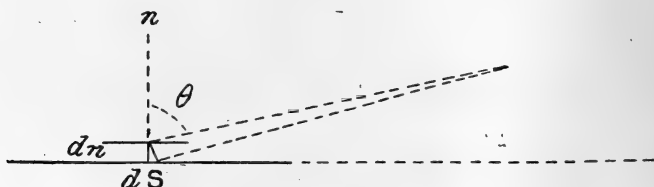
If the surface over which the integration is effected is part only of an infinite plane, the integral (a) can be written as

$$- \frac{1}{2\pi} \iint \frac{d}{dr} \left(\frac{e^{-ikr}}{r} \right) \phi dS \cos \theta. \quad \dots \quad (c)$$

At points on that part of the infinite plane over which the integration is *not* effected, the integral (c) is zero, for (fig. 3)

$\theta = \frac{\pi}{2}$ and $\cos \theta = 0$ for all such points. This part of the

Fig. 3.



infinite plane is therefore one of 'silence.' This is true whether ϕ is or is not constant over the whole of the surface of integration.

We shall now apply the solution (a) of the general differential equation to the following problems: it is understood that in each case the reflecting surface or transmitting aperture is of dimensions large compared with the wavelength.

Reflexion of plane aerial waves at an infinite rigid plane, whose position is given by $x=0$.

Let the plane waves be incident at an angle i on the positive of the plane, and the velocity-potential of the incident waves be the real part of

$$Ae^{ik(Vt+x\cos i-y\sin i)}.$$

Superpose upon this, the value of the integral

$$\frac{A}{2\pi} \iint \frac{d}{dx} \left(\frac{e^{-ikr}}{r} \right) e^{ik(Vt-Y\sin i)} dY dZ \dots (d)$$

taken over the whole of the infinite plane, Y and Z being the y and z coordinates at any point on the plane.

The value of the integral at any point in the space on the positive side of the infinite plane is

$$Ae^{ik(Vt-x\cos i-y\sin i)},$$

and on the negative side

$$-Ae^{ik(Vt+x\cos i-y\sin i)}.$$

The resultant disturbance on the positive side is

$$\begin{aligned} \phi &= A[e^{ik(Vt+x\cos i-y\sin i)} + e^{ik(Vt-x\cos i-y\sin i)}] \\ \frac{d\phi}{dx} &= Aik \cos i e^{ik(Vt-y\sin i)} [e^{ikx \cos i} - e^{-ikx \cos i}], \end{aligned}$$

which when $x=0$, is equal to zero.

On the negative side

$$\phi = 0 \quad \text{and} \quad \frac{d\phi}{dx} = 0.$$

The necessary conditions are thus satisfied. The normal velocity at the plane is zero on both sides of it, and the region on the negative side of the plane is entirely screened from disturbance. The same is true even if the reflecting surface occupies only part of the plane $x=0$, for at points close to the plane, on either side of it, the result of the integration (d) is practically the same as if it were extended over infinity.

In the case last mentioned of a finite reflecting surface, the effect of the reflected waves at points not near the reflecting

plane and on the positive side of it, is given by the integral (*d*) taken over the whole of the reflecting area. This integral may be written as

$$-\frac{A}{2\pi} \iint \frac{d}{dr} \left(\frac{e^{-ikr}}{r} \right) e^{ik(\sqrt{t}-Y \cos i)} \cos \theta dY dZ, \quad (f)$$

where θ is the obliquity of the point of observation viewed from the element $dY dZ$. The point of interest is, that for all points on the continuation of the reflecting sheet, θ being equal to $\frac{\pi}{2}$, the value of the integral is rigorously zero and the plane is therefore one of silence. This becomes of importance experimentally when the angle of incidence is such that the diffraction-pattern is formed near this plane of silence. It is not difficult to understand why the elements of a reflecting surface should be equivalent to double sources and not to simple sources of sound: for, if the reflecting plane be replaced by an *indefinitely thin* sheet in the same position, every point of which instead of being kept fixed is obliged to follow the vibration in the incident waves, then it is obvious that these waves would be transmitted without disturbance to the far side of the plane. The effect of the reflecting plane must therefore be equivalent to that of the reversed motion of the thin sheet in a medium entirely at rest. This involves periodic *compressions* and rarefactions on one side of the plane, and simultaneous *rarefactions* and compressions on the other: *i. e.*, periodic *introductions* and abstractions of fluid on one side, and simultaneous *abstractions* and introductions on the other. This is the equivalent of a sheet of double sources.

*Passage of Plane Aerial Waves through an Aperture
in a Thin Plate.*

This can be seen to be directly deducible from the preceding: the integral (*d*) would have to be taken over the reflecting plate (excluding the aperture), and the expression for the velocity potential of a system of plane aerial waves passing through an infinite medium superposed upon it. On the positive side of the reflecting sheet, the integral (*d*) gives the effect of the reflected waves, and since in the integration the area of the aperture is excluded, the effect of an element of the aperture in any direction on the positive side of the sheet is zero. On the negative side of the plate, the disturbance passing through the aperture appears as the difference of two quantities: if the integral (*d*) is taken over the whole

of the reflecting sheet, including the aperture in it, this difference is zero. It follows therefore that the disturbance passing through the aperture is given by the integral (d) taken over the area of the aperture alone, with its sign reversed. It follows therefore, that considerations such as those that apply to the case of reflexion at oblique incidences apply to this also.

Light incident on a perfectly reflecting screen, the waves being polarized in a plane at right angles to the plane of incidence.

The magnetic vector ζ in the incident waves is parallel to the axis of z and to the reflecting screen. Since ζ satisfies the general differential equation and also, at the screen, the condition $\frac{d\zeta}{dx} = 0$, the expression for ζ in the secondary waves

is exactly the same as that for ϕ in the preceding paragraphs. As for the other components of the magnetic vector in the secondary waves, they are zero at all points in the plane of incidence and we need not at present trouble about them.

Light incident on a perfectly reflecting screen, the waves being polarized in the plane of incidence.

The electric vector ζ in the incident waves is parallel to the axis of z and to the reflecting screen. The expression for the vector ζ in the secondary waves can still be deduced from the integral (d), if the sign of this is changed and the operand d/dx is replaced by d/dn , where dn is an element of the normal to the reflecting surface, both directions of this being regarded as positive. On the side of the screen on which the waves are incident and at points close to it,

$$\zeta = Ae^{ik(Vt+x\cos i-y\sin i)} - Ae^{ik(Vt-x\cos i-y\sin i)},$$

which is zero if $x = 0$.

On the other side, at points close to the reflecting sheet

$$\begin{aligned}\zeta &= Ae^{ik(Vt+x\cos i-y\sin i)} - Ae^{ik(Vt+x\cos i-y\sin i)} \\ &= 0.\end{aligned}$$

The other components of the electric vector in the secondary waves need not be considered here.

Apertures in Perfectly-Reflecting Plates.

The results for these cases can be deduced from the expressions for reflexion, in exactly the same way as was done for aerial waves. On the side on which the waves are incident,

the effect of an element of the aperture is zero. On the other side, the z component of the electric or (as the case may be) magnetic vector is given by the integral (f) taken over the aperture and with its sign reversed.

From the integral (f) it can be seen that the obliquity-factor, for the z component of the light-vector in the secondary waves, is simply the cosine of the obliquity and is independent of the angle of incidence of the waves on the reflecting screen or aperture. Differentiating and realizing the integral (f), it can be seen that for moderate incidences it is equivalent to the expression (1), for at such incidences, the diffraction-pattern is formed in a direction in which $\cos \theta$ does not vary rapidly with θ and may therefore be put equal to the mean value $\cos i$. At oblique incidences this is no longer true. The integral of (f) in the case of a rectangular aperture (sides l, m) held obliquely in a parallel beam of light, in front of a telescope, gives for the illumination in the diffraction-pattern

$$A^2 \frac{l^2 m^2}{\lambda^2 f^2} \cos^2 \theta \frac{\sin^2 \frac{\pi l}{\lambda} (\sin i - \sin \theta) \sin^2 \left[\frac{\pi m}{\lambda} \sin \psi \right]}{\left[\frac{\pi l}{\lambda} (\sin i - \sin \theta) \right]^2 \left[\frac{\pi m}{\lambda} \sin \psi \right]^2}, \quad (g)$$

θ and ψ being the angles of diffraction, in other words, the angles made by the diffracted 'ray' with the two planes normal to the plane of the aperture. The expression deduced from the non-analytical theory is

$$A^2 \frac{l^2 m^2}{\lambda^2 f^2} \cos^2 i \frac{\sin^2 \frac{\pi l}{\lambda} (\sin i - \sin \theta) \sin^2 \left[\frac{\pi m}{\lambda} \sin \psi \right]}{\left[\frac{\pi l}{\lambda} (\sin i - \sin \theta) \right]^2 \left[\frac{\pi m}{\lambda} \sin \psi \right]^2} \quad (h)$$

These two expressions give, for moderate incidences, practically identical results. At all incidences, they give the same positions for the minima of illumination in the diffraction-pattern. But as regards the distribution of illumination in the pattern, and as a minor point, as regards the positions of the maxima of illumination, they give at oblique incidences very different results. The expression (h), as has already been mentioned, makes it out that the illuminations at corresponding points on either side of the central band should be equal, and that at the limiting plane $\theta = \frac{\pi}{2}$ the curve of illumination should drop discontinuously from a finite to zero

value, both of which results are contradicted by experiment. The expression (g) shows that the illumination at points in the pattern on the side of the central band, nearer to the limiting plane $\theta = \frac{\pi}{2}$, is less than at the corresponding points on the farther side: and that since $\cos^2 \theta$ is zero if $\theta = \frac{\pi}{2}$, the curve of illumination falls continuously to zero at the limiting plane. Both these results are verified by experiment. Photographs (1) and (2) (see Plate III.) of the diffraction-bands formed by reflexion at the face of a prism, and photograph (3) by transmission through an aperture, all exhibit these effects.

The investigation of the intensity at different points on a secondary wave, given above, is for the case of the incidence of light on a perfectly reflecting screen. It can be shown that by a suitable modification of the integral (a) it can be made to cover the case of the reflexion and refraction of light-waves at a dielectric medium, and that the obliquity-factor in these cases, as given by theory and as applicable to experiment, is $\cos \theta$. As a matter of fact the photographs (1) and (2) in the Plate are of the diffraction-bands formed by reflexion at a glass prism.

Similar obliquity-effects should be obtained with other forms of aperture, for example with a reflexion or transmission grating, *with wide rulings*, if it is held obliquely and if points in the field of view receive no light from the *grooves* of the grating, the only effective parts being plane portions of the grating, these being parallel to, or coincident with, its general surface. These and other considerations will form the subject of a future paper.

Summary and Conclusion.

Each element of a reflecting surface, may, when waves are incident upon it, according to Huygens, be supposed to send out into each of the two media meeting at the surface, hemispherical secondary wavelets. The amplitude of the disturbance at points on these secondary waves may be investigated mathematically, as in this paper, and shown to be a maximum at the pole of the hemisphere and zero at points on its equator. A similar result for the case of an aperture in a screen can be deduced therefrom. Observation of diffraction-phenomena at oblique incidences confirms this result. The law of variation of the amplitude of the principal component of the light-vector is the cosine of the obliquity. This may be subjected to experimental investigation, and it is hoped that if the

necessary instruments are available, photometrical measurements to test this can be made.

The experiments and observations recorded in this note were made at the Physical Laboratory of the Indian Association for the Cultivation of Science, Calcutta, and formed the subject of a demonstration held at a Special Meeting of that body on the 18th of January last.

P.S. dated the 26th of Nov.—Photometrical measurements in verification of the above have been carried out. These will be dealt with in a future communication.

XIV. *The Effect of Pressure on the Natural Ionization in a Closed Vessel, and on the Ionization produced by the γ Rays.*
By W. WILSON, M.Sc., Graduate and Hatfield Scholar of the University of Manchester*.

THE natural ionization in gases has been very largely studied since the experiments of Geitel† and C. T. R. Wilson‡ in 1900. Experiments have been generally made with the object of determining whether ordinary matter is radioactive or not. Wilson observed the effect of diminishing the pressure of the gas inside the electroscope, and found that for pressures of 43 to 743 mm. of mercury, the ionization was proportional to the pressure. Patterson§ also studied the effect of varying the pressure and found that with large vessels (20 cms. long, 30 cms. diameter) the ionization increased with the pressure at first, but that at one atmosphere it had become sensibly constant. Strutt|| observed that the ionization varied for vessels of different materials; and McLennan¶ and Burton found that it was proportional to the pressure up to seven atmospheres.

Rutherford and Cooke** observed that the rate of leak was diminished on surrounding the electroscope with lead, thereby proving the existence of a penetrating radiation. Wood†† showed that the decrease in ionization depends both on the material of the screen and the material of the ionization vessel. He concludes that the natural ionization is due to an external radiation and its attendant secondary radiation, together with intrinsic rays from the sides of the

* Communicated by Prof. E. Rutherford, F.R.S.

† Geitel, *Phys. Zeit.* ii. p. 116 (1900).

‡ C. T. R. Wilson, *Proc. Camb. Phil. Soc.* xi. p. 32 (1900); *Proc. Roy. Soc.* lxxviii. p. 151 (1901). § Patterson, *Phil. Mag.* Aug. 1903

|| Strutt, *Phil. Mag.* June 1903.

¶ McLennan & Burton, *Phys. Review*, No. 4, 1903.

** Rutherford & Cooke, *Am. Phys. Soc.* Dec. 1902.

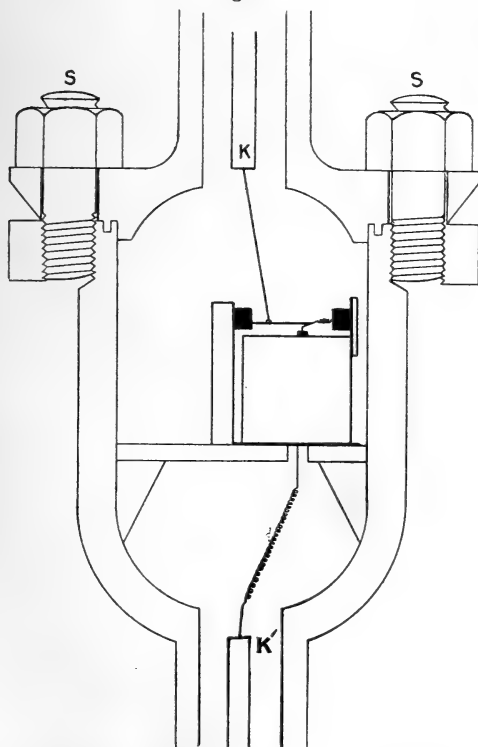
†† Wood, *Phil. Mag.* April 1905.

vessel. From some experiments with an ionization vessel with a movable side, Campbell * calculated the ranges of the particles given off by the walls, and has found that they are in general greater than those of the α particles from known radioactive elements ; but McLennan † has lately measured the natural ionization in vessels of different material, and attributes it to external radiation differences coming in on account of the various metals not giving out equal amounts of secondary radiation.

Apparatus.

It was decided to measure directly the ionization in an electroscope containing gas at the desired pressure. The pressure-chamber, which was kindly lent by Dr. Hutton and Prof. Petavel, has been described by them in Phil. Trans.

Fig. 1.



A. 207. p. 421. It consists briefly of a cylindrical iron vessel, a section of which is shown diagrammatically in fig. 1. The

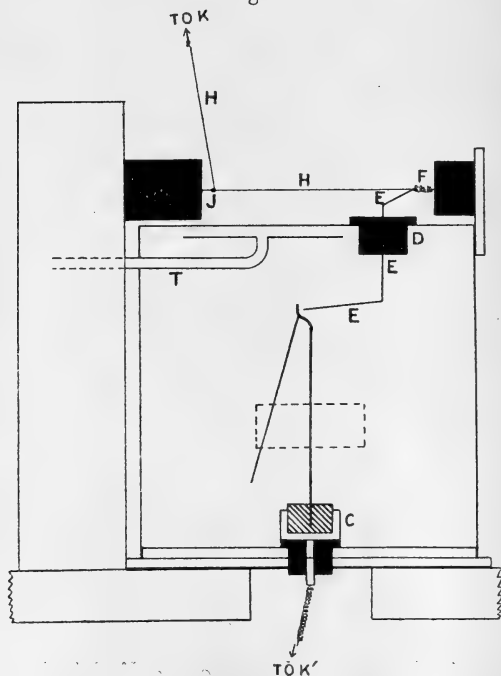
* Campbell, Phil. Mag. Sept. 1906.

† McLennan, Phil. Mag. Dec. 1907.

internal diameter is 10 inches, the height 16 inches, and the thickness of the walls 1·5 inch. This is surrounded by an iron water-jacket about ·5 inch thick. A massive lid which is made gas-tight by means of a lead joint can be fastened down to the main body by screws S S which are 10 in number. A rod K which projects into the chamber can be moved up and down by an insulated screw, and is itself insulated from the rest of the apparatus by mica. The air is prevented from escaping along the surface of the rod by means of a stuffing-box. The leak of the air from the vessel was quite small, not amounting to more than ·5 atmosphere per hour at 40 atmospheres, and even this could be largely stopped by tightening the stuffing-box.

The electroscope used consisted of an aluminium vessel, 10 cms. cube, with a base of brass. Aluminium was used because it has been found by other observers that the natural ionization in vessels made from it is small. It rested on a wooden stand securely fixed to the bottom of the pressure-

Fig. 2



chamber. A vertical section of the electroscope is shown in fig. 2. The brass base contained a central hole 1 cm. in

diameter, through which passed an ebonite stopper. A brass wire terminating in a brass cup, C, fitted centrally into the stopper. The wire was fastened by means of a metallic spring to a rod K' similar to K, at the bottom of the pressure-chamber. A sulphur plug in C supported the gold-leaf system, which could be viewed by means of a reading microscope, through two windows in the electroscope which could be placed in line with two others in the pressure-chamber.

The electroscope could be charged from the outside and the leaf left insulated as follows:—A brass wire passed through the ebonite stopper D, and was bent so that by moving it could be brought into contact with the gold-leaf system. The wire is shown at EEE, fig. 2. It was attached by a spring F to an ebonite plug and to the terminal K by a wire H, which passed through a copper ring J fixed into another ebonite plug. By raising the rod K, the wire E could be brought into contact with the gold-leaf system, and the leaf charged by connecting K to one end of a battery of 100 small storage-cells, the other end of which was to earth. On lowering K, while charged, the wire E was drawn back by the spring F, and the gold-leaf remained charged. In each experiment the rod K was lowered by the same amount and then connected to earth, in order to get the same electrostatic conditions each time. The pressure of the air was raised by connecting the pressure-chamber to a cylinder which had been charged to 100 atmospheres by the compressor of a liquid-air machine. The air passed into the pressure-chamber through two stopcocks, by means of which its rate of admission was regulated. It had to be allowed to enter very slowly or the gold-leaf was torn off its support. The air entered the electroscope through the copper tube T (fig. 2), which ended in a flat disk 2 inches in diameter, placed as shown. By this means the air was prevented from impinging directly on any one portion of the leaf and so injuring it.

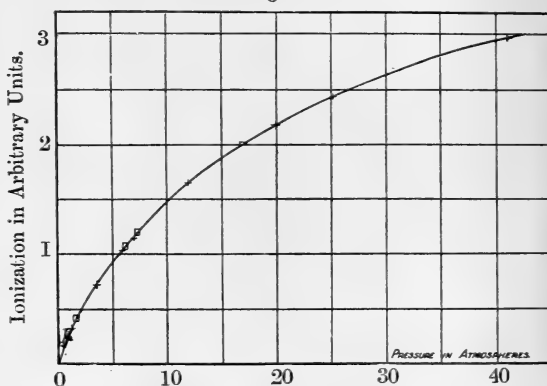
Experiments with γ Rays.

It will be more convenient to consider first some experiments on the effect of pressure on the ionization produced by γ rays. The experiments were made simultaneously with those on the natural leak, 1.09 m.g. radium bromide being fixed to a stand which could always be placed in a definite position outside the pressure-chamber by means of a hole and slot clamp. The measurements of the ionization were made by noting the time taken by the image of the gold-leaf to move over a certain number of divisions of the scale in the eyepiece of a reading-microscope.

The ionization was very weak, being only .25 scale-division per minute at atmospheric pressure. This corresponds to about 216 ions per c.c. per second. Such a low ionization was used in order that the comparatively low voltage applied would be sufficient to give the saturation current, without which of course the results would have no definite meaning. Some experiments on this subject have been already done by Erikson *, but he makes no statement as to whether he was using the saturation current or not. It was probably not in fact, for with an E.M.F. of 19 volts per cm. he obtained a maximum current at 70 atmospheres pressure, while with 1000 volts the maximum occurred at 140 atmospheres.

The results obtained are given in fig. 3, where the ionization is plotted against the pressure. It will be observed that the

Fig. 3.



ionization does not increase proportionally with the pressure, but less rapidly as the pressure rises. This confirms the fact, as McLennan has shown, that the secondary rays from the sides of a vessel exposed to γ rays contribute an appreciable amount to the ionization inside. Thus, since this secondary radiation is comparatively easily absorbed, the ionization due to it will approach a maximum value as the pressure rises, while that due to the γ rays alone will not do so, but will increase proportionally with the pressure, *i. e.* with the density of the gas.

An approximate equation to the curve can be obtained as follows:—Consider for simplicity of calculation that the secondary rays move perpendicular to the sides of the

* Erikson, Phys. Rev. Jan. 1908.

electroscope, and that they are absorbed according to an exponential law with the distance traversed.

Let dI be the ionization produced in a layer of thickness dx at a distance x from the side of the box, and let $I_0 dx$ be that produced in the layer adjacent to the side.

Then

$$dI = I_0 e^{-\lambda x} dx,$$

and the whole ionization due to the secondary rays, if the length of the box is a , is

$$\frac{I_0}{\lambda} (1 - e^{-\lambda a}).$$

Assuming the rays to be of the β ray type, λ will be proportional to the density of the gas, *i. e.* to the pressure, and so will I_0 , since $I_0 dx$ is the ionization in a thin layer of constant thickness dx .

Therefore the total ionization due to the secondary rays at different pressures is given by

$$B(1 - e^{-\lambda p}),$$

where B is a constant, p the pressure, and λ the coefficient of absorption at atmospheric pressure.

Since the γ rays are not appreciably absorbed in their passage across the vessel, the ionization due to them will be given by a term of the form $A p$, where A is a constant, having the value of the ionization due to the γ rays alone at atmospheric pressure.

We have then, if I is the whole ionization in the electroscope,

$$I = A p + B(1 - e^{-\lambda p}).$$

Now, Eve* gives the value of λ for aluminium for the secondary rays from aluminium as 24. Assuming the density law and taking the density of aluminium as 2.6, and that of air .00129, we obtain for our value of λ .0119.

It will be noticed that for high pressures the equation becomes

$$I = A p + B.$$

Therefore the intercept of this straight line with the ionization axis gives the value of B , and the tangent of the angle made by this straight line with the pressure-axis gives the value of A .

Producing the final portion of the curve backwards, we get then

$$A = .030, \quad B = 1.78.$$

The theoretical curve is drawn in the figure and the

* Eve, Phil. Mag. Dec. 1904.

experimental points are marked on it. The curve agrees much better than would be expected considering the number of assumptions and the uncertain values of the constants.

The current was tested for saturation at different pressures by noting the time taken by the leaf to move over the various divisions of the scale. The same time was taken to fall through a division at 190 volts as at 120 volts, except in the case of the last point at 40 atmospheres, when the time was slightly longer at the latter reading, *i. e.* the current was not quite saturated. The saturation current was further tested for by taking a few readings with the rays about twice as intense. On making one reading fit on the original curve, all the others on being reduced in the same ratio also fell on the curve. These points are shown by squares in the diagram.

We can from this formula deduce the relative values of the ionization due to the secondary and primary rays in the vessel at atmospheric pressure. The value found is 6.6, *i. e.* $\frac{B(1-e^{-\lambda a})}{A}$. This is greatly in excess of that obtained by

McLennan (*loc. cit.*), who found that in aluminium vessels the ionization produced by the secondary is half that due to the primary rays. This discrepancy cannot be accounted for by differences in the sizes of the vessels used, since it can be easily worked out approximately that, assuming the above formula to be correct and the amount of secondary radiation to be proportional to the exposed surface, and the ionization produced by the primary to be proportional to the volume, the ratio of the ionization produced by the primary rays to that produced by the secondary should be of the same order as that given above.

Experiments on the Natural Ionization.

We are now in a position to proceed with the experiments on the natural ionization, which were conducted as follows:—

The air was admitted till the required pressure was reached, which was recorded on a Schäfer and Budenberg gauge. The apparatus was then left until the gold-leaf came to rest, when the electroscope was charged, and the time taken for the image of the leaf to move over a certain number of scale-divisions in the microscope eyepiece was noted. From this the number of divisions per minute moved over by the image was calculated and the ionization is expressed in these units.

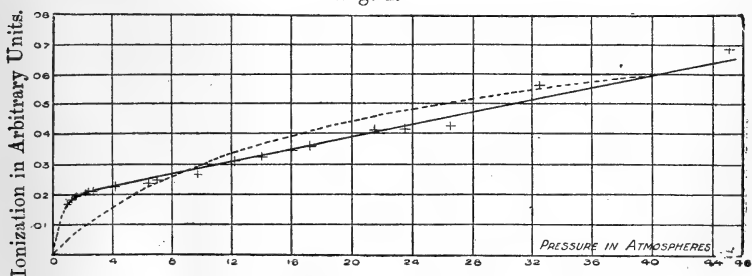
The first set of experiments was performed with the gas

originally at 21 atmospheres. The pressure was reduced from time to time and readings of the natural ionization taken. The leak decreased with pressure till four atmospheres was reached, when it commenced to rise. This effect appears to be due to an emanation given off by the walls of the pressure-chamber, for on filling to 15 atmospheres and then allowing the air to expand, filling again, and again allowing to expand, it was found that the ionization decreased by about 75 per cent. and fell in with the general trend of the curve. For the higher pressures, the emanation does not diffuse from the walls of the pressure-chamber into the electroscope so quickly on account of the great density of the gas. For the remaining experiments then the bomb was filled and emptied several times before taking each reading; this process taking so long that any active deposit from the emanation would (assuming it to be that from radium) have all disappeared.

The air used was always drawn from the outside of the laboratory, since it was found that air drawn from the inside was more active, due to the fact that experiments with radium were being carried on in the room in which the liquid-air machine was situated. It was found that leaving the air compressed in the cylinder for two weeks did not make any observable difference in the readings.

The insulation was tested from time to time by connecting the cup C to the same potential as the gold-leaf by means of the terminal K'. The rate of motion of the gold-leaf was never altered by more than 2 or 3 per cent. This small variation is to be expected, since when C is charged some ions will be drawn to it which before went to the gold-leaf system.

Fig. 4.



The final curve obtained connecting ionization with pressure is given in fig. 4. It is seen to be a straight line which curves over somewhat towards the pressure-axis at low pressures.

At the higher pressures, such as those above 20 atmospheres, the gold-leaf did not come to rest for some time after filling. At 30 atmospheres it moved irregularly to and fro, even after 8 hours. In such cases, if the leaf were not moving too irregularly, its position on the scale was read every quarter minute for two or three minutes and the mean was taken as the reading at the middle of this period.

If we reduce the γ -ray curve to the same dimensions as the natural ionization curve, as shown in the curve dotted in fig. 4, we see that the two curves are quite dissimilar, and therefore the natural ionization in the vessel cannot be wholly due to external γ rays. The natural leak, however, was found to be about 40 per cent. lower when in the pressure-chamber than when out, showing that the ionization in an unscreened vessel is partly due to this cause.

The number of ions produced per c.c. per second was deduced as follows:—Eve has shown that the number of ions produced per c.c. per second in an aluminium vessel by the γ rays from 14.3 m.g. radium bromide placed at a distance of 303 cms. is 297. In the present case 3.7 m.g. at a distance of 67 cms. produced an ionization of 5.50 scale-divisions per minute. The natural ionization at one atmosphere *on this scale* is .046 division per minute.

This corresponds then to a number of ions

$$\frac{.046}{5.50} \cdot \frac{297 \times 3.7 \times (303)^2}{14.3 \times (67)^2} = 13.1 \text{ per c.c. per second.}$$

This agrees very well with the corrected value of the number found by Cooke, who obtained 9.9 to 13.8, and confirms the test of the insulation.

The curve obtained if the ionization were due to the admixture of an emanation with the air, such as that of radium which emits rays of low penetrating power, would be a straight line passing through the origin within the limits of pressure examined; while if the walls of the vessel emit rays of low penetrating power, either intrinsic or due to radioactive impurities, the ionization should not vary with the pressure.

The experimental curve can be made up of two parts:—

(1) A constant part due to radiations from the sides of the vessel.

(2) A part which increases proportionally with the pressure, due for the most part to an emanation mixed with the gas, but in some part very likely also to penetrating radiation. The latter part must be very small, however, or the curve would be concave towards the pressure axis. The emanation,

if it is one of the known ones, must be that of radium, since the emanations of thorium and actinium have much too short a life to exist during the time required to take a reading.

The conditions of the experiment are not sufficiently well known to be sure of the explanation of the slight curvature of the lower portion of the curve, but it can very likely be ascribed to radiations shot out obliquely from the sides of the vessel, which even at atmospheric pressure have not completed their full path in air before striking another side.

The amount of emanation required to produce the sloping part of the curve can be calculated from these results and is found to be exceedingly small.

If we assume that the final portion of the curve is due exclusively to the radiation from the emanation, we see from the slope of the line that the ionization due to it is $\cdot 00104$ division per minute at atmospheric pressure. Now, $\cdot 017$ division per minute on the scale corresponds to an ionization of 13 ions per c.c. per second. Consequently, the ionization due to the emanation present is only $\cdot 80$ ion per c.c. per second.

Also, Rutherford has shown that one α particle from radium produces about 86,000 ions in its path through air. Taking into account the difference in the ranges, an α particle from the emanation must produce about 100,000 ions before it comes to rest.

Now, 800 ions are produced per c.c. per second by the emanations in the whole of the vessel which has a capacity of 1000 c.c. This corresponds to the passage of one α particle in every 125 seconds on the average.

And the radioactive constant of the radium emanation is $\frac{1}{463,000}$, that is, one atom in every 463,000 disintegrates per second.

Therefore, since we have only one atom breaking up in every 125 seconds, the number of emanation atoms present must be 3700 or $3\cdot 7$ per c.c.

This is an excessively small amount, and we can hardly expect to get rid of it, no matter how clean the vessel in which the air is stored. The effect of this small quantity is of course practically unmeasurable at atmospheric pressure, with the electroscope used, and only becomes at all marked when the pressure is raised considerably.

This research was suggested by Prof. Rutherford, and I wish to acknowledge his ever ready advice and encouragement.

Physical Laboratory,
The University, Manchester.

XVI. *Notices respecting New Books.*

Bulletin of the Bureau of Standards. Vol. IV. No. 4, May 1908.
(Washington : Government Printing Office.)

THE following are the titles of the articles appearing in this number:—Apparatus for Determination of the Form of a Wave of Magnetic Flux, by M. G. Lloyd and J. V. S. Fisher. Effect of Wave-form upon the Iron-losses in Transformers, by M. G. Lloyd. The Luminous Properties of Electrically Conducting Helium Gas, by P. G. Nutting. Function of a Periodic Variable given by the steady reading of an Instrument, with a Note on the use of the Capillary Electrometer with Alternating Voltages, by M. G. Lloyd. Selective Radiation from the Nernst Glower, by W. W. Coblentz. The Testing of Glass Volumetric Apparatus, by N. S. Osborne and B. H. Veazey.

XVII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from Vol. xvi. p. 832.]

June 3rd, 1908.—Prof. W. J. Sollas, Sc.D., LL.D., F.R.S.,
President, in the Chair.

THE following communication was read:—

‘On the Fossiliferous Rocks of the Southern Half of the Tortworth Inlier.’ By Frederick Richard Cowper Reed, M.A., F.G.S., and Prof. Sidney Hugh Reynolds, M.A., F.G.S.

This paper is a continuation of that on the igneous rocks of this area published in 1901 (Quart. Journ. Geol. Soc. vol. lvii. p. 267). The following succession is that adopted by the authors:—

OLD RED SANDSTONE.

LUDLOW ROCKS.

Thickness in feet.

LUDLOW ROCKS.		1 thickness in feet.
WENLOCK ROCKS.	{	5. Fissile red and yellow sandstones with gritty and calcareous bands... ?
		4. Upper limestone-band about 25
		3. Variable non-calcareous beds, according to Phillips about 500
		2. Highly-fossiliferous red clay or shale, with rubbly limestone and celestine- bands 12 (seen)
		1. Lower limestone-band, somewhat sandy thickly-bedded limestone ... 30
		625
LLANDOVERY ROCKS.	{	6. Shales with grit and highly-fossili- ferous calcareous sandstone ?
		5. Highly-fossiliferous ashy limestone . 2½ to 3½
		4. Upper trap-band about 60
		3. Sandy limestone and calcareous sand- stone and grit, crowded with fossils about 500
		2. Lower trap-band ... maximum about 185
		1. Micaceous sandstone, with <i>L. Symonds</i> i ?
		800 to 850

The rocks are affected by the Hercynian flexures which produced the Bristol coal-basin, and the outcrop of the beds in the main follows the horseshoe-shaped outcrop of the Old Red Sandstone. This regularity is lost at Daniel's Wood and Middlemill. Two important transverse faults traverse the outcrops, which are further obscured by the overlap of unconformable Trias. The trap-bands are found to be confined to the Llandovery, the number of recorded fossils has been largely added to, and previous statements as to the thinness and imperfect development of the Ludlow rocks and as to the probable exposure of the district to erosion in Ludlow and Lower Old-Red-Sandstone times are confirmed. The typical Ludlow fauna of Herefordshire and Shropshire has not been met with, and the series is clearly much attenuated. General remarks on the fossils are appended, and the paper contains lists of fossils in various collections (Bristol Museum, Sedgwick Museum, Earl Ducie's collection, and the Museum of Practical Geology, Jermyn Street), as well as those collected by the authors from the Llandovery and Wenlock formations.

November 4th.—Prof. W. J. Sollas, Sc.D., LL.D., F.R.S.,
President, in the Chair.

The following communications were read:—

1. 'The Relations of the Nubian Sandstone and the Crystalline Rocks of Egypt.' By Hugh John Llewellyn Beadnell, Assoc.Inst.M.M., F.G.S. (late of the Geological Survey of Egypt).

The paper opens with an account of the general conclusions of previous observers, which are mainly in favour of the view that the granites are not intrusive into the Nubian Sandstone but that the latter was deposited round denuded masses of the granite. The crystalline rocks south of the Oasis of Kharga are first dealt with. Eight exposures of crystalline rocks were met with. The sediments near the contact with the crystalline rocks are generally inclined at a high angle, and in some cases the former appear to undergo considerable alteration. The bedded rocks contain no fragments derived from the crystalline rocks. Hills of folded Eocene and Cretaceous strata seem to indicate that the intrusion of the granite may be of later date than Lower Eocene. These crystalline rocks do not appear to differ from those of the First Cataract and of the Eastern Desert and Sinai, where great vertical displacements have occurred, and where it seems likely that the sandstones were carried up when the great igneous core was elevated into its present position. Here too there seems to be no evidence of fragments of the crystalline rocks in question in the sediments. Thus the author concludes that the Nubian Sandstone was unconformably deposited, partly on pre-existing sedimentary formations, and partly on the planed-down surfaces of still older crystalline and metamorphic rocks. Subsequently it was invaded by outbursts from the underlying magma, the intrusions being probably connected with the elevation of the mountainous regions on the east side of the Nile.

2. 'On the Fossil Plants of the Waldershare and Fredville Series of the Kent Coalfield.' By E. A. Newell Arber, M.A., F.L.S., F.G.S.

November 18th.—Prof. W. J. Sollas, Sc.D., LL.D., F.R.S.,
President, in the Chair.

The following communication was read:—

‘On some Intrusive Rocks in the Neighbourhood of Eskdale (Cumberland).’ By Arthur Richard Derryhouse, D.Sc., F.G.S.

There appear to be five well-marked groups of intrusions in this district:—

- (a) The andesitic dykes in the neighbourhood of Allen Crag and Angle Tarn.
- (b) The dykes of the spherulitic and felsitic group on Yewbarrow and High Fell.
- (c) The dioritic (‘bastard granite’) bosses of Peers Gill, Lingmell Crag, and Bursting Knotts, with their associated dykes.
- (d) The Eskdale Granite, with the granite-porphry dyke running from Great Bank to Wasdale Head and thence to Kirkfell Crag.
- (e) The dolerite-dykes, having a general north-west to south-east trend.

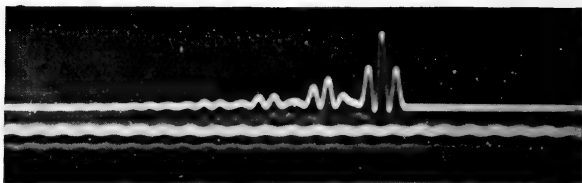
The dykes of series (a) bear a very strong petrological resemblance to the Borrowdale volcanic rocks, into which they were intruded. Furthermore, they are weathered to much the same extent and have developed the same secondary minerals, among which epidote is conspicuous. They appear to the author to be of Borrowdale age, and roughly contemporaneous with the lavas and ashes into which they are intruded. The spherulitic and more acid series (b) are considered to be also of Borrowdale age, though probably somewhat later than the andesitic series. The rocks of the dioritic group (c) are considered to be the holocrystalline and hypabyssal equivalents of the Borrowdale Lavas, and the author is of opinion that they also are of Ordovician age.

The Eskdale and Wasdale Granites (d) are much more acid, and show little sign of alteration except that due to weathering and dislocation. They are undoubtedly intrusive into the Borrowdale Series, but seem to be pre-Triassic. Thus the intrusion is probably Devonian, like the neighbouring granite of Shap, which, with the exception of its large phenocrysts of orthoclase, is not dissimilar to some of the varieties of the Eskdale Granite. The basic intrusions (e) have been examined only where they come into proximity to the granite. They may well be connected with the great Tertiary basic flows of Antrim, as has been suggested by Mr. Harker.

The granite becomes progressively more and more acid as its margin is approached, until in some places the percentage of silica amounts to 96·16. This is explained by the assumption that the magma as a whole was more acid than the eutectic mixture of quartz and orthoclase, and that consequently the excess of silica separated in the marginal portions, which were the first to solidify.

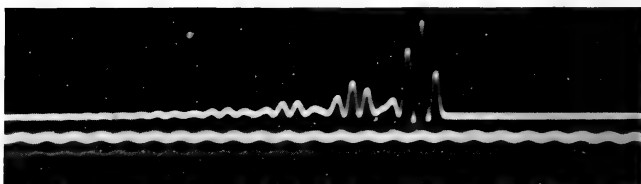


FIG. 5.



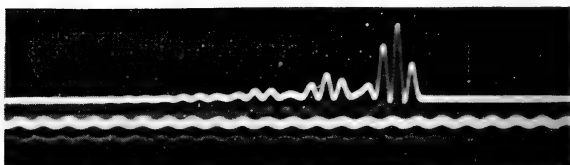
$C_1=9.55$, $C_2=.001063$ microfarad.

FIG. 6.



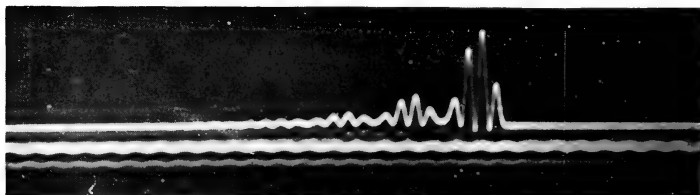
$C_1=9.55$, $C_2=.000875$ microfarad.

FIG. 7.



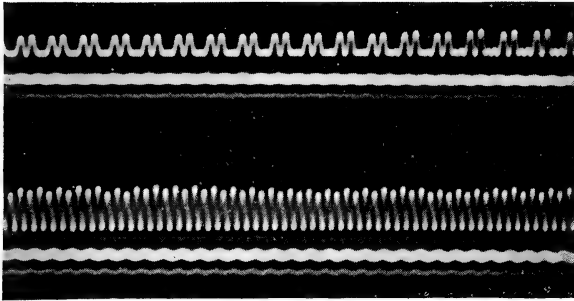
$C_1=11.87$, $C_2=.001063$ microfarad.

FIG. 8.



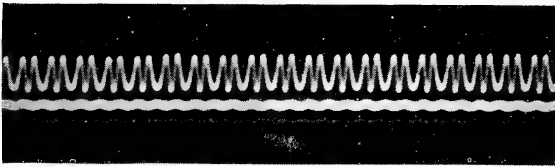
$C_1=11.87$, $C_2=.000875$ microfarad.

FIG. 9.



Musical arc oscillations in secondary circuit.

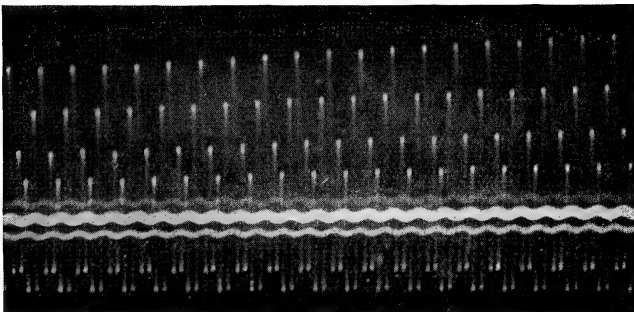
FIG. 10.



Lower note of musical arc.

$C_1=11.87$, $C_2=.000875$ microfarad.

FIG. 11.



Musical arc oscillations.

$C_1=9.55$, $C_2=.000275$ microfarad.

FIG. 5.

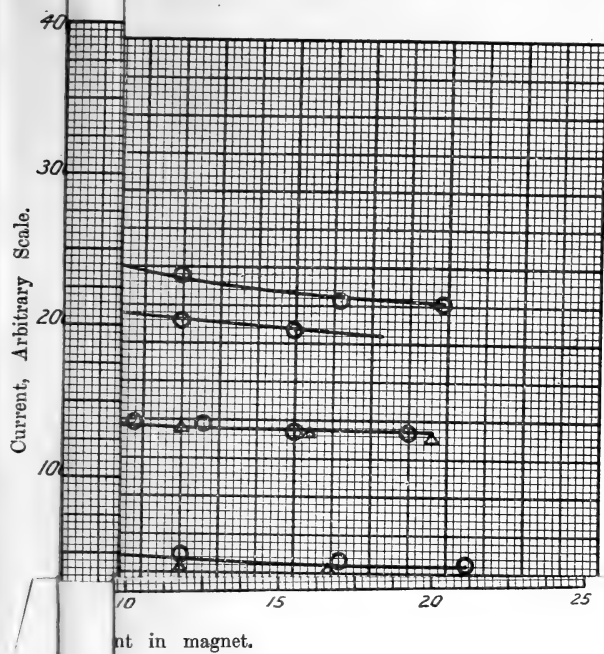


FIG. 18.

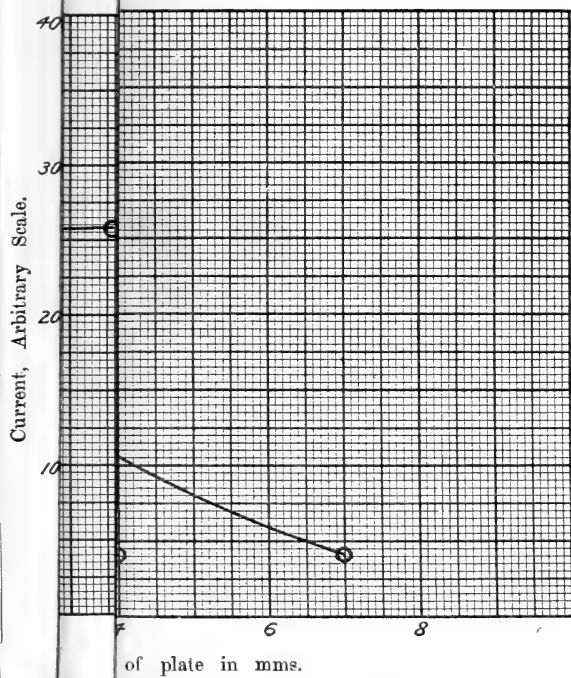


Fig. 2.

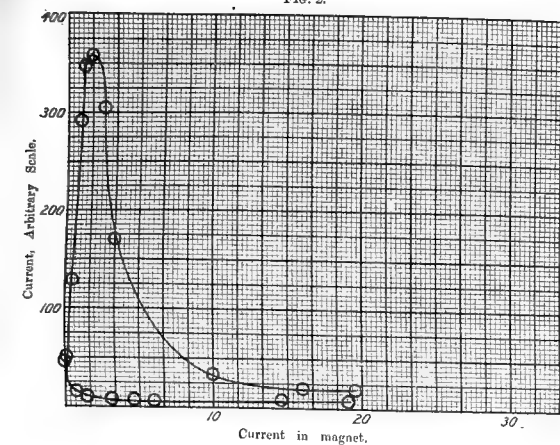


Fig. 3.

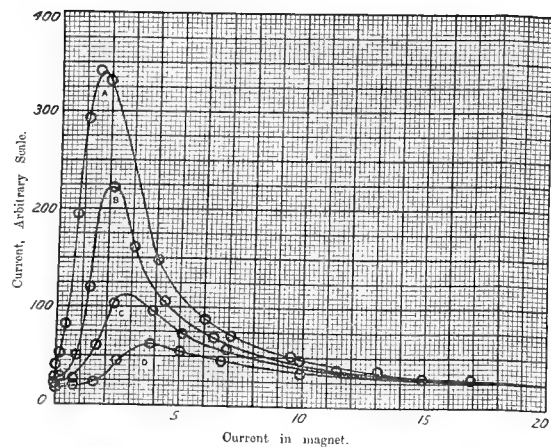


Fig. 4.

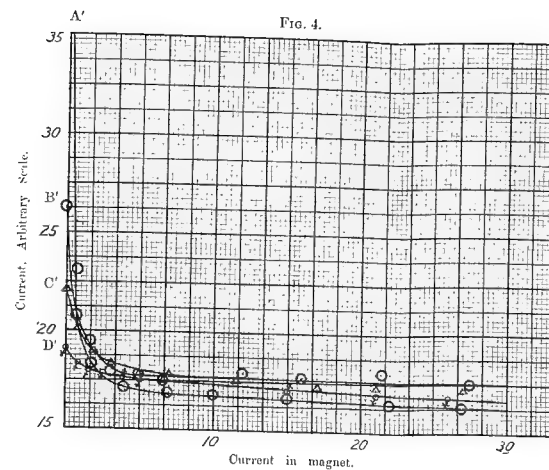


Fig. 5.

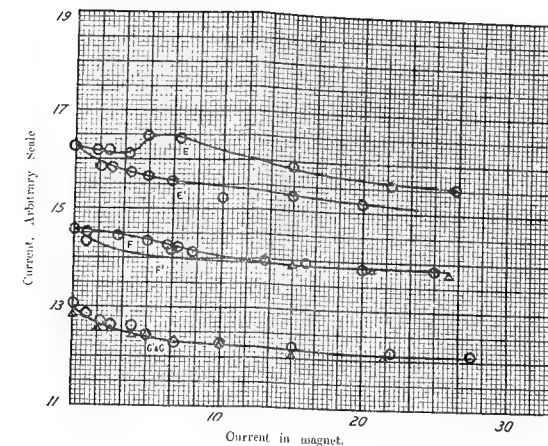


Fig. 6.

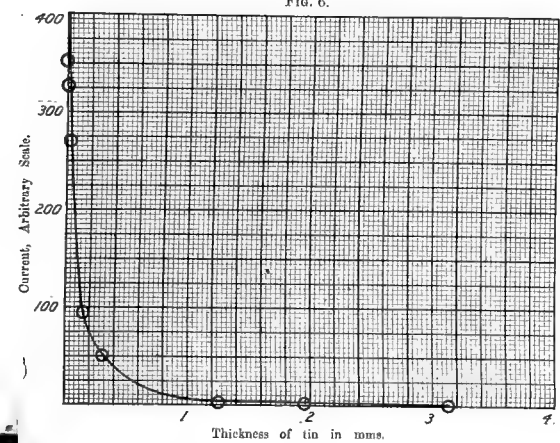


Fig. 7.

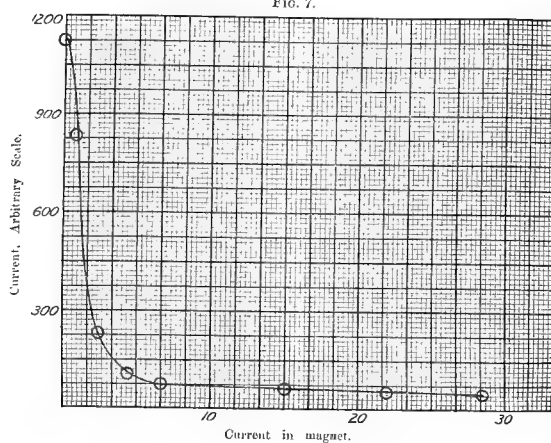


Fig. 8.

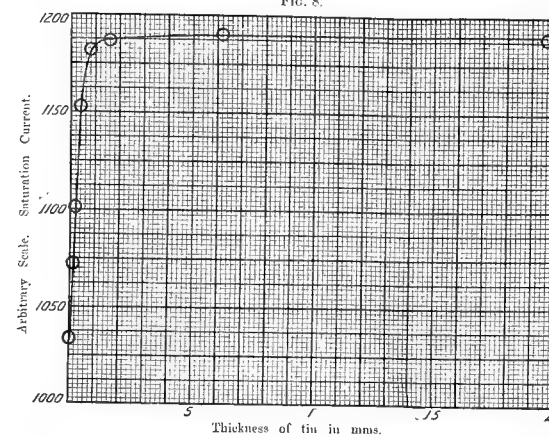


Fig. 9.

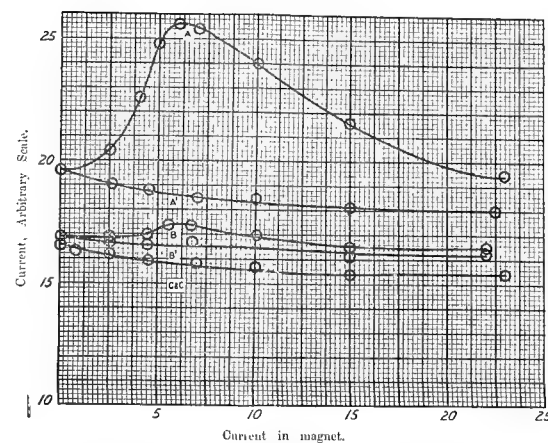


Fig. 10.

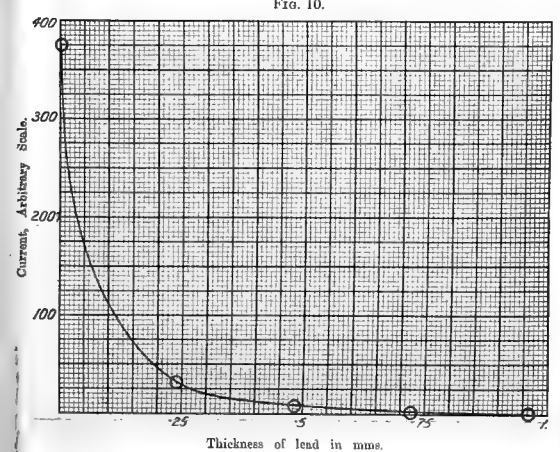


Fig. 11.

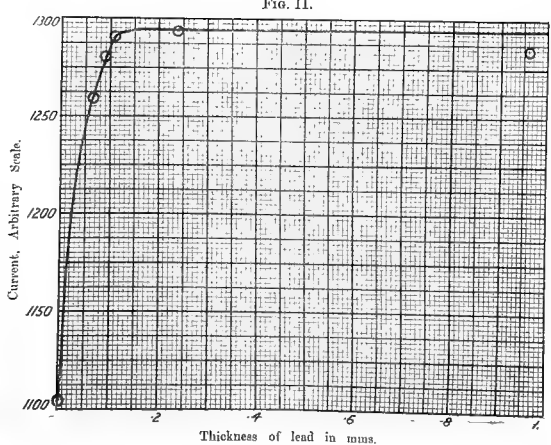


Fig. 12.

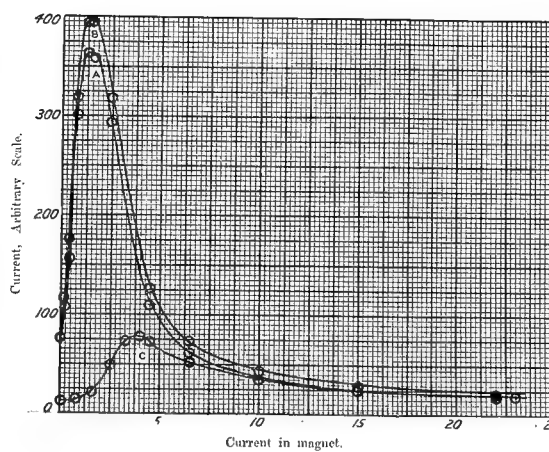


Fig. 13.

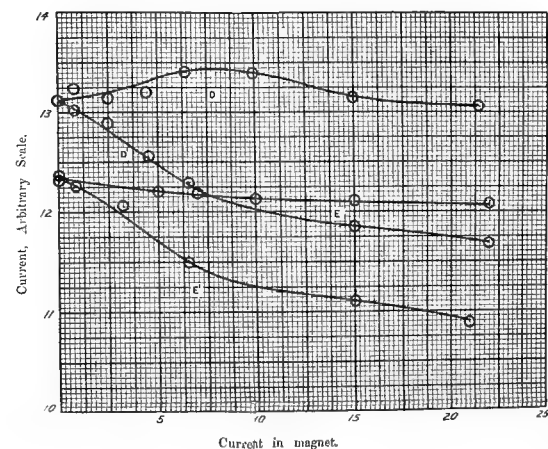


Fig. 14.

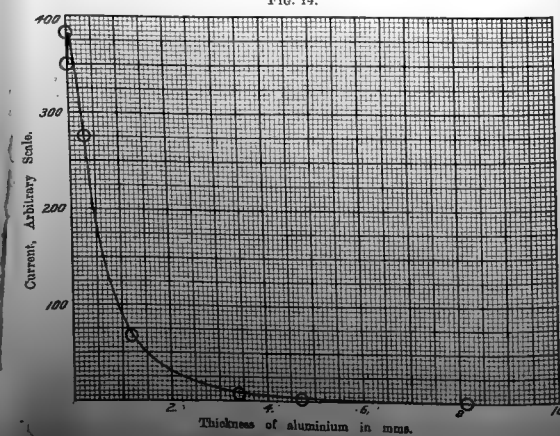


Fig. 15.

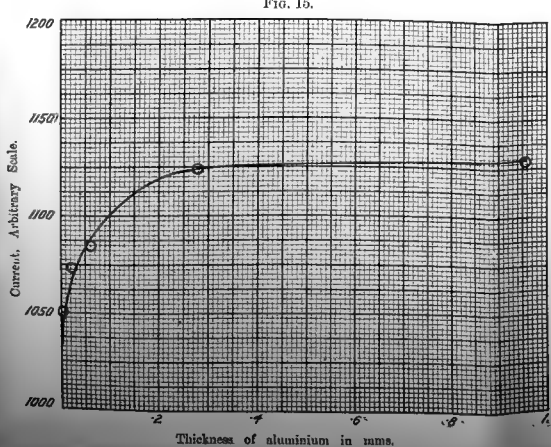


Fig. 16.

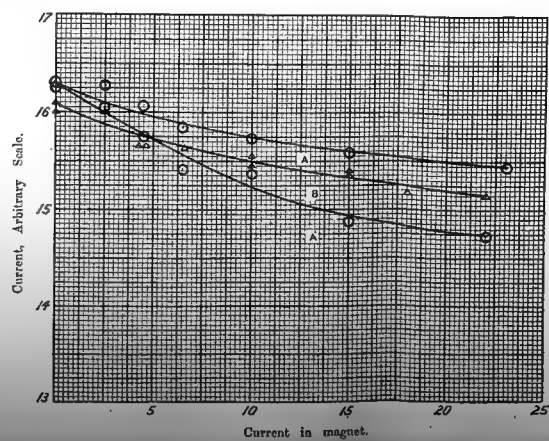
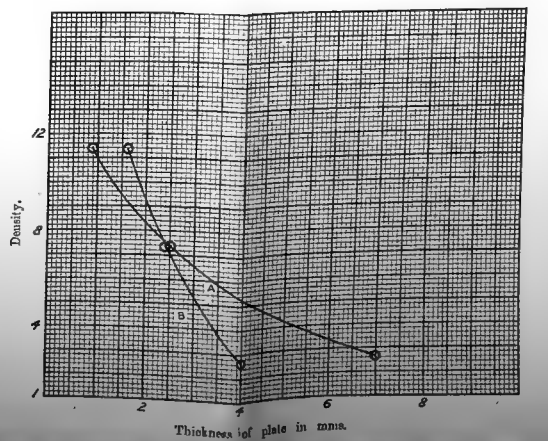


Fig. 18.



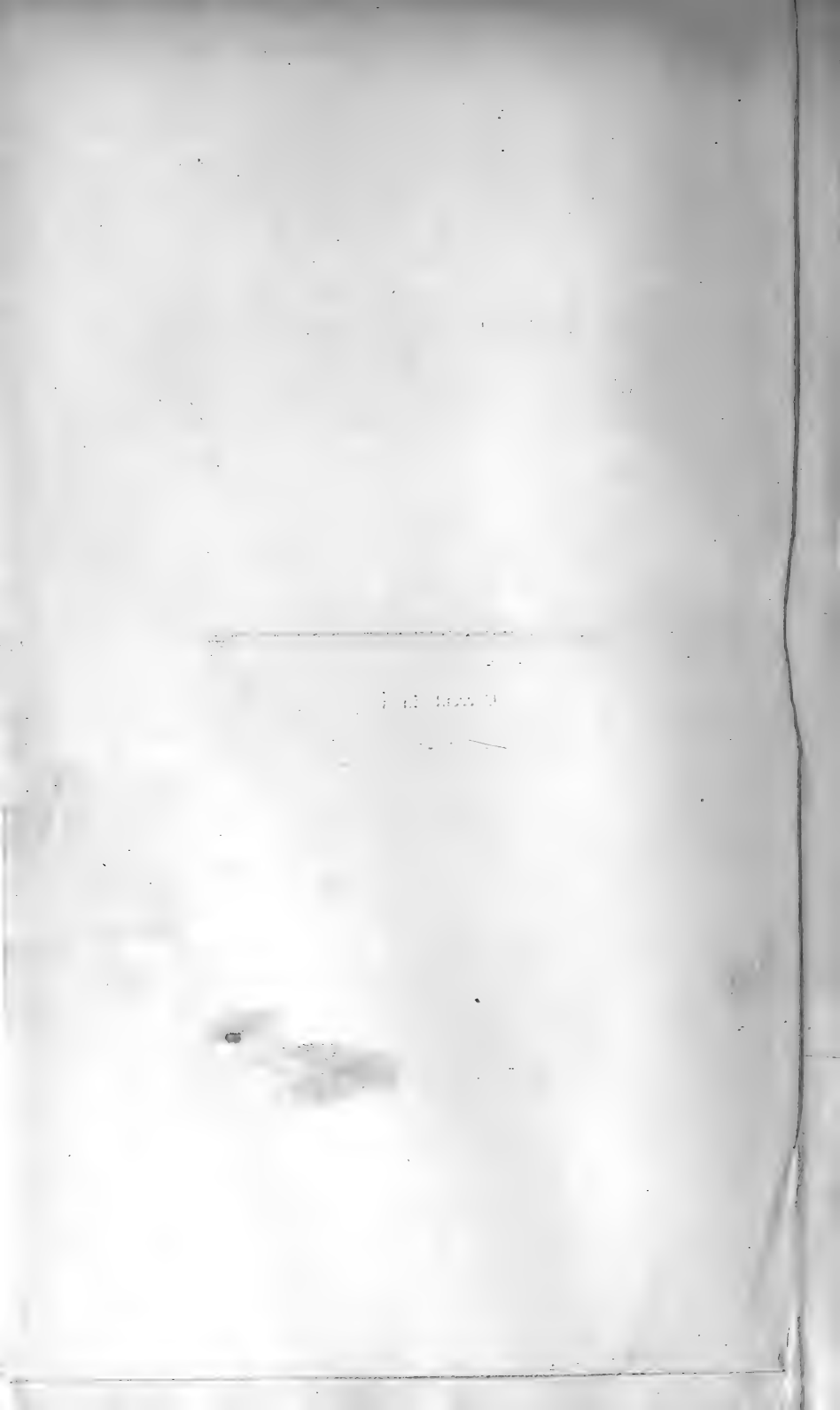


FIG. 1.

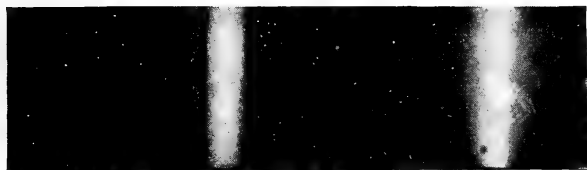


FIG. 2.

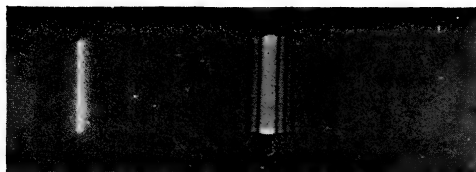
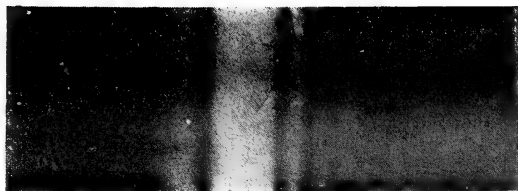
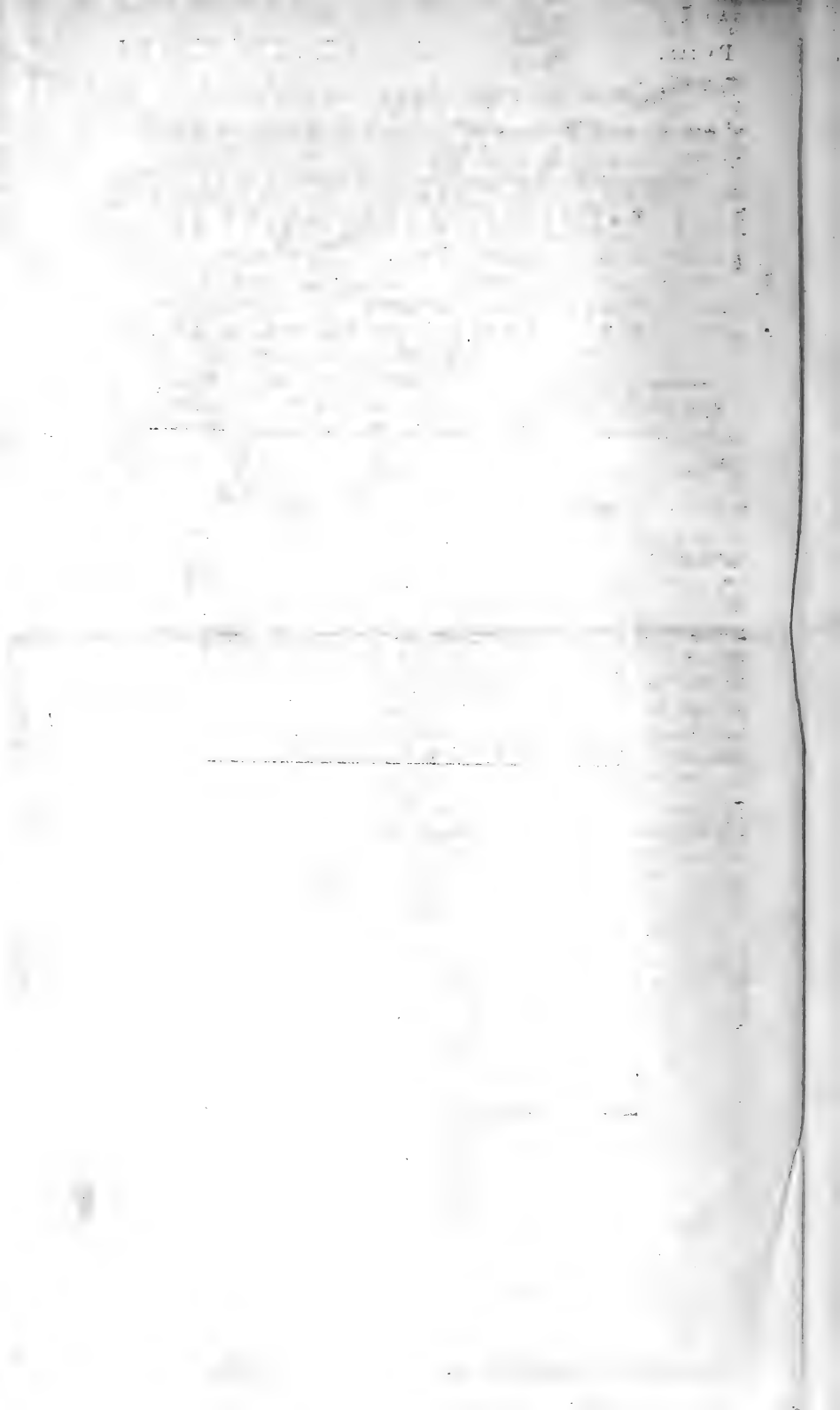


FIG. 3.





INDEXED

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

FEBRUARY 1909.

XVIII. *Temperature-Radiation and the Partition of Energy in Continuous Media.* By J. H. JEANS, M.A., F.R.S.,
Professor of Applied Mathematics in Princeton University.*

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II. A tube of air.	
III. A mechanical model of the <i>Æther</i> .	
§§ 22-25. The Rate of Approach to the Normal State.	248
§§ 26-30. Conclusion	250

INTRODUCTION.

1. **T**HE present paper attempts a critical examination of some of the difficulties which surround the application of the law of equipartition of energy to the energy of wave motion in the *æther* or other continuous media. These difficulties manifest themselves most clearly in connexion with the problem of determining the law of partition of radiant energy inside a uniform-temperature enclosure. Accordingly it is with this particular problem that we shall be mainly concerned.

2. Let two or more bodies, originally at different temperatures, be supposed placed inside a closed chamber whose

* Communicated by the Author.

walls reflect energy perfectly. These conditions cannot be realized experimentally; so that it is not known what would happen. But it is commonly supposed that the temperatures of the bodies would equalize* by radiation, so that after an infinite time the bodies would all be at the same temperature T ; also the æther, the vehicle of energy from one body to another, would be possessed of a certain amount of energy, and in spite of the continual transference and retransference of energy between matter and æther, the amount of energy in the æther, and the law of its distribution between different wave-lengths, would remain constant. In this supposed state, let the energy per unit volume of radiation of wave-lengths intermediate between λ and $\lambda + d\lambda$ be assumed to be

$$F(\lambda, T)d\lambda. \dots \dots \dots (1)$$

3. The position of the masses of matter inside the enclosure has been immaterial. Let them now be supposed spread over the walls, so that every part of the original perfectly-reflecting walls is covered. So far the whole system inside the reflecting walls has remained impervious to energy. Let a minute hole now be made at any point in one of the walls, and let the radiant energy stream through this hole into external space (which may, for simplicity, be thought of as devoid of radiant energy). At first the issuing stream of radiant energy will be of constitution given by formula (1); but in time, as the total amount of energy inside the enclosure diminishes, the constitution of the escaping energy will change. The flow of energy could be kept constant, provided energy of the appropriate amount *and constitution* could be supplied to the inside of the enclosure. There would then be a steady flow of energy through the aperture, of constitution given by formula (1).

4. Suppose that the walls are no longer perfectly-reflecting, but are provided with a mechanism which keeps them at a uniform temperature T . The flow of energy through the small aperture must now be "steady": experiment shows it to be independent of the nature and reflecting power of the

* This is the orthodox view. Equalization is commonly supposed to be a necessity, from the second law of thermodynamics. My own view is that equalization *would take place*, but only because the charges on the electric sources of radiation would be identical in the different masses of matter. If the electrons in one mass were more heavily charged than those in the other masses, I believe the temperatures would tend to become different, in opposition to the second law. (*Cf. Phil. Mag.* [6] xii. p. 57.)

walls. Let the constitution of this stream of energy be supposed given by the formula

$$f(\lambda, T)d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is commonly argued or assumed* that the constitution of the energy in this experiment must be the same as in the preceding imaginary experiment. If so, the functions F and f must be identical.

5. Assuming the legitimacy of using the theorem of Equipartition of Energy, the function F can be calculated at once from this theorem. It is found† that

$$F(\lambda, T) = 8\pi RT\lambda^{-4}, \quad . \quad . \quad . \quad . \quad (3)$$

where R is the universal gas-constant.

The function f can be determined experimentally. Planck‡, with the help of a mathematical argument, with the details of which we are not here concerned, arrives at the formula

$$f(\lambda, T) = \frac{8\pi c}{\lambda^5} \frac{1}{e^{\frac{c}{kT\lambda}} - 1}, \quad . \quad . \quad . \quad . \quad (4)$$

a form which agrees well§ with experimental readings, provided c and k may be treated as adjustable constants.

The values of c and k which Planck arrives at by comparing formula (4) with experiment are

$$c = 1.965 \times 10^{-16}, \quad k = 1.346 \times 10^{-16}.$$

On the other hand, Lorentz ||, using the form $8\pi k\lambda^{-4}T$, to which formula (4) reduces for long wave-lengths, obtains

$$k = 1.07 \times 10^{-16}$$

as the value of k given by experiments on light of great wave-length. According to Planck's analysis, the constant k ought to be identical with the gas constant R , of which the value is known to be $R = 10^{-16}$ to within a few per cent.

* The assumption is tacitly involved in the common employment of the expression "radiation appropriate to a given temperature." (Cf. Proc. Roy. Soc. A. lxxvi. p. 306, 1905.)

† Rayleigh, Phil. Mag. [5] xlix. p. 539 (1900), and Nature, lxxii. pp. 54, 243 (1905). Also J. H. Jeans, Phil. Mag. [6] x. p. 91 (1905).

‡ *Vorlesungen über Wärmestrahlungen* (1906), p. 157, and in earlier papers.

§ Planck, *Vorlesungen über Wärmestrahlungen*, p. 158, and Paschen, *Annalen d. Physik*, iv. p. 277.

|| Konink. Akad. van Wetenschappen (Amsterdam), April 24, 1903, p. 678.

Let us, for the present, continue to denote R and k by separate symbols, leaving open the question of whether or not the quantities they represent are identical. If these quantities are not identical, then the formulæ (3) and (4) are entirely different. But if they are found to be identical, then the formulæ will be seen to coincide for light of great wave-length, but to diverge widely for light of visible or very short wave-length.

6. The divergence between the two formulæ, whether complete or partial, raises various questions, to which the analysis of the present paper attempts to provide answers. It is natural to inquire—

(1) Is the use of the Theorem of Equipartition, and consequent derivation of formula (3), legitimate?

(2) If so, what is the essential difference between the physical conditions which lead to formula (3) and those which lead to formula (4)?

(3) If equation (1) is answered in the affirmative, do formulæ (3) and (4) become identical for long wave-lengths; and if so, why?

It may simplify what follows to state briefly in advance the conclusions arrived at.

7. It is found that question (1) can be definitely answered in the affirmative, but that the theorem of equipartition represents merely the tendency for energy to become degraded into irregular disturbances of the medium, and the utility of the theorem of equipartition (although not, of course, its truth) is limited by the circumstance that it represents a state attained only after enormous, or infinite, time.

The answer obtained to question (2) can be best explained by making use of an acoustical analogy. Let the æther be replaced by air: let waves of light in the æther be represented by waves of sound in air. Matter may be represented by a series of musical or noise-producing instruments: these will of course be capable of absorbing as well as emitting sound.

To represent the state of things considered in § 4, we have to imagine the walls of a room to be covered continuously with sound-instruments (to make the picture clearer, let us say telephone diaphragms), and we suppose that these are kept in vibration by agencies acting from outside the room. A person listening at an aperture in the wall of the room will hear the sound of the telephones (modified by the reflexion and absorption of the other diaphragms, perhaps): the

energy of these sounds corresponds to that given in formula (4).

To represent the state of things imagined in § 3, suppose that the walls behind the diaphragms are impervious to sound and to all kinds of energy, and that the aperture in the wall is stopped up with matter also impervious to energy. The diaphragms must be supposed to be initially set into vibration, and then left to themselves. The energy of the sounds they emit will be reflected, absorbed, re-emitted, and so on. Finally all sound will have become dissipated into heat. The energy of the original sounds—the total store of energy in the room—will figure as the heat of the air and of the diaphragms. A person outside the room who uncovers the aperture and listens will hear nothing at all, unless his ears are sufficiently acute to hear the waves of air originating in the random heat-motions of the molecules. For, as we shall see, this random motion of the molecules can be resolved, by Fourier's theorem, into the motion of trains of waves, and, in perfectly irregular heat motion, there is equipartition of energy between the different trains of waves, so that the law of partition according to wave-lengths is given by the formula $4\pi RT\lambda^{-4}d\lambda$. This agrees with formula (3), except for a numerical factor 2 which finds its origin in the different energy-capacities of transverse and longitudinal vibrations.

It is found that the third question must be answered in the affirmative. The reason for this answer will be found in the concluding sections (§§ 22–30), and the reader who is not interested in the abstract argument and analysis which follow is advised to pass at once to these sections.

The "Normal State" and Equipartition of Energy.

8. We begin the mathematical discussion by proving the law of equipartition in the form appropriate to the vibrational energy of continuous media. It is important to exhibit the proof in such a form as to make it clear that *the law rests on no assumptions of any kind, except the assumption that the motion of the medium obeys the laws of a conservative dynamical system.*

Let L be a function of variables $\theta_1, \theta_2, \dots, \theta_n, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_n$, and let these change in value so that $\frac{d\theta_1}{dt} = \dot{\theta}_1$, &c., while $\int L dt$ is stationary in value. Let L consist solely of terms of degrees two and zero in $\dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_n$.

New functions $u_1, u_2, \dots u_n$ and L' of these variables are introduced, defined by

$$u_1 = \frac{\partial L}{\partial \dot{\theta}_1}, \text{ \&c.}$$

$$L' = u_1 \dot{\theta}_1 + u_2 \dot{\theta}_2 + \dots + u_n \dot{\theta}_n - L;$$

and it is then proved in the usual way, by purely algebraic transformation*, that

$$\frac{du_1}{dt} = -\frac{\partial L'}{\partial \theta_1}; \quad \frac{d\theta_1}{dt} = \frac{\partial L'}{\partial u_1}; \quad . \quad . \quad . \quad (5)$$

whence it follows that

$$\frac{\partial}{\partial u_1} \left(\frac{du_1}{dt} \right) + \frac{\partial}{\partial \theta_1} \left(\frac{d\theta_1}{dt} \right) = 0. \quad . \quad . \quad . \quad (6)$$

The condition that $\int L dt$ is to be stationary determines uniquely the changes in $\theta_1, \theta_2, \dots \theta_n, \dot{\theta}_1, \dot{\theta}_2, \dots \dot{\theta}_n$, starting from given initial values. Now let a generalized space be constructed, having $\theta_1, \theta_2, \dots \theta_n, u_1, u_2, \dots u_n$, as its coordinates. Let this space be filled with "representative points" each of which is to move as directed by the condition that $\int L dt$ is to be stationary. If ρ denote the density of these "representative points" at any point in the generalized space, and if $\frac{D\rho}{Dt}$ denotes the rate of increase of ρ as we follow the "representative points" in their motion, we have†, as a matter of algebraic calculation,

$$\frac{D\rho}{Dt} = -\rho \sum_1^n \left\{ \frac{\partial}{\partial u_1} \left(\frac{du_1}{dt} \right) + \frac{\partial}{\partial \theta_1} \left(\frac{d\theta_1}{dt} \right) \right\} = 0, \quad . \quad (7)$$

by equation (6). If the "representative points" tended, in their motion, to concentrate onto any special points or regions in the generalized space, then $\frac{D\rho}{Dt}$ would be positive for those points or regions: similarly, if the "representative points" tended to scatter, $\frac{D\rho}{Dt}$ would be negative. The result obtained in equation (7), that $\frac{D\rho}{Dt}$ vanishes everywhere, shows that there is no tendency for the "representative points" to

* Routh, 'Elem. Rigid Dynamics,' chap. viii., or Jeans, 'Theoretical Mechanics,' chap. xii.

† Jeans, 'Dynamical Theory of Gases,' p. 63.

concentrate about any special points or regions in the generalized space, or the reverse.

We have so far merely studied the changes in the values of a system of algebraic variables, when they change as directed by a certain algebraic system of conditions (namely, $\delta \int L dt = 0$). The motion of points in the generalized space has merely provided a graphical representation of these changes.

9. Now let these variables $\theta_1, \theta_2, \dots, \theta_n, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_n$ be the coordinates and velocities of a dynamical system, and let L be its Lagrangian function. Then the motion of a "representative point" in the generalized space will represent the changes in the coordinates and velocities as these change in accordance with the principle of Least Action—i. e., as the system moves in accordance with the laws of nature. The proved fact that in this motion there is no tendency for the "representative points" to concentrate about any special points or regions of the generalized space leads at once to the following:—

THEOREM. *All properties (if any) which are such as to be finally acquired by the dynamical system, independently of the special state from which the system started, must be properties common to the whole of the generalized space.*

For, if the system must inevitably possess some property, this can only be either because its representative point tends inevitably to pass into the regions of the generalized space in which this property holds, or else because the property holds in all regions. The former alternative is disproved by equation (7): the latter alternative must accordingly be the true one.

10. It is found that, in general, there are no properties common to all regions in the generalized space (or rather, no properties of any importance for the present purpose). But when the system possesses an infinite (or very great) number of similar coordinates, there are certain statistical properties found to be common to the whole of the space except for infinitesimal regions of it. A system which possesses these statistical properties is said to be in the "Normal State."

A representative point may of course have its whole path in regions in which the "Normal State" does not obtain, or it may pass through these regions for periods, large or small, on its path. But we have the quite general theorem:—

THEOREM. *If a system tends to acquire definite properties, independently of its initial configuration, or if it tends to acquire*

these properties when it starts from any configurations except an infinitesimal few, then these properties must be properties of the "Normal State."

For, if not, suppose that the system tends to assume some property P which is not common to the whole of the generalized space, or even to the whole except for infinitesimal regions, but is confined to some region S of the generalized space. The "representative points" which at the beginning of the motion in the generalized space occupied the whole of this space (or the whole of it except for infinitesimal regions), must, by the end of the motion, all lie within the region S—a result which would be in opposition to equation (7), $\frac{D\rho}{Dt} = 0$.

11. Of the properties of the "normal state," that one which is of primary importance for the present investigation is the Equipartition of Energy*.

Suppose that the energy E of the system can be expressed as a function of the Lagrangian coordinates and velocities in the form

$$2E = \alpha_1 p_1^2 + \alpha_2 p_2^2 + \dots + \alpha_n p_n^2 + f(\theta_1, \theta_2, \dots, \theta_n),$$

where p_1, p_2, \dots, p_n are any quantities (coordinates, velocities, or momenta) and $\theta_1, \theta_2, \dots, \theta_n$ are other quantities which may or may not enter into $\alpha_1, \alpha_2, \dots, \alpha_n$. The law of equipartition states that if n is very great, the energy represented by any very great number s of the n first terms is, in the normal state, proportional to s . We assume it to be $\frac{1}{2}sRT$. Then, if part of the dynamical system consists of matter of any kind, T is the temperature of this matter†.

12. If the system is supposed, for the moment, to consist solely of a non-dissipative vibrating medium, free from disturbance by external agencies, the whole of the energy can be expressed in the form

$$2E = \alpha_1 p_1^2 + \alpha_2 p_2^2 + \dots + \alpha_n p_n^2,$$

in which p_1, p_2, \dots, p_n represent the normal coordinates and their rates of change. Thus each separate free vibration contributes two terms to the energy. In the normal state, the energy of any great number s of free vibrations must be sRT .

13. From a consideration of physical dimensions, it is clear that in any medium whatever, the number of free vibrations

* The proof that this is a property of the Normal State is purely algebraic in its nature: see 'The Dynamical Theory of Gases,' p. 67.

† Jeans, 'Dynamical Theory of Gases,' §§ 77, 124.

of wave-lengths intermediate between λ and $\lambda + d\lambda$ (where λ is large compared with the scale of structure of the medium, if the medium is coarse-grained) must be

$$\mathbf{C}\lambda^{-4}d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

per unit volume, where \mathbf{C} is a numerical quantity which depends on the structure of the medium. It follows that when the medium is in equilibrium with matter at temperature T , its vibrational energy of wave-lengths intermediate between λ and $\lambda + d\lambda$ must be

$$\mathbf{C}RT\lambda^{-4}d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

per unit volume.

If the medium is æther, it is easily found (§ 5) that the value of \mathbf{C} must be 8π , and formula (3) follows at once. If the medium is gaseous (so that all vibrations are longitudinal) the value of \mathbf{C} is 4π , while for an elastic solid medium $\mathbf{C} = 12\pi$.

14. If the medium is structureless, then formula (9) holds down to the very shortest wave-lengths. The energy corresponding to any finite value of T is infinite. Whatever the value of T , the whole energy (except for an infinitesimal fraction) is confined to vibrations of infinitesimal wave-length. In this case the value of the Law of Equipartition is not so much that it gives the final state of the medium (a state reached only after infinite time (*cf.* § 24, below)), as that it shows the tendency for the energy to run into vibrations of infinitesimal wave-length. Or, what is the same thing, it shows the tendency for regular trains of waves to become dissipated into irregular disturbances (subject to a certain limitation, *cf.* below).

15. If the medium is coarse-grained, then formula (9) is not applicable, when dealing with waves of length comparable with the scale of structure of the medium. Let λ_0 denote (loosely speaking) the smallest wave-length possible, so that λ_0 is a length comparable with the scale of coarse-grainedness of the medium. From formula (9), the total energy per unit volume of the medium is

$$\int_{\lambda=\lambda_0}^{\lambda=\infty} \mathbf{C}RT\lambda^{-4}d\lambda = \frac{\mathbf{C}RT}{3\lambda_0^3}.$$

If \mathbf{J} is the mechanical equivalent of heat, the "specific heat" of the medium must be

$$\frac{\mathbf{C}R}{3\mathbf{J}\lambda_0^3}$$

per unit volume. Denoting this specific heat by σ , and

replacing R/J by its known value 2×10^{-24} , we find that λ_0 must be of the order of $1.3 \times 10^{-8} \times \sigma^{-\frac{1}{2}}$. For instance, for water $\sigma=1$: it follows that the mean distance apart of the molecules of water must be comparable with 1.3×10^{-8} cm. For air at atmospheric pressure, $\sigma=.0002$: it follows that the mean distance apart of the molecules of the atmosphere must be comparable with 2×10^{-7} cm. If it were possible to measure the energy of æther in temperature-equilibrium with matter, we could determine the specific heat σ , and so obtain a knowledge of the scale of structure of the æther (if any).

Examples and Discussion of the "Normal State" in continuous media.

16. Before passing to further developments of the theory, I have thought it permissible to illustrate the foregoing ideas and concepts by a few mechanical illustrations.

I. *A Stretched String.*

17. Consider a dynamical system of which the kinetic and potential energies are respectively given by

$$2T = m(\dot{x}_1^2 + \dot{x}_2^2 + \dots + \dot{x}_n^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$2V = \mu\{(x_0 - x_1)^2 + (x_1 - x_2)^2 + \dots + (x_n - x_{n+1})^2\}, \quad . \quad (11)$$

and let n be very great.

If $x_0, x_1, x_2, \dots, x_n, x_{n+1}$ are coordinates of particles constrained to remain always in the same straight line, the system may be supposed to consist of a series of $n+2$ collinear particles, each attracting (or repelling) its neighbour according to the law of the direct distance, the two end particles being fixed in position. With a slight change in the meaning of the symbols, the system may be supposed to consist of heavy particles connected by elastic strings. In the limit when n is made infinite, the system will represent a continuous one-dimensional elastic medium, or a stretched string capable of performing longitudinal vibrations only.

Regarding the system as a collection of particles, the properties of the "normal state" can be seen at once from an examination of the energy-function. The velocities $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$ will be distributed according to Maxwell's law

$$Ae^{-\frac{1}{2}mu^2} du,$$

and, when n is infinite, the same can be shown to be true of the differences $x_0 - x_1, x_1 - x_2$, &c. Thus in the normal state,

each particle has, on the average, kinetic energy $\frac{1}{2}RT$, and each element of stretched string has, on the average, potential energy $\frac{1}{2}RT$. In the normal state, these kinetic and potential energies are distributed at random, and without correlation, about the common mean-value $\frac{1}{2}RT$.

In the present investigation, however, we are concerned with expressing the energy of the normal state in terms of the energy of trains of waves.

It is readily found that the free vibrations, subject to $x_0=0$, $x_{n+1}=0$, are given by

$$x_s = \sin \frac{sq\pi}{n+1} \quad (q=1, 2, \dots n), \quad . \quad . \quad . \quad (12)$$

and the frequency of this vibration (p) is given by

$$p = 2\sqrt{\frac{\mu}{m}} \sin \frac{q\pi}{2(n+1)}. \quad . \quad . \quad . \quad (13)$$

If $\xi_1, \xi_2, \dots \xi_n$ are the various principal coordinates, we may, from equation (12), write

$$x_s = \sum_{q=1}^{q=n} \xi_q \sin \frac{sq\pi}{n+1}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Expressed in terms of the principal coordinates, equations (10) and (11) become

$$2T = \frac{1}{2}m(n+1)(\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dots + \dot{\xi}_n^2),$$

$$2V = 2\mu(n+1) \left(\xi_1^2 \sin^2 \frac{\pi}{2(n+1)} + \xi_2^2 \sin^2 \frac{2\pi}{2(n+1)} + \dots + \xi_n^2 \sin^2 \frac{n\pi}{2(n+1)} \right).$$

We can now express the energy of the "normal state" in terms of the energies of free vibrations, or of trains of waves. Since each term in T and V has an average amount of energy $\frac{1}{2}RT$, it follows that each free vibration has average energy RT . From equation (12) it follows that the wave-length λ of the

q th free vibration is $\frac{2}{q} \times$ (length of system). Hence the number of free vibrations for which λ lies between λ and $\lambda + d\lambda$ is $2\lambda^{-2}d\lambda$ per unit length of the system. The energy per unit length, of wave-length intermediate between λ and $\lambda + d\lambda$, is accordingly

$$2RT\lambda^{-2}d\lambda. \quad . \quad . \quad . \quad . \quad (15)$$

This is the one-dimensional analogue of formula (9). It represents the energy of the random distribution of kinetic

and potential energies which we have already seen to obtain in the normal state, analysed into the energy of regular trains of waves.

We return to the discussion of this system in § 23.

II. *A Tube of Air.*

18. The next system to be considered will consist of the air inside a tube of uniform cross-section, closed at both ends. For simplicity, the molecules of air will be supposed to be similar and infinitely small spheres: let them be N in number.

The normal state of this system is one with which the Kinetic Theory of Gases has made us very familiar. As regards position, the molecules are distributed absolutely at random throughout the tube: as regards motion, each velocity-component is distributed according to Maxwell's law.

Let us consider the arrangement of positions first. Let us imagine the tube divided into a great number n of cells, each being of the same cross-section as the tube, and of volume ω .

An arrangement by which the molecules are placed at exactly equal distances apart, in some regular geometrical order, is of course a possible arrangement, but is no more typical of the normal state than would be a motion in which each molecule had exactly the same velocity. So also an arrangement in which each of the n cells into which the tube is divided contained exactly the same number N/n of molecules, is possible, but is not typical of the normal state. In the normal state, the numbers of molecules in the different cells will be distributed around the mean value N/n , according to a law which can be determined.

Consider a single arrangement in which the numbers of molecules in the n cells taken in order are $a_1, a_2, \dots a_n$. In the limit, when n is made infinite, a knowledge of the values of $a_1, a_2, \dots a_n$ will be equivalent to a knowledge of the density of the gas at every point of the tube. On expressing this, by Fourier's theorem, as a series of circular functions, we can represent the deviations from uniform density as due to the superposition of trains of waves.

Employing the conception of probability in the exact sense in which I have defined it elsewhere*, the probability that an

* Phil. Mag. [6] v. p. 597, or 'Dynamical Theory of Gases,' p. 53.

arrangement selected at random shall have $a_1, a_2, \dots a_n$ as the numbers of molecules in the successive n cells, is *

$$\frac{N! n^{-N}}{a_1! a_2! \dots a_n!}$$

Hence the probability that $\frac{a_1}{N}, \frac{a_2}{N} \dots$ shall lie within limits $d\left(\frac{a_1}{N}\right), d\left(\frac{a_2}{N}\right) \dots$, is

$$\frac{N! n^{-N} N^n}{a_1! a_2! \dots a_n!} d\left(\frac{a_1}{N}\right) d\left(\frac{a_2}{N}\right) \dots \quad (16)$$

Let us now transform variables from $a_1, a_2 \dots a_n$ to $\xi_1, \xi_2, \dots \xi_n$, where $\xi_1, \xi_2, \dots \xi_n$ are given by the equations

$$\frac{a_s}{\omega} = \frac{N}{n\omega} + \sum_{q=1}^{q=n} \xi_q \sin \frac{qs\pi}{n}, \text{ \&c. } (s=1, 2, \dots n). \quad (17)$$

In the limit, when n is made infinite, this equation becomes

$$\nu - \nu_0 = \sum_{q=1}^{q=\infty} \xi_q \sin \frac{x}{l} q\pi,$$

where ν is the molecular-density at a distance x along the tube (supposed of the total length l), and ν_0 is the average value of ν (cf. equation (14)). Thus $\xi_1, \xi_2, \dots \xi_n$ ultimately become proportional to the amplitudes of waves of wave-lengths $2l, \frac{1}{2}(2l), \frac{1}{3}(2l)$, &c.

Let expression (16), transformed to variables $\xi_1, \xi_2, \dots \xi_n$, be supposed to become

$$f(\xi_1, \xi_2, \dots \xi_n) d\xi_1, d\xi_2, \dots d\xi_n;$$

so that this expression will measure the probability that $\xi_1, \xi_2, \dots \xi_n$ shall lie within limits $d\xi_1, d\xi_2, \dots d\xi_n$. From equation (17.) we have

$$\frac{\partial}{\partial \xi_q} \left(\frac{a_s}{N} \right) = \frac{\omega}{N} \sin sq \frac{\pi}{n};$$

so that

$$\frac{\partial \left(\frac{a_1}{N}, \frac{a_2}{N}, \dots \right)}{\partial (\xi_1, \xi_2, \dots)} = \omega^n N^{-n} \Delta,$$

where Δ is the determinant whose (s, q) term is $\sin (sq\pi/n)$, a pure number and a constant.

* 'Dynamical Theory of Gases,' p. 39.

Thus we have

$$f(\xi_1, \xi_2, \dots, \xi_n) = \frac{N! n^{-N} N^n}{a_1! a_2! \dots a_n!} \frac{\partial \left(\frac{a_1}{N}, \frac{a_2}{N}, \dots \right)}{\partial (\xi_1, \xi_2, \dots)} \\ = \frac{N! n^{-N} \omega^n \Delta}{a_1! a_2! \dots a_n!}.$$

Using Sterling's approximation for the values of $N!$, $a_1!$, &c. this gives

$$\log f(\xi_1, \xi_2, \dots, \xi_n) = C - \sum_{s=1}^{s=n} (a_s + \frac{1}{2}) \log \frac{na_s}{N}, \quad (18)$$

where C denotes the constant

$$\frac{n}{2} \log n - \frac{n-1}{2} \log (2\pi N) + n \log \omega + \log \Delta.$$

Write equation (17) in the form

$$a_s = a_0 + S_s, \quad \dots \quad (19)$$

where

$$a_0 = \frac{N}{n}, \quad S_s = \omega \sum_{q=1}^{q=n} \xi_q \sin \frac{qs\pi}{n},$$

then, since S_s may be supposed small compared with a_0 , we have

$$(a_s + \frac{1}{2}) \log \frac{na_s}{N} = (a_0 + \frac{1}{2} + S_s) \log \left(1 + \frac{S_s}{a_0} \right) \\ = - \frac{2a_0 - 1}{4a_0^2} S_s^2,$$

as far as terms of order $(S_s/a_0)^2$. On summing, we obtain

$$\sum_{s=1}^{s=n} (a_s + \frac{1}{2}) \log \frac{na_s}{N} = - \frac{2a_0 - 1}{4a_0^2} \sum_{s=1}^{s=n} S_s^2 \\ = - \frac{2a_0 - 1}{4a_0^2} \frac{n\omega^2}{2} (\xi_1^2 + \xi_2^2 + \dots \xi_n^2).$$

Replacing $\frac{2a_0 - 1}{4a_0^2}$ by $\frac{1}{2a_0}$ in this, equation (18) becomes

$$\log f(\xi_1, \xi_2, \dots, \xi_n) = C - \kappa (\xi_1^2 + \xi_2^2 + \dots \xi_n^2),$$

where

$$\kappa = \frac{n\omega^2}{4a_0} = \frac{\Omega}{4\nu_0},$$

and so we obtain as the law of distribution

$$f(\xi_1, \xi_2, \dots, \xi_n) d\xi_1 d\xi_2 \dots d\xi_n = A e^{-\kappa(\xi_1^2 + \xi_2^2 + \dots \xi_n^2)} d\xi_1 d\xi_2 \dots d\xi_n \\ \dots \dots \dots (20)$$

where A is a constant. It follows at once that ξ_1^2, ξ_2^2, \dots each have the same mean value, given by

$$\overline{\xi_1^2} = \overline{\xi_2^2} = \dots = \frac{1}{2\kappa} = \frac{2\nu_0}{\Omega}. \quad (21)$$

The next step is to find the potential energy of the trains of waves $\xi_1, \xi_2, \dots \xi_n$. If p is the equilibrium pressure, and σ the condensation at any point, the potential energy V is given by

$$V = \frac{1}{2} p \iiint \sigma^2 dx dy dz,$$

where V is measured from the equilibrium configuration, and all heat-energy is treated as kinetic. The condensation in the s th cell, say σ_s , is, by equation (19),

$$\sigma_s = \frac{a_s - a_0}{a_0} = \frac{S_s}{a_0} = \frac{\omega}{a_0} \sum_{q=1}^{q=n} \xi_q \sin \frac{qs\pi}{n},$$

so that

$$\begin{aligned} V &= \frac{1}{2} p \sum \sigma_s^2 \omega \\ &= \frac{p\Omega}{4\nu_0^2} (\xi_1^2 + \xi_2^2 + \dots \xi_n^2), \end{aligned}$$

or, since $p = R\nu_0 T$,

$$V = \frac{1}{4} RT \frac{\Omega}{\nu_0} (\xi_1^2 + \xi_2^2 + \dots \xi_n^2).$$

Hence, by equation (21), each term in V has average energy $\frac{1}{2} RT$.

19. We next consider the partition of kinetic energy.

Suppose that of the N molecules, a number N' have x -components of velocity which are intermediate between u and $u + du$. In the "normal state" there is no correlation between velocity and positional coordinates, so that the N' molecules will be distributed between the n cells according to the same laws as the N molecules in the analysis just completed (§ 18).

Of these N' molecules, let the numbers in the different cells be $b_1, b_2, \dots b_n$, and let $\eta_1, \eta_2, \dots \eta_n$ be given (cf. equations (17)), by

$$\frac{b_s}{\omega} = \frac{N'}{n\omega} + \sum_{q=1}^{q=n} \eta_q \sin \frac{qs\pi}{n} \quad (s=1, 2, \dots n), \quad (22)$$

then the η 's will be distributed according to the law (cf. equation (20)),

$$A' e^{-\kappa'(\eta_1^2 + \eta_2^2 + \dots \eta_n^2)} d\eta_1 d\eta_2 \dots d\eta_n, \quad (23)$$

where $\kappa' = \Omega^2/4N'$.

That part of the x -momentum in the s th cell which arises from the motion of molecules with x -velocities between u and $u+du$, is

$$b_s mu = mu \frac{N'}{n} + \sum_{q=1}^{q=n} mu \omega \eta_q \sin \frac{qs\pi}{n}$$

The total momentum in the s th cell, obtained by summation, is accordingly

$$M_s = \sum_{u=-\infty}^{u=+\infty} b_s mu = \sum_{q=1}^{q=n} \zeta_q \sin \frac{qs\pi}{n}, \quad . \quad . \quad . \quad (24)$$

where

$$\zeta_q = \sum_{u=-\infty}^{u=+\infty} mu \omega \eta_q. \quad . \quad . \quad . \quad . \quad (25)$$

Equation (24) analyses the total x -momentum into the momenta of trains of waves. The total kinetic energy of these waves is

$$\frac{1}{2} \sum_{s=1}^{s=n} n \frac{M_s^2}{Nm} = \frac{n^2}{4Nm} (\zeta_1^2 + \zeta_2^2 + \dots \zeta_n^2). \quad . \quad . \quad (26)$$

To find the law of distribution of the ζ 's we return to equation (23). The law of distribution of $mu \omega \eta_1, mu \omega \eta_2, \dots$ may from this equation be expressed in the form

$$A'' e^{-\kappa'' \{ (mu \omega \eta_1)^2 + (mu \omega \eta_2)^2 + \dots \}} d(mu \omega \eta_1) d(mu \omega \eta_2) \dots,$$

where $\kappa'' = \Omega^2 / 4N' u^2 \omega^2 m^2$. It follows at once that the law of distribution of the quantities ζ_1, ζ_2, \dots given by equation (25) is*

$$A''' e^{-\kappa''' (\zeta_1^2 + \zeta_2^2 + \dots)} d\zeta_1 d\zeta_2 \dots, \quad . \quad . \quad . \quad (27)$$

where

$$\frac{1}{\kappa'''} = \sum_{u=-\infty}^{u=+\infty} \frac{1}{\kappa''} = \frac{4\omega^2 m^2}{\Omega^2} \sum_{u=-\infty}^{u=+\infty} N' u^2 = \frac{4mNRT}{n^2}.$$

* If $m_1 u_1, m_1 v_1, m_1 w_1$ are distributed according to the law

$$C e^{-\frac{\hbar}{m_1} \{ (m_1 u_1)^2 + (m_1 v_1)^2 + (m_1 w_1)^2 \}} d(m_1 u_1) d(m_1 v_1) d(m_1 w_1),$$

and $m_2 u_2, \dots$ according to the law

$$C' e^{-\frac{\hbar}{m_2} \{ (m_2 u_2)^2 + \dots \}} d(m_2 u_2) \dots,$$

then $m_1 u_1 + m_2 u_2 + \dots, \dots$ are distributed according to the law

$$C'' e^{-\frac{\hbar}{M} \{ (m_1 u_1 + m_2 u_2 + \dots)^2 + \dots \}} d(m_1 u_1 + m_2 u_2 + \dots) \dots,$$

where $M = m_1 + m_2 + \dots$. This result is obvious from physical considerations, or may of course be obtained by algebraic transformation.

Thus ζ_1, ζ_2, \dots are distributed according to the law of trial and error, and the mean values of their squares are given by

$$\overline{\zeta_1^2} = \overline{\zeta_2^2} = \dots = \frac{1}{2\kappa'''} = \frac{2mNRT}{n^2} \dots \dots (28)$$

The energy of the trains of waves being, as has been seen in equation (26),

$$\frac{n^2}{4Nm}(\zeta_1^2 + \zeta_2^2 + \dots + \zeta_n^2),$$

it now follows that the average value of each term is $\frac{1}{2}RT$.

20. Thus we have seen that in the normal state the random motions and positions of the molecules result in certain departures from uniformity, both of density and of momentum. We have seen how these departures from uniformity can be regarded as due to regular trains of waves, and have effected their analysis into such trains of waves.

In any general motion of the medium, the kinetic and potential energies T V are given by

$$2T = \alpha_1 \dot{\phi}_1^2 + \alpha_2 \dot{\phi}_2^2 + \dots$$

$$2V = \beta_1 \phi_1^2 + \beta_2 \phi_2^2 + \dots,$$

where $\phi_1, \phi_2 \dots$ measure the amplitudes of different trains of waves. We have found that in the normal state the average value of each of the terms

$$\frac{1}{2}\alpha_1 \dot{\phi}_1^2, \frac{1}{2}\alpha_2 \dot{\phi}_2^2, \dots \frac{1}{2}\beta_1 \phi_1^2, \dots$$

is the same, namely $\frac{1}{2}RT$, the value given by the theorem of equipartition of energy. Incidentally, we have also verified that the values of $\dot{\phi}_1, \dot{\phi}_2, \dots \phi_1, \dots$ are ranged round their mean values according to the law of trial and error, as they ought to be.

To put the matter in another way, we have found that the law of distribution

$$A e^{-hm(u_1^2 + v_1^2 + w_1^2 + u_2^2 + \dots)} du_1 dv_1 dw_1 du_2 \dots dx_1 dy_1 dz_1 dx_2 \dots$$

is identical with, and may be transformed into, the law of distribution

$$A' e^{-hm(\alpha_1 \dot{\phi}_1^2 + \alpha_2 \dot{\phi}_2^2 + \dots + \beta_1 \phi_1^2 + \beta_2 \phi_2^2 + \dots)} d\dot{\phi}_1 d\dot{\phi}_2 \dots d\phi_1 \dots$$

The former law regards the energy as that of a system of
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moving molecules: the latter regards the energy as that of a vibrating medium. Each law of distribution is that of the normal state. From the second, it follows at once (*cf.* equation (15)) that the law of partition of energy between vibrations of different wave-lengths is

$$2RT\lambda^{-2} d\lambda,$$

and it is easily seen that in the three-dimensional problem, the corresponding law is

$$4\pi RT\lambda^{-4} d\lambda.$$

These formulæ are of course true only for waves of length great compared with molecular distances. They require modification as we approach wave-lengths comparable with molecular distances.

III. *A Mechanical Model of the Æther.*

21. There appears to be no reason why the energy of the electromagnetic field cannot be treated similarly to that of a gas, except for the simplification that, so far as we know, no limitations need be introduced by the coarsegrainedness of the structure of the medium.

For simplicity let us consider a rectangular enclosure, and imagine it divided into equal cubical cells, each of edge l and volume ω . Let these cells be denoted by the numbers

$$000, 001, 002, \dots 010, 011, \dots$$

the cell pqr having as its Cartesian coordinates relative to the cell 000,

$$x = pl, \quad y = ql, \quad z = rl.$$

With any cell pqr we associate three coordinates ξ_{pqr} , η_{pqr} , ζ_{pqr} , and the three corresponding velocities $\dot{\xi}_{pqr}$, $\dot{\eta}_{pqr}$, $\dot{\zeta}_{pqr}$. We examine the motion of the system of which the energy function is given by

$$\begin{aligned} E = & \frac{\omega}{8\pi} \sum_p \sum_q \sum_r \left\{ \dot{\xi}_{pqr}^2 + \dot{\eta}_{pqr}^2 + \dot{\zeta}_{pqr}^2 \right. \\ & \left. + \frac{C^2}{l^2} \left[(\zeta_{p,q+1,r} - \zeta_{p,q,r} - \eta_{p,q,r-1} + \eta_{p,q,r})^2 + (\dots)^2 + (\dots)^2 \right] \right\}. \end{aligned} \quad \dots \dots (29)$$

This function E is of course identical with the Hamiltonian function L' of § 8, so that the equations of motion of the system can be written down at once (*cf.* equations (5)).

If we introduce new quantities $\alpha, \beta, \gamma, X, Y, Z$, associated with each cell and given by

$$\alpha_{pqr} = \dot{\xi}_{pqr}$$

$$X_{pqr} = \frac{C}{l} \left\{ \zeta_{p,q+1,r} - \zeta_{p,q,r} - \eta_{p,q,r+1} + \eta_{p,q,r} \right\}, \quad \dots \quad (30)$$

&c., then the energy-function becomes

$$E = \frac{\omega}{8\pi} \sum_p \sum_q \sum_r (\alpha^2 + \beta^2 + \gamma^2 + X^2 + Y^2 + Z^2). \quad \dots \quad (31)$$

If we make l and ω vanish in the limit, the value of E becomes

$$E = \frac{1}{8\pi} \iiint (\alpha^2 + \beta^2 + \gamma^2 + X^2 + Y^2 + Z^2) dx dy dz,$$

which is identical with the electromagnetic energy in free æther. The equations of motion obtained from equation (29) reduce, when l is made to vanish, to the equations

$$\frac{1}{C} \frac{dX}{dt} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$$

$$\frac{1}{C} \frac{d\alpha}{dt} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z},$$

which are the electromagnetic equations in free æther.

Thus the mechanical system now under discussion becomes identical dynamically with the electromagnetic field when $l=0$. Any dynamical property of the present system which is independent of l must accordingly be a property of the electromagnetic field.

The energy function (29) is the three-dimensional analogue of the one-dimensional energy-function given by equations (10) and (11). It can accordingly be expressed as the energy of trains of waves, following the method of § 17. From this it follows that in the "normal state" the law of partition of energy between waves of different wave-lengths must be

$$8\pi RT \lambda^{-4} d\lambda, \quad \dots \dots \dots (32)$$

so long as λ is large in comparison with l . Hence this must be the law of partition of energy in the "normal state" in an electromagnetic field, at any rate for waves which are long in comparison with the scale of structure (if any) of the æther.

The Rate of Approach to the "Normal State."

22. The "normal state" may be thought of as a sort of composite photograph of all possible states. Any features common to all states (or to all except an infinitesimal fraction of the whole), must be features of the normal state, and conversely. From this it follows (§ 10) that if there are any properties which a system tends to acquire, independently of the particular state from which the system starts, then these must be properties of the normal state. But it has not been proved, and cannot be proved, that a system will, in every case, tend to pass into the "normal state."

To take a well-known instance, the "normal state" of a gas inside a rectangular vessel is given by Maxwell's law, but if the system is started in such a way that the molecules all move on parallel paths perpendicular to one pair of faces, the system will not pass into the normal state at all.

23. Again, the energy of a non-dissipative medium, or conservative dynamical system capable of executing isochronous vibrations, can be expressed in the form

$$\left. \begin{aligned} 2T &= \alpha_1 \dot{\phi}_1^2 + \alpha_2 \dot{\phi}_2^2 + \dots \\ 2V &= \beta_1 \phi_1^2 + \beta_2 \phi_2^2 + \dots \end{aligned} \right\}, \quad \dots \quad (33)$$

where ϕ_1, ϕ_2, \dots are the coordinates of the separate free vibrations. In the "normal state," we have

$$\frac{1}{2} \alpha_1 \dot{\phi}_1^2 + \beta_1 \phi_1^2 = \frac{1}{2} \alpha_2 \dot{\phi}_2^2 + \beta_2 \phi_2^2 = \dots = RT.$$

But in any free motion of the system, the quantities

$$\frac{1}{2} (\alpha_1 \dot{\phi}_1^2 + \beta_1 \phi_1^2), \quad \frac{1}{2} (\alpha_2 \dot{\phi}_2^2 + \beta_2 \phi_2^2), \quad \&c., \quad \dots \quad (34)$$

retain through all time exactly those values with which they started. There is no tendency towards equalization of the values of these quantities, and therefore no tendency for the system to pass into the normal state.

For instance, in the system of § 17 (a string of particles) the motion consists of the propagation of trains of waves, without change of type or interchange of energy. As regards the practical problem of finding the final partition of energy, the existence of the "normal state" is of no account at all: the whole problem turns on the initial state of the system.

As regards the system of §§ 18-20 (the tube of air), we know that as a matter of fact the system does tend to assume the "normal state," which, as we have seen, is a state of random motion of the gas-molecules. The reason why this case differs from the last is that it is not permissible to

express the energy in the form (33), while at the same time treating the system as non-dissipative. If the energy is expressed in the form (33), the quantities (34) do not remain constant, for the energy they represent is subject to dissipation by viscosity. If we wish to treat the system as non-dissipative, we must regard it as an aggregation of molecules, and the energy can no longer be expressed in the form (33).

24. We come now to the electromagnetic system discussed in § 21. There seems to be no room for doubt that formula (32) accurately represents the partition of energy in the normal state, but the question of whether the system tends to pass into the normal state remains as yet unanswered.

Light reaches us from stars of which the parallax is too small to measure. The most refined measurements have never yet led us to suppose that its velocity depends either on its intensity or frequency. This and all other available evidence points to the fact that vibrations in free æther are isochronous and free from dissipation, or at least that they may be treated as such in the present investigation*. The energy may accordingly be expressed in the form (33), and the system treated as non-dissipative. It follows at once that the quantities (34) retain their original values—if not for ever, at least for a time incomparably greater than any that could be realized experimentally. It follows that a system consisting solely of free æther could never attain the normal state.

As soon, however, as matter is introduced into the enclosure, the problem assumes a different aspect. From the necessary interaction between matter and æther, it follows that the energy of the æther can no longer be expressed accurately in the form (33). The energies of the different vibrations into the æther no longer remain constant, for the matter supplies a means of interchange of energy between them. The question which now becomes of preponderating importance is that of the rate of transfer of energy.

25. It is known† that the rate at which energy is transferred to a vibration of frequency p is proportional to a

* A short calculation will show how safely we may neglect dissipation. Light reaches us from Arcturus, distant 2×10^{14} miles, and we have no reason to suppose that it is greatly dimmed on its way. However, as we can afford to be liberal in the allowance we make for dissipation, let us suppose that the light, by the time it reaches us, is dimmed to one-billionth (10^{-12}) of its original brightness. This means that the light has to travel 70,000 miles (a much greater distance than it could possibly be made to travel in any terrestrial experiment) before its energy is diminished even by one-millionth of one per cent.

† 'Dynamical Theory of Gases,' chap. ix.

factor $e^{-2p\beta}$, where β is a positive quantity. If the origin of heat-radiation is to be found in the collisions between electrons and atoms of matter, then β will be comparable with the time of collision, say, 7×10^{-14} sec. at ordinary temperatures. For yellow light $p = 3 \times 10^{15}$, so that $e^{-2p\beta} = e^{-420}$. Thus, purely as a matter of calculation, and apart from any special hypotheses or assumptions, we find that the rate at which visible light would be emitted, as the result of heat-radiation, from matter at ordinary temperatures, must necessarily be very slow. Thus, although the final law of partition of radiant energy in a perfectly reflecting enclosure containing some matter would be that of the "normal state," given by formula (32), yet it would require centuries to reach this final state, and the smallest departure from perfection in the reflecting power of the walls would result in this final state being impossible of attainment, even if infinite time were available.

So far from it being possible to assume infinite time or perfect reflecting power under experimental conditions, we find that we must assume exactly the reverse when light of short wave-length is concerned. For light of sufficiently small wave-length, the densest matter must be as transparent as is the atmosphere for ordinary light. Even for light of the wave-length of Röntgen rays, the walls of the experimental enclosure must be regarded as practically transparent. Thus energy of short wave-length disappears entirely from the enclosure within, say, 10^{-8} seconds after its emission, while the emissions of centuries would have to accumulate before the "normal state" could be established.

CONCLUSION.

26. It now appears certain that the observed partition of energy which Planck's formula (4) attempts to represent cannot be that of the "normal state"—at least for short wave-lengths.

We are at once confronted with the question: Why is it that the partition of energy given by this formula is, to all appearance at least, that of a final state which the system tends to assume, independently of the state from which it started? For the existence of such a final state, different from the "normal state," would at first sight seem contrary to the theorem of § 10.

In answering this question it is necessary to emphasize the distinction between "free" and "forced" oscillations. The "normal state" is one in which the oscillations are free;

the system neither gains nor loses energy, and is free from all external disturbance for an unlimited time. In the observed state there is both a loss and a gain of energy (the two being of course equal in amount), and the oscillations are "forced." Any dynamical system subject to a loss of energy will, after a sufficient time, reach a state from which all traces of the initial conditions have disappeared; its state will depend solely on the forces which act upon it from outside. There exists a final state of this kind, to which every system necessarily tends when acted on by definite external agencies, and this final state may be something entirely different from the "normal state": it is such a final state that Planck's formula represents. We may speak of this state as the "steady state": it is the state reached as soon as the influence of the initial conditions has been dissipated away.

27. The "normal state" can depend only on the æther, but the "steady state" will depend in addition on the material agencies which force the vibrations in the æther. Consequently, Planck's formula (4) contains more constants than the formula (3) of the "normal state."

There is no *a priori* reason why there should not be different "steady state" formulæ corresponding to different kinds of matter: the vibrations forced in the æther might reasonably be expected to depend on the nature of the forcing agency. In point of fact it is found that the specification of the "steady state" involves only the temperature, and not the nature or structure, of the matter by which the ætherial vibrations are forced. This is easily seen to be a necessary consequence of Kirchhoff's law*. Once the truth of this law is accepted it is readily seen that there can be only one "steady state" corresponding to a given temperature, so that this steady state must be the same for all kinds of matter. Thus the constants which enter into the steady-state formula must be quantities which are common to all kinds of matter.

In Planck's formula there are two constants, c and h . Of these h is either identical with the gas-constant R or is

* If the views of the present paper are sound, the structure commonly called "The Thermodynamics of Radiation" requires modification, and the usual theoretical proof of Kirchhoff's law with it. We then have to fall back on experimental investigation as to the truth of the law. The law seems undoubtedly to be true, whether the usual theoretical proof of it is valid or not.

Lorentz, on the basis of the electrical structure of matter, has verified the truth of Kirchhoff's law for radiation of long wave-length (*Konink. Akad. van Wetenschappen*, Amsterdam, 1903, p. 678).

some numerical multiple of it (*cf.* § 5 and § 29, *infra*). The constant c is, on inspection of the formula, found to be of the same physical dimensions as E^2/K , where E is an electric charge and K an inductive capacity. The only quantity of these physical dimensions which can be constructed out of quantities common to all matter is e^2/K_0 , where e is the charge of the electron, and K_0 is the inductive capacity of free æther. Thus we conclude that the c of Planck's formula must be some numerical multiplier of e^2/K_0 .*

28. Although the "steady state" is the same for all kinds of matter which actually exist, yet it depends on the quantity e , and so would be different for substances having different values of e , if such could exist†.

Imagine for the moment matter for which e is very small—matter for which we can say that $e=0$. For such matter the partition of energy in the æther in the steady state is obtained by putting $c=0$ in Planck's formula. It is therefore given by

$$f(\lambda, T) = 8\pi k T \lambda^{-4} \dots \dots \dots (35)$$

This same formula of course gives also the partition of energy for actual matter for large values of λ .

Planck's value of $f(\lambda, T)$ differs from this in that it falls off in value as we come to small values of λ . Let us examine the physical significance of this.

29. Planck's formula differs from formula (35) as soon as $\frac{e^2}{K_0 R T \lambda}$ becomes appreciable—i. e., as soon as λ is so small as to be comparable with $\frac{e^2}{K_0 R T}$. A physical interpretation of this last expression is readily found.

The motion of two electrons, or the motion of an electron and an atom (or molecule) can be regarded as the superposition of two motions. There is first the motion of the centre of gravity of the two bodies (which is unaffected by a collision or encounter between them), and second, there is the motion of the two bodies relative to their centre of

* It can be shown, from a consideration of physical dimensions (see *Proc. Roy. Soc.* lxxvi. p. 545), that, subject to a certain assumption, the "steady state" formula must be of the type

$$f(\lambda, T) = \frac{e^2}{K_0 \lambda^3} \phi \left(\frac{e^2}{K_0 R T \lambda} \right);$$

Planck's formula determines the form of the function ϕ .

† See *Phil. Mag.* [6] xii. p. 57.

gravity. The energy of the components of this last motion in any given direction is equal (on the average) to RT . Thus at the instant of closest approach in a collision or encounter, an amount of kinetic energy equal to RT must have been transformed into potential energy. If a is the distance of closest approach, the potential energy is $\frac{e^2}{K_0 a}$, so that

$$a = \frac{e^2}{K_0 RT} \cdot \cdot \cdot \cdot \cdot \cdot (36)$$

Defining a by this equation, we see that Planck's formula differs from formula (35) as soon as λ becomes comparable with a , a being the average distance of closest approach in a collision.

When we consider matter in which, in the limit, we take $e=0$, we have $a=0$, and formula (35) gives the partition of energy in the issuing radiation. This is as it should be; collisions in which the electrons approach indefinitely close to one another will in this matter be of frequent occurrence, and radiation, even of very small wave-lengths, is produced at a rapid rate.

30. We now come to the final problem. Is formula (35), which gives the "steady state" for large values of λ , identical with formula (3), which gives the "normal state"? Or, in other words, are k and R identical in § 5?

The work of Lorentz, already referred to (§§ 8, 27), provides incidentally an answer to this question. Lorentz finds, as the result of actual calculation, that what we have called the "steady state" formula, has for large values of λ the limiting form

$$8\pi RT\lambda^{-4}, \cdot \cdot \cdot \cdot \cdot \cdot (37)$$

and so establishes the identity of k and R . Lorentz's analysis, however, proceeds on certain definite assumptions, such as that the free electrons in a metal undergo instantaneous encounters, and that between these encounters they describe undisturbed free paths. It is therefore still necessary to examine whether, in order to compensate for the inaccuracy of these simplifying assumptions, it may not be necessary to correct Lorentz's expression by multiplication by a certain numerical factor*. If this is found to be necessary, then the identity of k and R will not have been proved.

* This is the only kind of correction which is possible without violating the physical dimensions of the formula in question.

31. Apart from this, the identity of k and R can, I think, be established by an argument of a very general nature, which does not depend on these special assumptions.

The continuous spectrum emitted by a solid must be granted to proceed from the motions of free electrons. A calculation, based on our knowledge of the total kinetic energy of these electrons and of the rate at which they radiate energy, will show that their rate of radiation of energy is very slow, when the time-scale is taken to be the average interval between two collisions. It follows that the kinetic energies of these electrons must be distributed in accordance with Maxwell's law about the mean value $\frac{3}{2}RT$. This theoretical result can be obtained independently of any assumptions as to the nature of free-paths, collisions, or forces by which the electrons are acted upon. Its truth has recently been confirmed in a very striking manner by the experiments of Richardson and Brown*.

It follows that the motion of the electrons can be analysed into the motion of trains of waves by the analysis of § 19. Each of the trains of waves, into which the motion of the electrons can be analysed, will have kinetic energy appropriate to the temperature T . There will obviously be extremely rapid transfer of energy between these waves of electrons and the æther in which they are imbedded. Thus the æther in the interior of matter or in a cavity made in the matter, will immediately take up its equilibrium partition of energy appropriate to temperature T , namely, that given by the formula

$$8\pi RT\lambda^{-4}d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

There is a limit to the applicability of this argument. The analysis of the electron-motion into regular trains of waves holds only for wave-lengths great compared with the distances apart of the nearest electrons. Thus formula (38) will hold only for values of λ which are great compared with a , where a is given by equation (36). This is exactly what is given by Planck's formula, if the k of his formula is identified with R .

Princeton, Oct. 8, 1908.

* Phil. Mag. xvi. p. 353.

XIX. *The Inductance of Two Parallel Wires.* By J. W. NICHOLSON, D.Sc., B.A., *Isaac Newton Student in the University of Cambridge* *.

WHEN direct and return currents flow in two wires of great length, and the alternation is not rapid, the effective self-induction L per unit length of the system may be calculated readily by the method of geometric mean distances† or by simple integration‡. If the wires have radii (a , b) and permeabilities (μ_1 , μ_2), and if c be the distance between their axes,

$$L = 2 \log \frac{c^2}{ab} + \frac{1}{2}(\mu_1 + \mu_2). \quad . \quad . \quad . \quad (1)$$

But this formula ceases to be of any practical utility in many cases when the frequency of alternation is several thousands per second. Such frequencies are of constant use in practical work. For example, in the measurement of small inductances by Mr. Albert Campbell's method§, it is necessary to employ long leads in order to keep them at some considerable distance from bridge and other circuits. The self-induction of these leads must be small, and a calculation of its value is very desirable. It was therefore suggested to me that I should attack this problem. The general case presents apparently insuperable mathematical difficulty, but the solutions given below appear to include all cases of practical importance. A short statement of these results was given by the author in 'Nature,' Jan. 30th, 1908, but the limitations were not emphasized.

Let the axis of z be chosen parallel to those of the two wires. Any point in a section defined by a constant value of z may be conveniently specified by means of polar coordinates in two ways. Let these coordinates be (r, θ) and (ρ, ϕ) , where (r, ρ) are the distances of the point from the two axes respectively, and (θ, ϕ) its orientations measured from a line perpendicular to both axes.

In the figure A and B are the projections of the axes of the two wires, and $AB = c$. A and B will be referred to as the

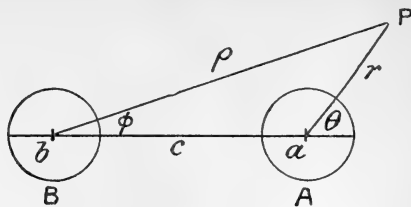
* Communicated by the Physical Society: read June 12, 1908.

† Maxwell, 'Electricity and Magnetism.'

‡ Russell, *Alt. Currents*, i. p. 56.

§ *Phil. Mag.* Jan. 1908.

first and second wires respectively. If the first wire were not disturbed by the presence of the second, the electric and



magnetic vectors would be derivable from a vector potential H parallel to the axis, independent of θ , and proportional to a time factor e^{pt} , if $2\pi/p$ be the frequency of alternation.

The electric force in the general case is axial, and will be denoted by R . There is no axial magnetic component. Let the magnetic components radial and transversal with respect to an origin in the axis of the first wire be $(\alpha'\beta')$.

Then in a medium of permeability μ ,

$$\alpha' = \frac{1}{\mu r} \frac{\partial H}{\partial \theta}, \quad \beta' = -\frac{1}{\mu} \frac{\partial H}{\partial r} \dots \dots \dots (2)$$

All vectors are independent of z . The current is parallel to the axis, and of magnitude

$$\begin{aligned} w &= \frac{1}{4\pi r} \left\{ \frac{\partial}{\partial r} (r\beta') - \frac{\partial \alpha'}{\partial \theta} \right\} \\ &= -\frac{1}{4\pi\mu} \left\{ \frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} \right\}. \end{aligned}$$

If the medium have specific inductive capacity K , and specific resistance σ ,

$$\begin{aligned} w &= \frac{R}{\sigma} + \frac{K}{4\pi} \frac{\partial R}{\partial t} \\ &= \left(\frac{1}{\sigma} + \frac{iKp}{4\pi} \right) R \end{aligned}$$

for the periodic vibration.

Moreover, so far as inductive action is concerned,

$$R = -\frac{\partial H}{\partial t}, \quad \dots \dots \dots (3)$$

and thus, H being entirely due to inductive agency,

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} - 4\pi\mu p \left\{ \frac{1}{\sigma} + \frac{iKp}{4\pi} \right\} H = 0. \quad (4)$$

In the wire, the current may be regarded as due to ohmic conductivity alone, and writing

$$k^2 = -4\pi \mu p / \sigma, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then
$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} + k^2 H = 0, \quad . \quad . \quad . \quad (6)$$

whose solution finite at the axis of the wire is

$$H = \sum_0^\infty A_n J_n(kr) \cos n\theta, \quad . \quad . \quad . \quad (7)$$

where

$$J_n(kr) = \frac{k^n r^n}{2^n n!} \left\{ 1 - \frac{k^2 r^2}{2^2 1! n + 1} + \frac{k^4 r^4}{2^4 2! n + 1 n + 2} \dots \right\}, \quad (8)$$

and is the ordinary Bessel function of the first kind. In the surrounding medium, $\frac{1}{\sigma}$ may be neglected, and writing $1/K\mu = C^2$, where C is the velocity of propagation in the medium,

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} + h^2 H = 0, \quad . \quad . \quad . \quad (9)$$

where $h = p/C$.

The solution proper to the space outside the wire, satisfying the condition of being finite at infinity, is

$$H = \sum_0^\infty B_n K_n(hr) \cos n\theta, \quad . \quad . \quad . \quad (10)$$

where

$$K_n(hr) = \int_0^\infty e^{-hr \cosh \phi} \cosh n\phi d\phi. \quad . \quad . \quad (11)$$

This function is connected with the more usual Bessel function of the second kind, introduced by Hankel, by the formula*

$$K_n(hr) = -\frac{1}{2} i^{-n} \{ Y_n(hr) + i\pi J_n(hr) \}, \quad . \quad (12)$$

h being real, and, if $\gamma = .577 \dots$

$$Y_n(x) \equiv 2J_n(x) \left\{ \log \frac{x}{2} + \gamma \right\} - \left(\frac{x}{2} \right)^n \left\{ n-1! + \frac{n-2!}{1!} \left(\frac{x}{2} \right)^2 \dots \right\} \\ - \left(\frac{x}{2} \right)^n \left\{ \frac{1}{n!} S_n - \frac{1}{1! n+1!} \left(\frac{1}{1} + S_n \right) \left(\frac{x}{2} \right)^2 + \frac{1}{2! n+2!} \left(\frac{1}{1} + \frac{1}{2} + S_n \right) \left(\frac{x}{2} \right)^4 + \dots \right\}, \quad (13)$$

where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}. \quad . \quad . \quad . \quad (14)$$

* *Vide e. g.* Somme, Math. Ann. 16, 1880.

The Single Wire.

We consider first the distribution of vectors in and about a single wire conveying an alternating current, whose distribution is undisturbed by the presence of a second wire. All vectors are independent of orientation, and the impressed electric force may be supposed equal at all points of the wire, the resistivity (σ) of the latter being constant. Let Ee^{ipt} be the line integral of this force per unit length. Then the total force, inductive and non-inductive, in the wires is

$$\begin{aligned} R &= E - \frac{\partial H}{\partial t} \\ &= E - ip AJ_0(kr), \end{aligned}$$

since H does not depend on θ . The time factor is ignored throughout.

The corresponding force outside is

$$R = -\frac{\partial H}{\partial t} = -ip BK_0(\iota hr),$$

and by continuity, if a be the radius of the wire,

$$E - ip AJ_0(ka) = -ip BK_0(\iota ha).$$

The tangential magnetic force (β) at the surface is also continuous, and therefore, if μ be the permeability of the wire,

$$\frac{1}{\mu} Ak J_0'(ka) = \iota h Bk_0'(\iota ha),$$

we deduce that

$$A = \frac{-E}{ip} K_0'(\iota ha) / (K_0'(\iota ha) J_0(ka) - \frac{k}{\mu h \iota} J_0'(ka) K_0(\iota ha)). \quad (15)$$

$$B = \frac{kE}{ph\mu} J_0'(ka) / (K_0'(\iota ha) J_0(ka) - \frac{k}{\mu h \iota} J_0'(ka) K_0(\iota ha)), \quad (16)$$

and the vector potential in the wire is

$$H = A J_0(kr), \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and outside is

$$H = B K_0(\iota hr). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The current crossing any section is the line integral of magnetic force round the section, divided by 4π , and becomes

$$\begin{aligned} \varpi &\doteq \frac{1}{4\pi} 2\pi a (\beta)_{r=a} \\ &= Aka J_0'(ka) / 2\mu, \quad . \quad . \quad . \quad . \quad (19) \end{aligned}$$

and with the above value of A,

$$\frac{E}{\pi i p} = -\frac{2\mu}{k} \left\{ \frac{J_0(ka)}{J_0'(ka)} - \frac{k}{\mu h i} \frac{K_0(\iota ha)}{K_0'(\iota ha)} \right\}, \quad (20)$$

and the effective self-induction is the real part of the expression on the right. The effect due to variability of current in the section, deduced from the first term, is in accord with the result obtained by Lord Kelvin* and otherwise by Lord Rayleigh†. Maxwell and Heaviside have also given other forms of this solution.

Transformation of Bessel Functions to a new origin.

We have by the integral formula for the function finite at infinity,

$$K_n \{ \iota(\overline{c+r}) \} = \int_0^\infty e^{-\iota \overline{c+r} \cosh \phi} \cosh n\phi \, d\phi.$$

But

$$e^{\frac{1}{2}c(z-\frac{1}{z})} = J_0(c) + \sum_1^\infty J_n(c) \left\{ z^n + \frac{(-)^n}{z^n} \right\}, \quad (21)$$

by an ordinary definition of Bessel functions of the first kind whose order is an integer.

Writing $z = -\iota e^\phi$,

$$e^{-\iota c \cosh \phi} = J_0(c) + 2 \sum_1^\infty (-\iota)^n J_n(c) \cosh n\phi. \quad (22)$$

if the series be convergent.

Thus

$$\begin{aligned} & K_n(\iota \cdot \overline{c+r}) \\ &= \int_0^\infty d\phi \cdot e^{-\iota r \cosh \phi} \cosh n\phi \cdot \{ J_0(c) + 2 \sum_1^\infty (-\iota)^s J_s(c) \cosh s\phi \} \\ &= J_0(c) \int_0^\infty e^{-\iota r \cosh \phi} \cosh n\phi \, d\phi + \sum_1^\infty (-\iota)^s J_s(c) \int_0^\infty e^{-\iota r \cosh \phi} \left\{ \cosh \overline{n+s\phi} \right. \\ & \quad \left. + \cosh \overline{n-s\phi} \right\} d\phi \\ &= J_0(c) K_n(\iota r) + \sum_{s=1}^\infty (-\iota)^s J_s(c) \{ K_{n+s}(\iota r) + K_{n-s}(\iota r) \}, \quad (23) \end{aligned}$$

where $K_{n-s}(\iota r)$ is identical with $K_{s-n}(\iota r)$.

* Presidential Address to Inst. of Elect. Engineers, 1889. Math. and Phys. Papers, vol. iii. p. 492.

† Vide Phil. Mag. 1886, where a complete practical solution may be found.

This may also be written

$$K_n(\iota \cdot c + r) = J_0(r) K_n(\iota c) + \sum_{s=1}^{\infty} (-\iota)^s J_s(r) \{K_{n+s}(\iota c) + K_{n-s}(\iota c)\} \quad (24)$$

Now the series

$$K_n(\iota hc) J_0(hr) + \sum_{s=1}^{\infty} (-\iota)^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(hr) \cos s\theta$$

is a solution of the equation (9) which, when $\theta=0$, takes the form of the right-hand side of (24). Such a solution is also $\cos n\phi \cdot K_n(\iota h\rho)$, where ρ and ϕ are connected with r and θ as in the figure.

$$\text{Thus} \quad \rho^2 = r^2 + c^2 + 2cr \cos \theta. \quad (25)$$

$$\tan \phi = \frac{r \sin \theta}{c + r \cos \theta}, \quad (26)$$

and when $\theta=0$, $\cos n\phi K_n(\iota h\rho) = K_n\{\iota h(c+r)\}$,

Thus the two solutions only differ by a function vanishing with θ , and therefore capable of expression in a series of sines. This function is at once seen to be zero, by an application of Fourier's theorem. The function $\cos n\phi K_n(\iota h\rho)$ and the series last written do not change sign with θ . Thus if (ρ, ϕ) are connected with (r, θ) as before,

$$\begin{aligned} & \cos n\phi \cdot K_n(\iota h\rho) \\ &= K_n(\iota hc) J_0(hr) + \sum_{s=1}^{\infty} (-\iota)^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(hr) \cos s\theta \quad (27) \end{aligned}$$

and similarly

$$\begin{aligned} & \cos n\theta \cdot K_n(\iota hr) = K_n(\iota hc) J_0(h\rho) \\ & + \sum_{s=1}^{\infty} \iota^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(h\rho) \cos s\phi. \quad (28) \end{aligned}$$

Effect of the Return Wire.

It has been shown that if an impressed force E (ignoring a time-factor e^{pt}) acts per unit length in a single wire of radius a , the vector potential produced outside is

$$H = BK_0(\iota hr),$$

where $h = p/C$, C being the velocity of propagation of electromagnetic disturbances in the medium, and

$$B = \frac{kE}{ph\mu} J_0'(ka) / (K_0'(\iota ha) J_0(k\alpha) - \frac{k}{\mu hi} J_0'(k\alpha) K_0(\iota ha)),$$

where the origin is in the axis of the wire, and

$$k^2 = -4\pi\mu\nu\rho'/\sigma.$$

It will be convenient to write

$$\alpha = K_0'(\iota ha) J_0(ka) - \frac{k}{\mu h \iota} J_0'(ka) K_0(\iota ha). \quad (29)$$

where α depends entirely on the first wire.

The corresponding function of the second wire will be called β , and its radius b .

Thus
$$B = \frac{kE}{ph\mu\alpha} J_0'(ka). \quad (30)$$

Writing $n=0$ in (28), it appears that

$$K_0(\iota hr) = K_0(\iota hc) J_0(h\rho) + 2 \sum_{s=1}^{\infty} \iota^s K_s(\iota hc) J_s(h\rho) \cos s\phi, \quad (31)$$

and if hb is small, as in all applications proposed in this paper, this series is rapidly convergent in the neighbourhood of the second wire.

We may write

$$\theta_0 = K_0(\iota hc), \quad \theta_s = 2\iota^s K_s(\iota hc) \quad s \neq 0, \quad (32)$$

and the vector potential H , which may be regarded as incident on the second wire, becomes

$$H = B\theta_0 J_0(h\rho) + \sum_1^{\infty} B \cdot \theta_s \cdot J_s(h\rho) \cos s\phi. \quad (33)$$

We neglect, in the present section, the terms dependent on orientation. This will afterwards appear to be equivalent to a neglect of terms of the order $(a/c)^2$, so that the wires are not close together. Thus

$$H = B\theta_0 J_0(h\rho), \quad (34)$$

is incident on the second wire. A secondary disturbance will be thrown off, of the form

$$H = CK_0(\iota h\rho), \quad (35)$$

zero at infinity, and inside the wire, a disturbance

$$H = DJ_0(k'\rho), \quad (36)$$

finite at the origin, will be introduced, where

$$k'^2 = -4\pi\nu\rho/\sigma', \quad (37)$$

ν being the permeability of the second wire.

Since the electric force R is continuous at the surface, H is also continuous (for this disturbance due to induction), and therefore

$$DJ_0(k'b) = CK_0(\iota hb) + B\theta_0 J_0(hb),$$

and because the magnetic force is also continuous,

$$\frac{1}{\nu} k' D J_0'(k'b) = \iota h CK_0'(\iota hb) + B h \theta_0 J_0'(hb).$$

Adopting the notation

$$\left. \begin{aligned} \delta_1 &= J_0(ka) J_0'(ha) - \frac{k}{\mu h} J_0(ha) J_0'(ka) \\ \delta_2 &= J_0(k'b) J_0'(hb) - \frac{k'}{\nu h} J_0(hb) J_0'(kb) \end{aligned} \right\}, \quad (38)$$

it appears that

$$\left. \begin{aligned} C &= \iota B \theta_0 \delta_2 / \beta \\ D &= \iota B \theta_0 / h b \beta \end{aligned} \right\}, \quad (39)$$

where the numerator of D has been simplified by the reduction

$$\begin{aligned} & \iota K_0'(\iota hb) J_0(hb) - K_0(\iota hb) J_0'(hb) \\ &= \frac{1}{2} (J_0'(hb) Y_0(hb) - J_0(hb) Y_0'(hb)) \\ &= -\frac{1}{hb}, \end{aligned}$$

by the usual property of the Hankel solution Y_0 .

The second wire, therefore, throws out a disturbance of vector potential

$$H = \frac{\iota B \theta_0 \delta_2}{\beta} K_0(\iota h \rho). \quad (40)$$

Potential independent of Orientation.

In the calculation just made, the terms of a transformed series of Bessel functions involving orientation were neglected. We now trace the series of compensating disturbances which may be regarded as thrown off by the wires, and form their sum under the same condition. This leads to a very accurate result if the wires are not very close together.

A vector potential $H = BK_0(\iota hr)$ leaving the first wire has, for its main term near the second, a magnitude $B\theta_0 J_0(h\rho)$, and causes (1) an inside potential

$$H = -\frac{B\theta_0}{\iota hb\beta} J_0(k'\rho), \quad (41)$$

and (2) a reflexion of vector potential from that wire, of

$$H = \frac{iB\theta_0\delta_2}{\beta} K_0(ih\rho). \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This is again incident on the first wire, having for a leading term there,

$$H = \frac{iB\theta_0\delta_2}{\beta} \cdot \theta_0 J_0(hr), \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and causing an inside potential

$$H = \frac{iB\theta_0^2\delta_2}{\beta} \cdot \frac{i}{ha\alpha} \cdot J_0(hr), \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and a reflexion

$$\left. \begin{aligned} H &= \frac{iB\theta_0^2\delta_2}{\beta} \cdot \frac{i\delta_1}{\alpha} \cdot K_0(ihr) \\ &= -B\theta_0^2 \cdot \frac{\delta_1\delta_2}{\alpha\beta} \cdot K_0(ihr) \end{aligned} \right\} \quad . \quad . \quad . \quad (45)$$

This process may be continued. The successive waves of reflexion form a geometric series of common ratio

$$\theta_0^2\delta_1\delta_2/\alpha\beta.$$

The subsequent reduction indicates that in general, with a frequency of the magnitude dealt with in this paper, this series is convergent. The effect inside the second wire, of a potential

$$H = BK_0(ihr),$$

originally supposed to leave the first, is a potential

$$\begin{aligned} H &= \frac{iB\theta_0}{hb\beta} J_0(k'\rho) \left\{ 1 - \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta} + \theta_0^4 \frac{\delta_1^2\delta_2^2}{\alpha^2\beta^2} - \dots \right\} \\ &= \frac{i\theta_0}{hb\beta} \frac{B}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} J_0(k'\rho). \quad . \quad . \quad . \quad . \quad . \quad (46) \end{aligned}$$

Moreover, the effect on the potential in the first wire, due to the waves caused by the potential originally leaving that wire, is

$$\begin{aligned} H &= -\frac{B\theta_0^2\delta_2}{ha\alpha\beta} \left\{ 1 - \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta} + \dots \right\} J_0(hr) \\ &= -\frac{B\theta_0^2\delta_2}{ha\alpha\beta} \frac{J_0(hr)}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} \quad . \quad . \quad . \quad . \quad . \quad (47) \end{aligned}$$

A similar potential $H = B'K_0(\iota h\rho)$ leaving the second wire produces secondary potentials

$$H = \frac{\iota\theta_0}{ha\alpha} \frac{B'}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} J_0(kr) \quad . \quad . \quad . \quad (48)$$

$$H = -\frac{B'\theta_0^2\delta_1}{hb\alpha\beta} \frac{J_0(k'r)}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}}, \quad . \quad . \quad . \quad (49)$$

in the first and second respectively, and as

$$B = \frac{k}{ph\mu} \frac{EJ_0'(ka)}{\alpha} \text{ by (16),}$$

so

$$B' = \frac{k'}{ph\nu} \frac{EJ_0'(k'b)}{\beta}, \quad . \quad . \quad . \quad (50)$$

so that the secondaries may be exactly analogous to the first group.

The value of H in the first wire is therefore, to this order,

$$H = \left\{ A - \frac{B\theta_0^2\delta_2}{ha\alpha\beta} \frac{1}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} + \frac{\iota\theta_0 B'}{ha\alpha} \frac{1}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} \right\} J_0(kr) \quad (51)$$

with a corresponding total current

$$\varpi_1 = \frac{ka}{2\mu} J_0'(ka) \cdot \frac{H}{J_0(kr)}. \quad . \quad . \quad . \quad (52)$$

As previously indicated, ha and hb are to be supposed very small, so that $haJ_0'(ha)$ is of order $(ha)^2$.

Neglecting this order of magnitude, a justification for the process appearing later,

$$\left. \begin{aligned} \delta_1 &= -\frac{k}{\mu h} J_0'(k\bar{a}) \\ \delta_2 &= -\frac{k'}{\nu h} J_0'(k'b) \end{aligned} \right\}, \quad . \quad . \quad . \quad (53)$$

and therefore

$$B = \frac{-E}{p\alpha} \delta_1, \quad B' = \frac{-E}{p\beta} \delta_2; \quad . \quad . \quad . \quad (54)$$

and in the first wire, on reduction,

$$\Pi = \frac{-EJ_0(kr)}{pha\alpha} \left(1 + \frac{\iota\theta_0\delta_2}{\beta} \right) \left/ \left(1 + \frac{\theta_0^2\delta_1\delta_2}{\alpha\beta} \right) \right., \quad . \quad (55)$$

and in the second,

$$H = \frac{-E J_0(k' \rho)}{p h b \beta} \left(1 + \frac{\iota \theta_0 \delta_1}{\alpha} \right) / \left(1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right). \quad (56)$$

The total currents are

$$\varpi_1 = \frac{E \delta_1}{2 p \alpha} \left(1 + \frac{\iota \theta_0 \delta_2}{\beta} \right) / \left(1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right), \quad (57)$$

$$\varpi_2 = \frac{E \delta_2}{2 p \beta} \left(1 + \frac{\iota \theta_0 \delta_1}{\alpha} \right) / \left(1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right), \quad (58)$$

by (52).

If L is the self-induction of the system per unit length, $\iota L p$ is the imaginary part of $\frac{E}{w_1} + \frac{E}{w_2}$, or of

$$2p \left(1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right) \left\{ \frac{1}{\frac{\delta_1}{\alpha} \left(1 + \frac{\iota \theta_0 \delta_2}{\beta} \right)} + \frac{1}{\frac{\delta_2}{\beta} \left(1 + \frac{\iota \theta_0 \delta_1}{\alpha} \right)} \right\}. \quad (59)$$

In particular, if the wires be equal and of the same material, as will usually occur in practice, ιL is the imaginary part of

$$\frac{4\alpha}{\delta_1} \left(1 + \frac{\theta_0^2 \delta_1^2}{\alpha^2} \right) / \left(1 + \frac{\iota \theta_0 \delta_1}{\alpha} \right), \text{ or of } \frac{4\alpha}{\delta_1} \left(1 - \frac{\iota \theta_0 \delta_1}{\alpha} \right). \quad (60)$$

Case of a Small Frequency.

When p is small, $ha = pa/C$ is exceedingly small for a wire not of excessive thickness, and may be safely ignored as in the above calculation, even in its first power, and thus

$$\begin{aligned} K_0(\iota ha) &= -\frac{1}{2} \{ Y_0(ha) + \iota \pi J_0(ha) \} \\ &= - \left(\gamma + \frac{\iota \pi}{2} + \log \frac{ha}{2} \right) \\ &= -\log ha \quad \text{practically.} \end{aligned} \quad (61)$$

$$K_0'(\iota ha) = -\frac{1}{\iota ha}, \quad (62)$$

so that on reduction,

$$\alpha = \frac{k}{h\mu} J_0'(ka) \left\{ \log ha - \frac{\mu J_0(ka)}{ka J_0'(ka)} \right\}, \quad (63)$$

and

$$-\frac{\iota \delta_1}{\alpha} = 1 / \left\{ \log(ha) - \frac{\mu J_0}{ka J_0'} \right\}. \quad (64)$$

$$\theta_0 = K_0(\iota hc) = -\log hc.$$

If hc is not too great

$$1 + \frac{\iota\theta_0\delta_1}{\alpha} = \frac{\log \overline{h^2ac} - \mu J_0 / ka J_0'}{\log ha - \mu J_0 / ka J_0'},$$

and has the value 2 for a small value of h .

$$1 + \frac{\theta_0^2\delta_1\delta_2}{\alpha\beta} = - \left\{ \log \frac{c^2}{ab} + \epsilon_1 + \epsilon_2 \right\} / \log h, \quad (65)$$

to the same order, where

$$\epsilon_1 = \mu J_0(k\alpha) / ka J_0'(ka), \quad (66)$$

and ϵ_2 is the same function of (k', b) ,

$$\frac{\delta_1}{\alpha} = \frac{\iota}{\log h} = \frac{\delta_2}{\beta},$$

and ϵ_1 reduces practically, under the present circumstances, to

$$-2\mu / k^2 a^2 \left\{ 1 - \frac{k^2 a^2}{4} \right\},$$

and quoting the value of k , the real part is $\frac{1}{2}\mu$, and the self-induction becomes

$$L = 2 \log \frac{c}{ab} + \frac{1}{2}(\mu + \nu), \quad (67)$$

in accord with the results deduced from a distribution assumed uniform.

Case of two equal wires.

When the wires are equal, ιL is the imaginary part of

$$\frac{4\alpha}{\delta_1} \left(1 - \iota\theta_0 \frac{\delta_1}{\alpha} \right).$$

When $h^2 c^2$ and $h^2 a^2$ are both neglected, this leads to

$$L = 4 \log \frac{c}{a} + \frac{4\mu}{x} \frac{\text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2}, \quad (68)$$

where, if $\frac{n}{2\pi}$ is the frequency of alternation per second,

$$x = 2a \sqrt{\frac{\pi \mu n}{\sigma}}. \quad (69)$$

The functions $\text{ber } x$, $\text{bei } x$, are those introduced by Lord Kelvin*, and subsequently tabulated.

* *Loc. cit. ante.*

To this order of approximation, the result might have been written down at once. For c/a has already been assumed so great that the distribution may be regarded as made up of cylindrical current sheets. The number of linkages of the lines of force outside the cylinders* is $4 \log c/a$, and the linkages due to the lines of force wholly in the metal give rise to the second part of (68) †.

To the same order the approximation for two wires becomes

$$L = 2 \log \frac{c^2}{ab} + \frac{2\mu}{x} \frac{\text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} + \frac{2\nu}{y} \frac{\text{ber } y \text{ber}' y + \text{bei } y \text{bei}' y}{(\text{ber}' y)^2 + (\text{bei}' y)^2} \quad (70)$$

where

$$x = 2a \sqrt{\frac{\pi \mu n}{\sigma}}, \quad y = 2b \sqrt{\frac{\pi \nu n}{\sigma}}, \quad (71)$$

and may also be written down at once.

When the vibration becomes more rapid, even though $h^2 a^2$ may be neglected in comparison with unity, the latter cannot itself be neglected in comparison with $\log(ha)$. Thus (61) requires modification, and it appears that

$$\frac{-\iota \delta_1}{\alpha} = 1 \left/ \left\{ \log \frac{(ha)}{2} + \gamma + \frac{\iota \pi}{2} \right\} - \mu J_0 / ka J_0' \right\},$$

and ιL is the imaginary part of

$$\frac{4\alpha}{\delta_1} \left(1 - \iota \theta_0 \frac{\delta_1}{\alpha} \right)$$

or of

$$-4\iota \left\{ \log \frac{ha}{2} + \gamma + \frac{\iota \pi}{2} - \frac{\mu J_0}{ka J_0'} \right\} - 4\iota \cdot \left\{ -\log \frac{h\iota}{2} - \gamma - \frac{\iota \pi}{2} \right\}$$

leading to the same value as before.

But when the wires are different, the new approximation thus calculated differs from the old, and is very cumbrous. It is therefore to be desired that, in practical work involving a knowledge of the inductance of parallel leads, the leads should be of the same size and material. The formula for inductance is then of the same accuracy over a wider range of frequency.

Cases may arise in practice in which c is so great in comparison with a that the neglect of $h^2 a^2$ may not involve that

* Russell, *Alternating Currents*, i. p. 56 *et seq.*

† *Vide* Gray and Matthews, *Treatise on Bessel Functions*.

of h^2c^2 . Tables of the Bessel functions of the second kind must then be employed. L is the real part of

$$-4 \left\{ \log \frac{ha}{2} + \gamma + \frac{i\pi}{2} - \frac{\mu J_0}{ka J'_0} \right\} - 4K_0(ihc),$$

and becomes

$$L = 2Y_0(hc) - 4 \left\{ \log \frac{ha}{2} + \gamma \right\} + \frac{4\mu}{x} \frac{\text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} \quad (72)$$

where x has its previous value, and Y_0 represents the function introduced by Hankel. Tables of this function are given in Gray and Matthews' Treatise.

If $\frac{n}{2\pi}$ is the frequency,

$$h = \frac{n}{3} \cdot 10^{-10}. \quad . \quad . \quad . \quad . \quad . \quad (73)$$

A general idea of the range of frequency with which these formulæ may be used, can be obtained at once. The neglected terms are of order h^2a^2 in comparison with those retained. The radius of a wire is rarely greater than two millimetres, and thus with a frequency of a hundred million per second, h^2a^2 becomes $1.6 \cdot 10^{-5}$, which may be neglected at once in comparison with unity. Most frequencies dealt with in practice are therefore within the scope of these formulæ, when the wires are not of large cross-section. But another source of error is present, owing to the neglect of terms depending on orientation. Upon examination, the error is found to be of relative order a^2/c^2 . Even if $c = 5a$, the formulæ are sufficiently accurate for most applications. The error is about ten per cent. if $c = 3a$, which is the limiting closeness for the majority of practical purposes. More accurate results suited to this case are obtained in the next section.

Higher Approximation.

The terms depending on orientation in the vector potential do not contribute to the total current across any section of a wire, which is proportional to a line integral round the boundary. In considering the supposed waves of potential, it is therefore only necessary to calculate the successive coefficients of the Bessel functions of zero order.

A potential

$$H = BK_0(\iota hr), \dots \dots \dots (74)$$

leaving the first wire, reaches the second as

$$H = B\theta_0 J_0(h\rho) + B\sum_1^\infty \theta_s J_s(h\rho) \cos s\phi$$

where

$$\theta_s = 2\iota^s K_s(\iota hc).$$

If the reflected and internal potentials be respectively

$$\left. \begin{aligned} H &= c_0 K_0(\iota h\rho) + \sum_1^\infty c_s K_s(\iota h\rho) \cos s\phi \\ H &= d_\infty J_0(k'\rho) + \sum_1^\infty d_s J_s(k'\rho) \cos s\phi \end{aligned} \right\}, \dots (75)$$

the surface conditions readily give

$$c_s = \iota B\theta_s \delta_2^s / \beta_s, \quad d_s = \iota B\theta_s / hb\beta_s, \dots (76)$$

where

$$\delta_1^s = J_s(ka) J_s'(ha) - \frac{k}{\mu h} J_s(ha) J_s'(ka), \dots (77)$$

$$\alpha_s = K_s'(\iota ha) J_s(ka) - \frac{k}{\mu h \iota} J_s'(ka) K_s(\iota ha), \dots (78)$$

and (δ_2^s, β_s) are corresponding functions for the second wire. The potential

$$H = c_0 K_0(\iota h\rho) + \sum_1^\infty c_s K_s(\iota h\rho) \cos s\phi$$

leaves the second wire. But, if the argument of the functions K be ιhc in all cases,

$$K_1(\iota h\rho) \cos \phi = K_1 J_0(hr) + \sum_{s=1}^\infty (-\iota)^s \{K_{s+1} + K_{s-1}\} J_s(hr) \cos s\theta,$$

$$K_2(\iota h\rho) \cos 2\phi = K_2 J_0(hr) + \sum_{s=1}^\infty (-\iota)^s \{K_{s+2} + K_{s-2}\} J_s(hr) \cos s\theta,$$

and so on,

Thus the potential becomes

$$H = A_0 J_0(hr) + \sum_1^\infty A_s J_s(hr) \cos s\theta, \dots (79)$$

where

$$A_0 = C_0 K_0 + C_1 K_1 + C_2 K_2 + \dots$$

$$A_1 = -\iota C_0 \cdot 2K_1 - \iota C_1 (K_2 + K_0) - \iota C_2 (K_3 + K_2) \dots$$

$$A_2 = -C_0 \cdot 2K_2 - C_1 (K_3 + K_1) - C_2 (K_4 + K_0) \dots (80)$$

The first term alone contributes to the current crossing any section of the first wire. It causes an internal potential

$$H = -\frac{A_0}{i h a \alpha} J_0(kr) \quad . \quad . \quad . \quad . \quad (81)$$

in accordance with previous calculation, and a reflected wave

$$H = -\frac{A_0 \delta_1}{\alpha} K_0(ihr) + i \sum_{s=1}^{\infty} \frac{A_s \delta_1^s}{\alpha_s} K_s(ihr) \cos s\theta. \quad (82)$$

We now introduce the fact that the squares of ha , hb , hc may be ignored. Thus

$$K_0 = -\log(hc),$$

$$K_n = -\frac{i^{-n}}{2} \{Y_n(hc) + i\pi J_n(hc)\}$$

$$= \frac{1}{2} \cdot n-1! \left(\frac{2}{i hc}\right)^n \quad n \neq 0,$$

$$J_s(ha) = \left(\frac{ha}{2}\right)^s \cdot \frac{1}{n!},$$

$$K_s'(iha) = -2^{n-1} n! / (iha)^{n+1}.$$

If

$$\lambda_s = \left(1 - \frac{\mu s J_s(ka)}{ka J_s'(ka)}\right) / \left(1 + \frac{\mu s J_s(ka)}{ka J_s'(ka)}\right), \quad . \quad . \quad (83)$$

and μ_s be the corresponding quantity for the second wire, it appears on reduction that

$$\begin{aligned} C_s &= -\frac{B \theta_s \mu_s (hb)^{2s} i^s}{2^{2s-1} s! s-1!} \\ &= -\frac{B \mu_s h^s b^{2s} i^s}{s! 2^{s-1} c^s} \cdot \quad . \quad . \quad . \quad . \quad (84) \end{aligned}$$

The coefficients of type c_s therefore rapidly become small, especially if c/b is large. If hc is large and hb small the convergence is still more rapid, but the first form of c_s is to be used. θ_s itself decreases rapidly as hc increases.

Accordingly, when c is so large that the approximate values above cannot be used for the functions $K_n(ihc)$, the result will be correct to the extent of an error of order $h^2 a^2$, and this case needs no further consideration.

When hc is small, on reduction

$$\left. \begin{aligned} A_0 &= C_0 K_0 - B \Sigma_1^\infty \frac{\mu_s}{s} \left(\frac{b}{c}\right)^{2s}, \\ A_1 &= -\frac{2C_0}{hc} + \frac{2B}{hc} \Sigma_1^\infty \mu_s \left(\frac{b}{c}\right)^{2s}, \\ A_2 &= \frac{4C_0}{h^2 c^2} + \frac{4B}{h^2 c^2} \Sigma_1^\infty (s+1) \mu_s \left(\frac{b}{c}\right)^{2s} \end{aligned} \right\}, \quad \dots \quad (85)$$

and so on, very convergent series being obtained if $\frac{b}{c}$ is small.

The reflected wave of (82) becomes, near the second wire,

$$H = B_0 J_0(h\rho) + \Sigma_1^\infty B_s J_s(h\rho) \cos s\phi, \quad \dots \quad (86)$$

where

$$\begin{aligned} B_0 &= \frac{\iota A_0 \delta_1}{\alpha} K_0 + \iota \Sigma_1^\infty \frac{A_s \delta_1^s}{\alpha_s} K_s, \\ B_1 &= 2\iota K_1 \cdot \frac{\iota A_0 \delta_1}{\alpha} - \Sigma_1^\infty A_s (K_{s+1} + K_{s-1}). \end{aligned}$$

The ratios a/c , b/c , will be retained to the second power only. To this order,

$$B_0 = \frac{\iota \delta_1 K_0}{\alpha} \left\{ C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right\} + \lambda_1 C_0 \frac{a^2}{c^2} \quad \dots \quad (87)$$

B_1 will only subsequently be required to the first order, and becomes

$$B_1 = \frac{\iota \delta_1}{\alpha} 2\iota K_1 \cdot C_0 K_0. \quad \dots \quad (88)$$

We note that the change in B_0 due to taking account of $c_1 c_2 \dots$ is of order a^2/c^2 as stated in the previous section.

So far as the current is concerned, the internal potential produced in the second wire is

$$H = -\frac{B_0}{\iota h b \beta} J_0(k\rho), \quad \dots \quad (89)$$

and a potential

$$H = C_0 J_0(hr) + \Sigma_1^\infty C_s J_s(hr) \cos s\theta \quad \dots \quad (90)$$

is incident again on the first wire, where

$$\begin{aligned} C_0 &= \frac{\iota \delta_2}{B} \cdot B_0 K_0 + \iota \Sigma_1^\infty B_s \frac{\delta_2^s}{\beta_s} K_s \\ &= \frac{\iota \delta_2}{B} \cdot B_0 K_0 + \frac{\iota \delta_2'}{\beta_1} B_1 \quad \text{to order} \quad \frac{a^2}{c^2} \\ &= \rho_1 \rho_2 \left(C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right) + \lambda_1 \rho_2 C_0 \frac{a^2}{c^2} - \mu_1 \rho_1 C_0 \frac{b^2}{c^2} \quad (91) \end{aligned}$$

where

$$\rho_1 = \frac{\iota \delta_1 K_0}{\alpha} \quad \rho_2 = \frac{\iota \delta_2 K_0}{\beta} \quad . \quad . \quad . \quad (92)$$

and

$$C_1 = -2\iota K_1 \rho_1 \rho_2 C_0. \quad . \quad . \quad . \quad (93)$$

If the next wave meets the second wire in the form

$$H = D_0 J_0(h\rho) + \sum_1^\infty D_s J_s(h\rho) \cos s\phi, \quad . \quad . \quad (94)$$

then in a similar manner,

$$\begin{aligned} D_0 &= -\rho_1 C_0 - \lambda_1 \frac{a^2}{c^2} \rho_1 \rho_2 (-C_0) \\ &= \rho_1^2 \rho_2 \left(C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right) + 2\lambda_1 \rho_1 \rho_2 C_0 \frac{a^2}{c^2} - \rho_1^2 \mu_1 C_0 \frac{b^2}{c^2} \\ D_1 &= 2\iota K_1 \rho_1^2 \rho_2 C_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (95) \end{aligned}$$

The leading coefficient in the potential next meeting the first wire is

$$E_0 = \rho_1^2 \rho_2^2 \left(C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right) + 2\lambda_1 \rho_1 \rho_2^2 C_0 - 2\rho_1^2 \rho_2 \mu_1 C_0 \frac{b^2}{c^2}, \quad (96)$$

and so on, the mode of formation being now obvious.

The potentials containing the coefficients $A_0, C_0, E_0, G_0 \dots$ contribute to the current in the first wire, and those with $d_0, B_0, D_0 \dots$ to that in the second.

Now

$$\begin{aligned} A_0 + C_0 + E_0 + \dots &= \left(C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right) (1 + \rho_1 \rho_2 + \rho_1^2 \rho_2^2 + \dots) \\ &\quad + \left(\lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right) C_0 (1 + 2\rho_1 \rho_2 + 3\rho_1^2 \rho_2^2 + \dots) \\ &= \left(\rho_2 B K_0 - B \mu_1 \frac{b^2}{c^2} \right) / (1 - \rho_1 \rho_2) \\ &\quad + \rho_2 B \left\{ \lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right\} / (1 - \rho_1 \rho_2)^2, \quad . \quad (97) \end{aligned}$$

since

$$C = \iota B \theta_0 \frac{\delta_2}{\beta} = \rho_2 B,$$

and

$$\begin{aligned} d_0 + B_0 + D_0 + F_0 + \dots &= \left(B K_0 - \rho_1 B \mu_1 \frac{b^2}{c^2} + \lambda_1 \frac{a^2}{c^2} B K_0 \right) / (1 - \rho_1 \rho_2) \\ &\quad + B \rho_1 \rho_2 \left(\lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right) / (1 - \rho_1 \rho_2)^2 \quad . \quad (98) \end{aligned}$$

on summation.

Similarly, due to $B'K_0(ih\rho)$ originally leaving the second wire, we have in the first an addition to $A_0 + C_0 + \dots$ of magnitude

$$\left(B'K_0 - \rho_2 B' \lambda_1 \frac{a^2}{c^2} + \mu_1 \frac{b^2}{c^2} B'K_0 \right) / (1 - \rho_1 \rho_2) \\ + B' \rho_1 \rho_2 \left(\mu_1 \rho_1 \frac{b^2}{c^2} - \lambda_1 \rho_2 \frac{a^2}{c^2} \right) / (1 - \rho_1 \rho_2)^2. \quad (99)$$

But

$$B = -\frac{E\delta_1}{p\alpha} = \frac{eE\rho_1}{p}, \quad B' = -\frac{E\delta_2}{p\beta} = \frac{eE\rho_2}{p} \text{ by (54).}$$

Adding the potential $H = AJ_0(kr)$ obtained in a previous section, and reducing, the total potential in the first wire, which contributes to the current, becomes

$$H = \frac{-EH_1}{p\hbar\alpha} J_0(kr), \quad . \quad . \quad . \quad (100)$$

where

$$H_1 = 1 + \left(\rho_2 K_0 - \mu_1 \frac{b^2}{c^2} \right) \frac{\rho_1}{K_0} / (1 - \rho_1 \rho_2) \\ + \frac{\rho_1 \rho_2}{K_0} \left(\lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right) / (1 - \rho_1 \rho_2)^2 \\ + \frac{\rho_2}{K_0} \left(K_0 - \rho_2 \lambda_1 \frac{a^2}{c^2} + \mu_1 K_0 \frac{b^2}{c^2} \right) / (1 - \rho_1 \rho_2) \\ + \frac{\rho_2^2}{K_0} \left(\mu_1 \rho_1 \frac{b^2}{c^2} - \lambda_1 \rho_2 \frac{a^2}{c^2} \right) / (1 - \rho_1 \rho_2)^2, \quad . \quad . \quad (101)$$

and the current is

$$\varpi_1 = \frac{E\delta_1}{2p\alpha} \cdot H_1. \quad . \quad . \quad . \quad (102)$$

A similar expression may be written down at once for the current in the second wire, and a formula for the self-induction obtained. When the wires are similar in all respects,

$$H = 1 + \left(\rho_1 K_0 - \lambda_1 \frac{a^2}{c^2} \right) \frac{\rho_1}{K_0} / (1 - \rho_1^2) \\ + \frac{\rho_1}{K_0} \left(K_0 - \rho_1 \lambda_1 \frac{a^2}{c^2} + K_0 \lambda_1 \frac{a^2}{c^2} \right) / (1 - \rho_1^2) \\ = \frac{1 + \rho_1}{1 - \rho_1^2} + \lambda_1 \frac{a^2}{c^2} \cdot \frac{\rho_1}{1 - \rho_1^2} - \frac{\lambda_1}{K_0} \cdot \frac{a^2}{c^2} \frac{\rho_1}{1 - \rho_1}.$$

If L be the self-induction of the system, $\epsilon L p$ is the imaginary part of

$$\frac{4p\alpha}{\delta_1} \left\{ \frac{1}{1-\rho_1} - \frac{\lambda_1 a^2}{K_0 c^2} \cdot \frac{\rho_1}{1-\rho_1} + \lambda_1 \frac{a^2}{c^2} \cdot \frac{\rho_1}{1-\rho_1^2} \right\}^{-1},$$

or of

$$\frac{4p\alpha}{\delta_1} \cdot (1-\rho_1) \cdot \left\{ 1 + \frac{\lambda_1 a^2}{K_0 c^2} \cdot \rho_1 - \lambda_1 \frac{a^2}{c^2} \cdot \frac{\rho_1}{1+\rho_1} \right\}. \quad (103)$$

to the same order.

The first term of this expression leads to the results already given. The second term, writing

$$\rho_1 = -K_0 \left/ \left(\log ha - \frac{\mu J_0}{ka J_0'} \right) \right.,$$

$$1 + \rho_1 = 2 \text{ practically,}$$

becomes

$$4\epsilon p \frac{a^2}{c^2} \lambda_1 \cdot \frac{\log\left(\frac{a}{c}\right) - \frac{\mu J_0}{ka J_0'}}{\log(ha) - \frac{\mu J_0}{ka J_0'}}.$$

If R be the real part of

$$4 \frac{a^2}{c^2} \lambda_1 \cdot \frac{\log\frac{a}{c} - \frac{\mu J_0}{ka J_0'}}{\log ha - \frac{\mu J_0}{ka J_0'}} \quad . \quad . \quad . \quad (104)$$

where

$$\lambda_1 = \left(1 - \frac{\mu}{ka} \cdot \frac{J_1}{J_1'} \right) \left/ \left(1 + \frac{\mu}{ka} \cdot \frac{J_1}{J_1'} \right) \right., \quad . \quad . \quad (105)$$

and the argument of the Bessel functions is in all cases

$$ka = x\epsilon^{3/2} = 2a\sqrt{\frac{\pi\mu n}{\sigma}} \cdot \epsilon^{3/2} \text{ as before.} \quad . \quad (106)$$

Then the self-induction is

$$L = 4 \log \frac{c}{a} + \frac{4\alpha}{x} \cdot \frac{\text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} + R. \quad (107)$$

This formula may be used even when $c = 3a$, to obtain a result not in error by more than one per cent. For the application of this result, the functions analogous to $\text{ber } x$, and $\text{bei } x$ derived from the Bessel function of order 1, require tabulation.

When the resistivity of the wires is not very great, and the frequency is large, the Bessel functions in these results may usually be replaced by their ordinary asymptotic expansions for large argument, and tables cease to be necessary. The solutions then approximate to that for an infinite frequency, which is well known*. The formula (72), applicable when the wires are very far apart in comparison with their radii, is chiefly useful in determining a differential effect, for the ends of the wires will have a great influence in most cases to which the formula might be applied if the wires were really infinite.

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XX. *The Treatment of Electrodynamics.*

By R. A. LEHFELDT, *D.Sc.*†

IT has long been customary in English books to treat magnetism before current electricity, and "explain" the latter by analogy with the action of magnetic shells. This mode of presentation has always seemed to the writer unsatisfactory, putting the more obscure phenomenon as an explanation of the simpler; and now that the electron theory has shown itself capable of application to all branches of electricity, it is most inappropriate, since magnetism is looked upon as an effect of molecular electric currents. An attempt has therefore been made to put the treatment of electrodynamics, including magnetism, on a more logical basis.

The experimental starting-point is the brilliant investigation of Ampère on the forces between conductors carrying currents. The electrostatic forces, although out of all proportion greater in amount, produce no effect, on account of the equal quantities of positive and negative electricity in each wire. Again, a charge in motion produces no effect, so far as is known, on a charge at rest. There remain, therefore, only the purely electrodynamic forces—the action of one moving charge on another moving charge. These forces can be explained by means of the concept of a magnetic field, but it is better to begin with the forces themselves, although their spatial relations are somewhat complicated.

Let us suppose, then, a charge e , located at the origin and moving with the vector velocity \mathbf{V} , whose components are $(u_1, v_1, 0)$, *i. e.* the axis of z is chosen perpendicularly to the

* Russell, *l. c. ante*.

† Communicated by the Author. From the Transactions of the South African Association for the Advancement of Science, Grahamstown, 1908.

motion. Another charge e_2 is located at the point $(r, 0, 0)$, and has the velocity V_2 with components (u_2, v_2, w_2) . Then the first charge exerts on the second a force F , which has components in the x and y directions, but not in z , and its components are

$$F_x = -\frac{e_1 v_1 e_2 v_2}{c^2 r^2}, \quad F_y = +\frac{e_1 v_1 e_2 u_2}{c^2 r^2},$$

where the charges are expressed in electrostatic units and c is the velocity of light. The reaction exerted by e_2 on e_1 is not equal and opposite to this, because the momentum of the field must be taken into account if we wish to apply Newton's laws of motion. That is of no consequence for the present purpose, however, as the laws just stated enable one to calculate the forces and hence the motions, whether of the conductors as a whole or of the electrical charges within them.

Now it is of assistance to regard this action as taking place in two stages, and say that the movement of the first charge produces a magnetic field, and that this field acts on the second charge, causing a mechanical force.

Calling the magnetic field H , we may define it as a vector of the value

$$H_x = -\frac{e_1 v_1}{c r^2}$$

in the direction of x and without components in the other directions, and can then say that the field H acts on the charge e_2 moving with velocity (u_2, v_2, w_2) with the force components

$$F_x = +\frac{H e_2 v_2}{c} \quad \text{and} \quad F_y = -\frac{H e_2 u_2}{c},$$

i. e. with a resultant force $\frac{H e_2 \sqrt{u_2^2 + v_2^2}}{c}$ perpendicular to the plane containing the field and the velocity, and proportional to their vector product.

The signs occurring in the above equations may be determined by the well-known "right-hand screw" and "three-finger" rules.

It is, of course, open to dispute whether the above statements form a complete account of the action of one moving electron on another, since that action is not directly accessible to experiments; it might be proposed to employ the "rationalized current element," consisting of the system of lines of force produced by a moving electric charge, in addition to the charge itself. But again this is unnecessary here, as we

are only concerned to find the simplest means of calculating observed effects of electric currents and magnets. We will therefore take the above rules and apply them to certain cases.

A conduction current consists in a continuous succession of moving charges. Let there be a short element of wire, of length dl , in which there are on the average n' electrons per unit length, each of charge e , and travelling with the average velocity V . Then neV is the total charge conveyed per second across a section of the wire, *i. e.* the current, in electrostatic measure. In electrodynamic units the current is $\frac{neV}{c} = i$.

In the short length considered, however, there will be $n dl$ electrons, so that the total electrodynamic effect of that length will be proportional to $\frac{neV dl}{c} = i dl$. We may accordingly write the former equations in a form suitable for conduction currents as :—

$$H_x = -\frac{i_1 dl_1}{r^2},$$

$$F_x = +Hi_2 dl_2 \sin \theta_2, \quad F_y = -Hi_2 dl_2 \cos \theta_2,$$

where θ_2 is the angle whose tangent is $v_2 \div u_2$.

The resultant force is therefore $F = Hi_2 dl_2$.

The equations may, of course, be transformed, without difficulty, into the usual polar coordinates, or into the language of vector algebra.

It is now convenient to deal with the magnetic action of a rectangular elementary circuit. Two principal cases arise according as the field is required at a point lying in the axis of the small circuit or in its plane. In the first case, let $dy dz$ be the area of the circuit, and the point considered be at a distance r along the axis of x . Then one of the sides, of length dy , situated $\frac{1}{2} dz$ from the axis of y , will produce a field $\frac{i dy}{r^2}$, of which the component in the axial direction is $\frac{i dy}{r^2} \times \frac{dz}{2r}$. The transverse components of the opposite sides neutralize, leaving an axial resultant amounting to four times the above, or

$$dH = 4 \times \frac{i dy}{r^2} \times \frac{dz}{2r} = \frac{2i dA}{r^2}.$$

In the same way it may be shown that for a point in the plane of the elementary circuit the field $dH = \frac{i dA}{r^3}$.

The product $i dA$ may then be defined as the magnetic moment of the elementary circuit.

It is easy to show that, for a point in any direction, the area—or magnetic moment—of the elementary circuit may be resolved into axial and equatorial components, and the effect of the two parts calculated according to the preceding rules. We have, thus, a complete picture of the magnetic field round an element of a circuit.

The next step is to show that the field can be derived from a potential. The necessary integration is given in all treatises on electricity, and shows that the magnetic potential in the axial direction has the value

$$d\Omega = \frac{i dA}{r^2},$$

while in the equatorial direction it vanishes; and at any point making an angle χ with the axis of the elementary circuit the magnetic potential is $d\Omega = \frac{i dA}{r^2} \cos \chi$. Now the solid angle subtended by the circuit at the point is $\frac{dA}{r^2} \cos \chi = dw$. Hence $d\Omega = i dw$. If, however, this relation is integrated, bearing in mind the way in which a circuit may be supposed built up of small plane elements, the sides of which cancel in their magnetic effects (as demonstrated by Ampère) we conclude that

$$\Omega = iw;$$

that is, the magnetic potential produced by a current is equal to the strength of the current multiplied by the solid angle subtended by the circuit in which it flows.

It then follows, by considering the changes of magnetic potential along any closed path, that

$$\Omega_{\text{cyclic}} = 4\pi i,$$

the first circuital relation of electrodynamics.

At this point it is convenient—still considering non-magnetic media—to calculate the field in the interior of a solenoid, either infinitely long or finite, and, if desired, to calculate the forces and couples exerted on circuits—plain or solenoidal—when placed in a magnetic field.

It may then be pointed out that, on the view that magnets

are a collection of molecular circuits, the same formulæ are applicable to them, and the usual deductions as to magnetic moments, the Gauss positions, &c., can be made.

This, however, leads to the question of the field in the interior of a magnet. The total field there varies from point to point, very rapidly and irregularly; but the average is constant, and is called the magnetic induction; while the term "field" is usually restricted to that part of the total effect which is not due to currents in the immediate neighbourhood of the point considered. The usual procedure is to elucidate (?) this distinction by considerations about a tube or crevasse in the interior of the iron; an argument in the highest degree artificial, and very commonly unintelligible to students. It is also quite unnecessary, as may be seen from the following:—

To determine the magnetic induction in the interior of a magnet. Let the magnet be of 1 square cm. cross-section, and infinitely long. Let there be m circuits per cubic centimetre, each of a small area A with current i circulating in it; and let χ be the angle between the axis of one of these circuits and the axis of the magnet. Then the moment of this circuit, resolved in the length of the magnet, is $i A \cos \chi$ (which may be positive or negative), and the intensity of magnetization of the bar (or magnetic moment per c. c.) is $miA \cos \chi = I$.

Now consider a straight line drawn at random down the length of the magnet. This will cut through some of the molecular circuits, and the number of those cut will bear to the total the ratio of the projected area of a circuit $A \cos \chi$ to the area of the bar, or unity. The number cut is therefore $m A \cos \chi$. Hence, by the first circuital relation the average change of magnetic potential per unit length is

$$4\pi i m A \overline{\cos \chi} = 4\pi I.$$

This is in addition to any rate of change of magnetic potential due to outside causes, say H . Hence we have, for the magnetic induction (per square cm.) as defined above—

$$B = H + 4\pi I.$$

The same argument, applied to a bar of finite length, gives, in the simplest possible manner, the relation between field and induction, in that case, *i. e.* the "demagnetizing effect of the ends."

To deal with the induction of currents, we will take, first, an element of a circuit moved at right angles to itself, in a magnetic field. Let the wire—of length dl —lie in the

direction of x , and the field, of intensity H , be in the positive direction of z ; and the wire be moved, with velocity V , in the positive direction of y . Then any electron in the wire, being carried with the velocity V , will constitute a current in that direction, and will consequently suffer a mechanical force, perpendicular to its velocity and to the magnetic field, *i. e.* in the direction of x , and as the three-finger rule shows, in the positive sense. It will accordingly be driven along the wire, and there will be an "induced" current in the wire. The mechanical force on each electron amounts to

$$\frac{evH}{c},$$

which is the same as if it were placed in an electric field amounting to $\frac{vH}{c}$. As every electron in the wire is subjected to a similar force, it is as if there were an electric field of this intensity prevailing in the wire, so that between the ends of it there was a difference of potential $\frac{vHdl}{c}$ —or, in electromagnetic units, $vHdl$.

Since, however, we shall need to deal with induction in cases in which the magnetic metals are present, it would be well to write B instead of H , the latter symbol having been applied above to a part only of the magnetic induction. But $vBdl$ is the rate at which the element of circuit cuts the total magnetic induction. We may then say that the "electromotive force" generated in the element is the rate at which it crosses magnetic induction. By integrating this result over a whole circuit we arrive at the second circuital relation

$$E = -\frac{dN}{dt}.$$

N being the total magnetic induction through the circuit, the negative sign follows from the above case because a positive motion (in y) into a field decreases N and causes a current in the positive direction of x .

The above treatment has the advantage of deriving the two integral relations of electrodynamics from one, more easily comprehended, differential relation, and puts magnetism in its place as a special case of currents.

Transvaal University College,
Johannesburg, 1908.

XXI. *The Nature of the α Particle from Radioactive Substances.* By Professor E. RUTHERFORD, F.R.S., and T. ROYDS, M.Sc., 1851 Exhibition Science Scholar*.

THE experimental evidence collected during the last few years has strongly supported the view that the α particle is a charged helium atom, but it has been found exceedingly difficult to give a decisive proof of the relation. In recent papers, Rutherford and Geiger† have supplied still further evidence of the correctness of this point of view. The number of α particles from one gram of radium have been counted, and the charge carried by each determined. The values of several radioactive quantities, calculated on the assumption that the α particle is a helium atom carrying two unit charges, have been shown to be in good agreement with the experimental numbers. In particular, the good agreement between the calculated rate of production of helium by radium and the rate experimentally determined by Sir James Dewar‡, is strong evidence in favour of the identity of the α particle with the helium atom.

The methods of attack on this problem have been largely indirect, involving considerations of the charge carried by the helium atom and the value of e/m of the α particle. The proof of the identity of the α particle with the helium atom is incomplete until it can be shown that the α particles, accumulated quite independently of the matter from which they are expelled, consist of helium. For example, it might be argued that the appearance of helium in the radium emanation was a result of the expulsion of the α particle, in the same way that the appearance of radium A is a consequence of the expulsion of an α particle from the emanation. If one atom of helium appeared for each α particle expelled, calculation and experiment might still agree, and yet the α particle itself might be an atom of hydrogen or of some other substance.

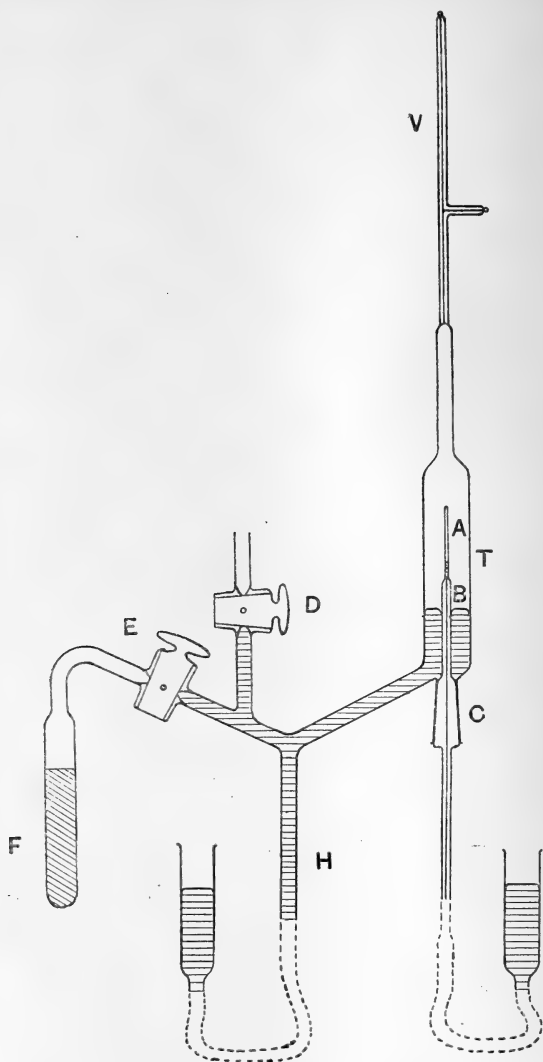
We have recently made experiments to test whether helium appears in a vessel into which the α particles have been fired, the active matter itself being enclosed in a vessel sufficiently thin to allow the α particles to escape, but impervious to the passage of helium or other radioactive products.

* Communicated by the Authors.

† Proc. Roy. Soc. A. lxxxi. pp. 141-173 (1908).

‡ Proc. Roy. Soc. A. lxxxi. p. 280 (1908).

The experimental arrangement is clearly seen in the figure. The equilibrium quantity of emanation from about 140 milligrams of radium was purified and compressed by means of a



mercury-column into a fine glass tube A about 1.5 cms. long. This fine tube, which was sealed on a larger capillary tube B, was sufficiently thin to allow the α particles from the emanation and its products to escape, but sufficiently strong to

withstand atmospheric pressure. After some trials, Mr. Baumbach succeeded in blowing such fine tubes very uniform in thickness. The thickness of the wall of the tube employed in most of the experiments was less than $\frac{1}{100}$ mm., and was equivalent in stopping power of the α particle to about 2 cms. of air. Since the ranges of the α particles from the emanation and its products radium A and radium C are 4.3, 4.8, and 7 cms. respectively, it is seen that the great majority* of the α particles expelled by the active matter escape through the walls of the tube. The ranges of the α particles after passing through the glass were determined with the aid of a zinc-sulphide screen. Immediately after the introduction of the emanation the phosphorescence showed brilliantly when the screen was close to the tube, but practically disappeared at a distance of 3 cms. After an hour, bright phosphorescence was observable at a distance of 5 cms. Such a result is to be expected. The phosphorescence initially observed was due mainly to the α particles of the emanation and its product radium A (period 3 mins.). In the course of time the amount of radium C, initially zero, gradually increased, and the α radiations from it of range 7 cms. were able to cause phosphorescence at a greater distance.

The glass tube A was surrounded by a cylindrical glass tube T, 7.5 cms. long and 1.5 cms. diameter, by means of a ground-glass joint C. A small vacuum-tube V was attached to the upper end of T. The outer glass tube T was exhausted by a pump through the stopcock D, and the exhaustion completed with the aid of the charcoal tube F cooled by liquid air. By means of a mercury column H attached to a reservoir, mercury was forced into the tube T until it reached the bottom of the tube A.

Part of the α particles which escaped through the walls of the fine tube were stopped by the outer glass tube and part by the mercury surface. If the α particle is a helium atom, helium should gradually diffuse from the glass and mercury into the exhausted space, and its presence could then be detected spectroscopically by raising the mercury and compressing the gases into the vacuum-tube.

In order to avoid any possible contamination of the apparatus with helium, freshly distilled mercury and entirely new glass apparatus were used. Before introducing the emanation into A, the absence of helium was confirmed

* The α particles fired at a very oblique angle to the tube would be stopped in the glass. The fraction stopped in this way would be small under the experimental conditions.

experimentally. At intervals after the introduction of the emanation the mercury was raised, and the gases in the outer tube spectroscopically examined. After 24 hours no trace of the helium yellow line was seen; after 2 days the helium yellow was faintly visible; after 4 days the helium yellow and green lines were bright; and after 6 days all the stronger lines of the helium spectrum were observed. The absence of the neon spectrum shows that the helium present was not due to a leakage of air into the apparatus.

There is, however, one possible source of error in this experiment. The helium may not be due to the α particles themselves, but may have *diffused* from the emanation through the thin walls of the glass tube. In order to test this point the emanation was completely pumped out of A, and after some hours a quantity of helium, about 10 times the previous volume of the emanation, was compressed into the same tube A.

The outer tube T and the vacuum-tube were removed and a fresh apparatus substituted. Observations to detect helium in the tube T were made at intervals, in the same way as before, but no trace of the helium spectrum was observed over a period of eight days.

The helium in the tube A was then pumped out and a fresh supply of emanation substituted. Results similar to the first experiment were observed. The helium yellow and green lines showed brightly after four days.

These experiments thus show conclusively that the helium could not have diffused through the glass walls, but must have been derived from the α particles which were fired through them. In other words, the experiments give a decisive proof that the α particle after losing its charge is an atom of helium.

Other Experiments.

We have seen that in the experiments above described helium was not observed in the outer tube in sufficient quantity to show the characteristic yellow line until two days had elapsed. Now the equilibrium amount of emanation from 100 milligrams of radium should produce helium at the rate of about .03 c.mm. per day. The amount produced in one day, if present in the outer tube, should produce a bright spectrum of helium under the experimental conditions. It thus appeared probable that the helium fired into the glass must escape very slowly into the exhausted space, for if the helium escaped at once, the presence of helium should have

been detected a few hours after the introduction of the emanation.

In order to examine this point more closely the experiments were repeated, with the addition that a cylinder of thin sheet lead of sufficient thickness to stop the α particles was placed over the fine emanation tube. Preliminary experiments, in the manner described later, showed that the lead-foil did not initially contain a detectable amount of helium. Twenty-four hours after the introduction into the tube A of about the same amount of emanation as before, the yellow and green lines of helium showed brightly in the vacuum-tube, and after two days the whole helium spectrum was observed. The spectrum of helium in this case after one day was of about the same intensity as that after the fourth day in the experiments without the lead screen. It was thus clear that the lead-foil gave up the helium fired into it far more readily than the glass.

In order to form an idea of the rapidity of escape of the helium from the lead some further experiments were made. The outer cylinder T was removed and a small cylinder of lead-foil placed round the thin emanation-tube surrounded the air at atmospheric pressure. After exposure for a definite time to the emanation, the lead screen was removed and gested for helium as follows. The lead-foil was placed in a glass tube between two stopcocks. In order to avoid a possible release of the helium present in the lead by pumping out the air, the air was displaced by a current of pure electrolytic oxygen*. The stopcocks were closed and the tube attached to a subsidiary apparatus similar to that employed for testing for the presence of neon and helium in the gases produced by the action of the radium emanation on water (Phil. Mag. Nov. 1908). The oxygen was absorbed by charcoal and the tube then heated beyond the melting-point of lead to allow the helium to escape. The presence of helium was then spectroscopically looked for in the usual way. Using this method, it was found possible to detect the presence of helium in the lead which had been exposed for only four hours to the α rays from the emanation. After an exposure of 24 hours the helium yellow and green lines came out brightly. These experiments were repeated several times with similar results.

A number of blank experiments were made, using samples of the lead-foil which had not been exposed to the α rays, but in no case was any helium detected. In a similar way,

* That the air was completely displaced was shown by the absence of neon in the final spectrum.

the presence of helium was detected in a cylinder of tinfoil exposed for a few hours over the emanation-tube.

These experiments show that the helium does not escape at once from the lead, but there is on the average a period of retardation of several hours and possibly longer.

The detection of helium in the lead and tin foil, as well as in the glass, removes a possible objection that the helium might have been in some way present in the glass initially, and was liberated as a consequence of its bombardment by the α particles.

The use of such thin glass tubes containing emanation affords a simple and convenient method of examining the effect on substances of an intense α radiation quite independently of the radioactive material contained in the tube.

We can conclude with certainty from these experiments that the α particle after losing its charge is a helium atom. Other evidence indicates that the charge is twice the unit charge carried by the hydrogen atom set free in the electrolysis of water.

University of Manchester,
Nov. 13, 1908.

XXII. *A Note on the Photoelectric Properties of Potassium-Sodium Alloy.* By J. A. FLEMING, M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London *.

IN connexion with further researches on my oscillation valve or glow-lamp radiotelegraphic wave-detector, I was led to examine instances of electronic emission other than that due to the incandescence of metals or carbon *in vacuo*, to ascertain how far rectifying effects with high frequency currents could be obtained by them†. It is well known that under the action of ordinary and ultra-violet light the electropositive metals lose a negative charge of electricity, and it was shown by Elster and Geitel that this photoelectric effect is most pronounced in the case of rubidium, potassium, and the liquid alloy of potassium and sodium.

With the object of examining this effect, experiments were

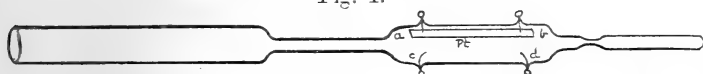
* Communicated by the Physical Society : read November 13th, 1908.

† See J. A. Fleming, "On the Conversion of Electric Oscillations into Continuous Currents by means of a Vacuum Valve," *Proc. Roy. Soc. Lond.* vol. lxxiv. p. 476 (1905). Also "The Construction and Use of Oscillation Valves for rectifying High Frequency Currents," *Phil. Mag.* May 1906, p. 659.

made with various forms of apparatus, and, as the outcome of these, it was found that a convenient mode of preparing a suitable specimen of highly photoelectric metal was as follows :—

A tube of lead glass about 50 cms. long, and 1.5 to 2 cms. internal diameter has a constriction made at one place, and on one side of this a couple of platinum wires are sealed through the glass which are welded to a slip of platinum foil *a, b*, about 5 cms. long and 1 cm. wide (see fig. 1). This foil is

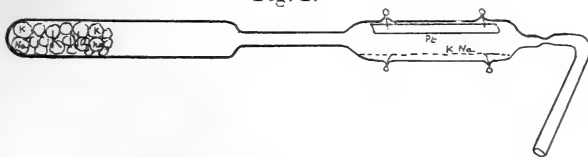
Fig. 1.



fixed close to the inside surface of the tube and opposite to it two other platinum wires *c d* are sealed through the tube. The end of this part of the tube is then drawn off and sealed to a length of smaller tube with a constriction in it by which the tube can ultimately be closed. A quantity of metallic potassium and sodium is then cut up into small cubes under naphtha, so as to give a number of clean morsels of the metal.

The operation of filling the glass tube is then as follows :—The fine quill tube at one end is first bent over at an angle of about 75° (see fig. 2), and by means of an indiarubber tube, coal-gas, which has been dried by bubbling through strong sulphuric acid, is led into it. The glass tube as a whole must have previously been well cleansed and carefully dried. When all the air is expelled, the fragments of sodium and potassium are dropped in at the wide open end of the tube, and about equal volumes of the two metals introduced. The open end is then loosely stopped with a plug of asbestos, and the glass tube heated in the blowpipe, and the open end so plugged is melted and sealed. If this is done skilfully, the tube can be sealed off with a rounded end by utilizing the gas-pressure to blow out the softened glass.

Fig. 2.



When cool, the lumps of metallic potassium and sodium are tilted over and should fill up about 10 to 15 cms. of the length of the so sealed off tube (see fig. 2). The tube is

then attached to a mercury pump for the purpose of making a high vacuum in it, and should be supported in a slanting position on a sheet of asbestos cardboard and covered by an iron trough. The tube should then be exhausted, and at the same time the metallic potassium and sodium heated so as to melt the lumps into a mass of liquid alloy which, however, will be covered with crusts of oxide.

When a good vacuum has been produced, the whole tube being hot and dry, the constriction in the quill tube may be melted with the blowpipe and the glass tube sealed off from the pump. If this is properly done, the part of the tube having in it the platinum plate will remain quite clean, and the molten metal or liquid alloy will be contained in the other portion of the tube. It is then easy to tilt the tube and transfer the clean mercury-like alloy of K and Na into the part of the tube containing the platinum plate, so as to make a pool of liquid alloy on the bottom of the tube having an electrical connexion with the outside by means of the platinum wires sealed through the tube, and having over it, and not far from its surface, a platinum plate also in connexion with the outside by sealed-in wires. In so doing, care should be taken that the liquid alloy is not splashed upon the platinum plate, but that the latter is kept quite clean and free from adherent drops of potassium-sodium.

It is better not to attempt to seal off that part of the tube in which the alloy is melted from that part to which it is decanted over, because the glass nearly always cracks and spoils the apparatus in so doing if there is the slightest particle of alloy smeared on it at the part heated in the flame.

A tube so prepared should be carefully handled, as if it is broken the liquid K and Na alloy is spontaneously inflammable and deflagrates violently on coming in contact with moisture. It is convenient to mount it on a wooden stand, which should be placed in an iron tray in case of an accidental breakage. By the aid of such a specimen of photoelectric alloy we can show many interesting experiments. If the tube is supported in a horizontal position, an electric arc contained in a projection lantern, equipped with the ordinary condenser-lens, can be so placed and tilted downwards as to converge on to the brilliant mercury-like surface of the pool of alloy a very concentrated beam of light. If, then, we connect the platinum plate and the alloy by means of wires with a sensitive mirror galvanometer, we find that the impact of the light upon the surface of the alloy not merely facilitates the escape of negative electricity from

it, but actually creates an electromotive force and current in the galvanometer circuit, which is in such a direction as to indicate that negative electricity is caused to move, by the action of the light, through the vacuum from the surface of the alloy to the platinum plate*.

It has been pointed out by Professor Sir J. J. Thomson in his book 'The Discharge of Electricity through Gases' (see p. 73) that air traversed by ultra-violet light behaves like an electrolyte; and it is also stated that Stoletow has shown by direct experiment that two different metals immersed in such illuminated air are brought down to the same potential just as when they are immersed in a liquid electrolyte. Again, it is said that when plates of two different metals are opposed to each other and ultra-violet light allowed to fall on one of them, whilst the terminals of a battery of electromotive force E are connected to them so as to force a current across the space between them, the total E.M.F. in the circuit is not E but is $E + M/M'$, where M/M' is the contact difference of potential between the metals. There is, however, here an unexplained discrepancy. If metallic potassium or sodium were placed with a platinum plate in a liquid aqueous electrolyte, and the two metals connected by a wire outside so as to construct a voltaic cell, the current through the cell, that is the movement of positive electricity, would be from the potassium or sodium to the platinum through the electrolyte, and therefore the movements of negative electricity would be in the opposite direction. In the case of the photoelectric cell, the movement of negative electricity in the cell is from the K-Na alloy to the platinum. Hence the physical operation of the photoelectric cell is not identical with that of an ordinary voltaic cell. Stoletow has described (*Physikalische Revue*, i. p. 765, 1892) experiments in which two plates of different metals, one perforated with holes, were placed parallel and at a little distance from each other, and ultra-violet light allowed to fall through the openings of the perforated plate on to the other plate. The plates, when connected together, were found to produce a current of electricity in the connecting circuit. Prof. Sir J. J. Thomson, however, remarks (*loc. cit.* p. 73) that for this to happen in accordance with voltaic principles, the perforated plate must be made of the more electropositive metal, for then only

* In the case of one tube of alloy prepared as above, my assistant, Mr. G. B. Dyke, noticed that the galvanometer deflexion was largely increased for a time by tilting the tube so as to make momentary contact between the alloy and the platinum, but the increased effect is not permanent.

would the negative electricity move towards it across the interspace between the plates. As a matter of fact, the best photoelectric metals are the most electropositive metals. Accordingly, if we imagine a sheet of potassium opposed to a sheet of zinc, then to make the photoelectric current agree in direction with the volta-electric current, we should have to illuminate the zinc plate with ultraviolet light, but keep the potassium plate in the dark. In the present case, the photoelectric current is in the opposite direction to the volta-electric current, assuming the ionized gas replaced by an aqueous electrolyte.

It seems therefore that in the case considered there are two separate sources of electromotive force, viz.: the volta contact-difference of potential of the metals in the cell and a photoelectromotive force due to the illumination; and to these may be added an external electromotive force due to any battery inserted in the circuit. In two tubes, made as above, I found that the electromotive force produced by the incident light was equal to 0.45 volt in one, and in the other tube to 0.6 volt. This E.M.F. was measured by the counter E.M.F. (produced by a shunted cell) which had to be introduced into the photoelectric cell circuit to reduce the galvanometer deflexion to zero. These differences are no doubt due to small differences in composition of the alloy in the two cases, to differences in the intensity of the light used at the time of making the measurements, or to differences in the pressure of the residual gas in the tube. The current produced by the tube of E.M.F. equal to 0.6 volt through a galvanometer of resistance 180 ohms, was found to be 5.4 microamperes. This indicates that the equivalent resistance of the tube is 74,000 ohms on the assumption that the effective E.M.F. is the same when the circuit is open and closed. The current increases very rapidly at first with the intensity of the incident light, but the author has not yet been able to find time to make measurements of the relation between the illumination per square cm. of the surface and the E.M.F. created in the circuit.

By the use of two such cells placed in series, I have found that the separate photoelectromotive forces are additive, and that if such photoelectric cells are joined in series like voltaic cells and separately illuminated, the individual E.M.F.'s are added together in an external circuit connecting the first and last plate. In the case of the two tubes mentioned above, giving separately 0.45 and 0.6 volt E.M.F., the E.M.F. when in series was found to be 1.0 volt.

At one time I contemplated constructing a photoelectric

battery of such cells which should give a high E.M.F. by the mere impact of light upon its most electropositive elements.

With the above described cells we can also verify easily a number of interesting observations made by previous experimentalists.

As regards the nature of the radiation which is chiefly effective, we find that for potassium alone, or for the potassium-sodium alloy, the effective rays are the most refrangible ones of the visible spectrum. The ultra-violet light in the arc is considerably filtered out, if not altogether stopped, by the thick glass condenser-lens; and by the glass walls of the tube. We can, however, effect a further separation of visible rays by screens of coloured glass, or solutions or stained gelatine films.

If we interpose in the path of the incident light a very thin film of gelatine stained with a yellow dye or a sheet of ordinary yellow glass, the deflexion of the galvanometer drops almost to zero. The same reduction is effected by a sheet of ruby glass or gelatine film stained red. A green glass cuts off a good deal of the deflexion, but a sheet of cobalt glass reduced it only to about two-thirds, showing that the blue glass is fairly transparent to those rays which can produce this electromotive force.

The galvanometer deflexions increase very rapidly with the intensity of the incident light, and unless an extremely sensitive galvanometer is used it is necessary to throw a very intense beam of light upon the surface of the metal to obtain any marked evidence of the production of electromotive force by the mere action of light upon the surface of the alloy. If the energy required to produce this photoelectric current comes from the incident light, then the latter must in some degree be absorbed, and the active rays must therefore be those which are absorbed by the photoelectric metal or conversely emitted by it when it is heated. We know well that the light due to the violet line in the flame spectrum of potassium salts passes easily through cobalt glass. Hence the fact that light which has passed through cobalt glass is still exceedingly active in producing the photoelectric effect with potassium, may indicate that it is this ray which is absorbed in its production. In the case of rubidium, Elster and Geitel showed that the yellow and orange rays were relatively more active in the production of the photoelectric effect, and a glance at the flame spectrum of rubidium salts shows that the lines are most numerous in the red, yellow, and green region.

Also, we can confirm easily another observation of the

effect of the plane of polarization of the incident light. If we polarize the incident light by a Nicol's prism, and connect the metals of the photoelectric cell with a galvanometer either with or without a battery of 1 or 2 secondary cells inserted in the circuit, then we find at once that the galvanometer deflexion is much greater when the plane of polarization is at right angles to the plane of incidence, than when these planes coincide.

On the electromagnetic theory of light, this may be stated by saying that the effect is a maximum when the electric vector of the plane-polarized light is normal to the surface of the photoelectric metal, and a minimum when it is parallel to it. The theory that fits in best with the above facts, is that the light facilitates or causes an escape of negative corpuscles or electrons from the surface of the photoelectric metal, and to do this it must of course impart to them energy sufficient to give them a velocity enough to carry them beyond the range of attraction of the positive charge which remains behind on the metal.

Since the electropositive metals are those which most easily lose electrons from their atoms, this is in accordance with the observed fact that the most electropositive metals are the most highly photoelectric substances. Again, if there is an emission under the action of light of electrons from the surface, we have an explanation of another fact easily proved with these tubes, viz. that the formation of a magnetic field parallel to the surface of the alloy greatly reduces the photoelectric current. We can show this easily by the great reduction which occurs in the galvanometer deflexion when even an ordinary horseshoe magnet is placed with its poles across the tube. Again, we can explain on this hypothesis the unilateral conductivity of the vacuous space over the illuminated alloy surface. If we connect the negative pole of a secondary cell to the external terminal of the alloy and the positive to the platinum plate, and insert a galvanometer as usual in the circuit, we greatly increase the galvanometer deflexion, which occurs when no cell is inserted. On the other hand, if the cell is reversed, then, provided the platinum is quite free from splashes of alloy, we cannot reverse the deflexion of the galvanometer.

It is obvious that if a magnetic field is created parallel to the surface of the alloy, a force will be exerted on the negative ion as it moves normally away from the surface, deflecting it from its path, and this will reduce the number of ions which in any time reach the platinum plate, and therefore will diminish the current. Again, the negative electrification

of the photoelectric metal will increase the force of propulsion on the ion, and hence increase the emission per unit of time, and therefore increase the observed current.

It is obvious, then, that the ionized gas over the illuminated alloy has a unilateral conductivity, and will conduct the current from a voltaic cell in one direction, but not in the opposite. Negative electricity can be conveyed from the alloy to the platinum across the rarefied air-space, but not in the opposite direction. Hence, an alternating current whether of low or of high frequency, can be rectified, and by interposing such a light-cell in the circuit of a galvanometer, in which circuit high-frequency oscillations are also created by the inductive action of a discharging condenser, I have been able to rectify these oscillations. The action, however, proved to be much more feeble and irregular than the similar rectifying effect which can be produced by a glow-lamp, and as at that time I had succeeded in finding a far more efficient glow-lamp rectifier or oscillation detector, the experiment with the rectifying effects of potassium-sodium alloy were not continued.

The interesting question, however, still remains as to the source and nature of this photoelectromotive force which is produced by the absorption of light by the surface of highly electropositive metals.

It is much affected by temperature, being increased by heating the alloy and by exposing it to powerful radiation for some time. If a galvanometer is used as described to detect the photoelectric effect, the current is found to be dependent upon the pressure and nature of the residual gas in the tube.

This was confirmed as follows:—A tube was prepared as above described with potassium-sodium alloy. Before sealing it off the pump, dry hydrogen gas was admitted and pumped out and then admitted again, and the tube exhausted to a pressure of about 0.01 mm. This tube was compared with others, in which the residual gas was air, but the exhaustion carried below 0.001 mm. The former tube exhibited only the very smallest evidence of photoelectric effect as measured by the galvanometer current, whereas the highly vacuous tube with residual air under very small pressure exhibited the effects well. The case of other tubes made as described, in which water vapour was accidentally present, which in a short time liberated hydrogen under the action of the metal; these after an interval ceased to give any current, even under the action of the strongest illumination, when the platinum plate was connected with the potassium-sodium

alloy through a galvanometer. These experiments and others show that to obtain the effect well a very high vacuum is necessary. In the process of preparing the tube the potassium and sodium should be melted, and the alloy and tube well heated for some time whilst the pump is going to drive off all traces of water vapour, also of hydrocarbon derived from the naphtha in which the metallic potassium is commonly preserved, and especially to drive off the hydrogen which seems to be occluded in considerable quantity by the alkaline metals, no doubt derived from the decomposition of water vapour.

The effect of variation of pressure of the gas upon the photoelectric effect was first investigated by Stoletow, and an account of his researches is given in Prof. Sir J. J. Thomson's treatise on the "Conduction of Electricity through Gases" (p. 224). He found that the current increased rapidly as the pressure diminished, which continued until the current reached a maximum value, after which it began to decline, but had a finite value at the lowest attainable pressures.

Stoletow's experiments appear to have been made with zinc plates and ultra-violet light. Owing to the greater manipulative difficulty when dealing with the more oxidisable and electropositive metals no attempt has been made to push these investigations with potassium-sodium alloy very far. The primary object in view in conducting them was to ascertain if the photoelectric effects could be utilized as an oscillation detector in radiotelegraphy; but as a type of glow-lamp detector has now been found by the writer far more efficient than that originally proposed by him, involving the employment of a carbon filament, these photoelectric experiments have not been pursued. They are put on record here merely for the sake of aiding any who may wish to show them as interesting lecture or class experiments, or pursue the purely physical investigation of the effect itself still further.

The question of photoelectric effects is not without interest in connexion with long distance radiotelegraphy. It has been shown that perfectly dust-free air is not ionized by ultra-violet light. If, therefore, the absorption of long radiotelegraphic electric waves which is found to exist when they pass through considerable distances of sunlit air is due to the presence of free ions in the air, these may arise from the photoelectric action of the light upon the dust particles. This suggests the question whether these particles may not be the same that create the blue colour of the sky. We know that whenever photoelectric effects take place, light must be

absorbed, and light of the same kind as that emitted by the photoelectric substance if it radiates. Thus zinc is rendered photoelectric under the action of ultra-violet light, but zinc when heated, whether by being used as spark-balls for an electric spark or as arc terminals for an electric arc, radiates much ultra-violet light.

In the same manner, the photoelectric effect of potassium appears to be due to the absorption of that violet ray which potassium itself emits if heated. If, then, these atmospheric particles absorb ultra-violet light, that would account for the relatively small percentage of ultra-violet light found in sunlight at the earth's surface, and also for the ionization found to exist in the atmosphere. This suggests the need for further observation on the number of ions present in the terrestrial atmosphere at various heights above the sea-level.

In conclusion, I have pleasure in mentioning the aid rendered in these experiments by my assistant, Mr. G. B. Dyke.

XXIII. *On Variations in the Conductivity of Air enclosed in Metallic Receivers.* By C. S. WRIGHT, B.A., *Exhibition Scholar, University of Toronto, and Wollaston Student, Gonville and Caius College, Cambridge* *.

[Plates IV. & V.]

1. INTRODUCTION.

IN a paper in the Phil. Mag. of December 1907, Professor McLennan records some observations made on the ionization of air enclosed in cylindrical receivers of lead, zinc, and aluminium. For " q ," the number of ions generated per c.c. per sec. in these receivers he obtained the value 15, when they were made of zinc and aluminium; while with lead, values were found ranging all the way from 23 to 160 ions per c.c. per sec., depending on the sample of lead from which the cylinder was made. From these results Prof. McLennan drew the conclusion that ordinary commercial lead contained in general varying amounts of some active impurity.

From these and other experiments he pointed out also, that four possible causes must be considered as contributing to the ionization in the cylinders, viz.:—(1) penetrating radiation from the earth; (2) secondary rays excited by this

* Communicated by Prof. J. C. McLennan, and read before the Royal Society of Canada, May 28, 1908.

type of radiation in the metal of the receivers; (3) radioactive impurities present in the metals; and (4) a possible intrinsic radiation from the metals themselves.

In view of the theoretical importance of ascertaining whether metals generally possessed any specific activity, it was thought highly desirable to endeavour to obtain metals as free as possible from active impurities, and also to take observations on the conductivity of air enclosed in vessels made from them, in localities and under conditions, in which the penetrating radiation from the earth, and the corresponding radiation excited by it in the receiver, was very largely cut off or at least reduced to a minimum.

Several attempts have been made during the last few years to find some efficient screen for the earth's radiation. Cooke*, while making some measurements on the conductivity of the air enclosed in a brass cylinder, found a decrease of about 30 per cent. when the cylinder was completely surrounded by large masses of lead. Shortly afterwards Elster and Geitel† observed a fall of 28 per cent. in the conductivity of air enclosed in an aluminium receiver, when the apparatus was set up in a rock-salt mine. Later still Cooke‡ was able to reduce the ionization 12 per cent. by immersing his cylinder in a reservoir of water to a depth of several feet. The greatest decrease, however, was that observed by McLennan and Burton§, who cut off 37 per cent. of the conductivity of the air enclosed in a galvanized iron cylinder, by surrounding it with a layer of water drawn from Lake Ontario, 60 cms. thick. This last result, taken in conjunction with an observation by Prof. McLennan that Lake Ontario water contains no appreciable radioactive impurities, gave rise to the idea that possibly a large body of water such as the lake itself might furnish an efficient screen for the earth's radiation.

The object of the present investigation, therefore, was to observe the conductivity of the air confined in metallic vessels possessing little if any radioactive impurity, and from the results to determine, if possible, what portion of the ionization was due to an intrinsic activity in the metal.

In carrying out the investigation, the conductivity of air confined in vessels of lead, zinc, and aluminium was measured

* H. L. Cooke, *Phil. Mag.* [6] vi. p. 403 (1903).

† Elster and Geitel, *Phys. Zeit.* Nov. (1) 1905, p. 733.

‡ H. L. Cooke, *Phil. Mag.* 1903, *l. c.*

§ McLennan and Burton, *Phys. Rev.* 3. 1903; Burton, *Phys. Rev.* 3. 1904.

at a number of points in the neighbourhood of Toronto, both on land and over the water of Lake Ontario; and it was found that while a uniformly low and steady value was obtained for the conductivity over the water at all depths beyond a few metres, values varying over a wide range were obtained for the ionization, in measurements made on land at different places and on different soils.

The lowest values for " q ," the number of ions generated per c.c. per sec. in air confined in the metallic cylinders, were obtained in measurements on the surface of the lake and on the top of large masses of sand on the lake-shore. With a lead receiver under these circumstances the value 8.6 ions per c.c. per sec. was obtained for " q ," and with zinc and aluminium cylinders under the same conditions the values 6.00 and 6.55 respectively. These values, it will be seen from Table I., are considerably below those hitherto recorded for the conductivity, obtained under any circumstances, of air contained in closed metallic receivers.

TABLE I.

Receiver.	Observer.	q .	Conditions.
Lead	Eve ¹ .	96	} Observations in Physical Lab., McGill, Mon- treal.—Unscreened.
Zinc	"	24	
Aluminium ...	"	24	
Brass	H. L. Cooke ² .	13.6	} Unscreened. Screened by large masses of lead. Observations taken in basement of Univer- sity Library. McGill.
"	"	9.1	
Lead	McLennan ³ .	23	} Unscreened. Measurements made in old Physics department, Toronto University.
Zinc	"	15	
Aluminium ...	"	15	
Lead	Wright.	8.6	} Measurements made over the surface of Lake Ontario. — Receivers otherwise unscreened.
Zinc	"	6.0	
Aluminium ...	"	6.55	

¹ Eve, Phil. Mag. Sept. 1906.

² H. L. Cooke, Phil. Mag. vi. p. 403 (1903).

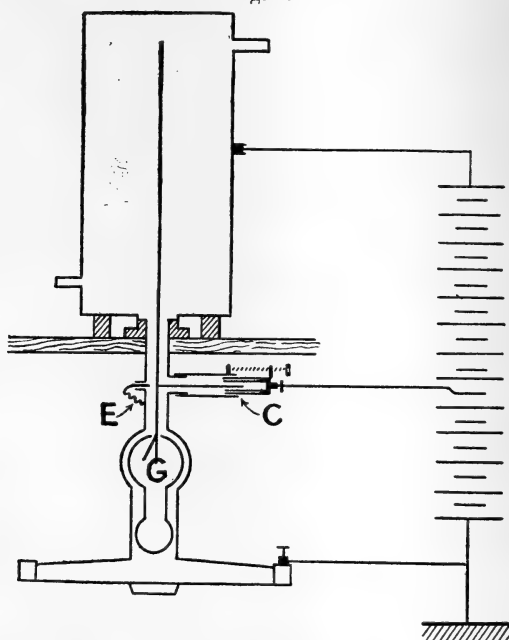
³ McLennan, Phil. Mag. Dec. 1907.

2. APPARATUS.

Measurements such as were contemplated on the ionization in metal receivers over the surface of the lake required the use of some instrument which would be portable, and at the same time not easily put out of adjustment. The electro-scope recently devised by C. T. R. Wilson was found to fulfil all the requirements.

The instrument, fig. 1 and fig. 1a (Pl. IV.), consisted essentially of a gold-leaf system G, insulated from the outer case by a quartz ring and suspended inside a similarly insulated inner case connected with a quartz leyden-jar of about 100 cms. capacity charged to a potential of +50 volts.

Fig. 1.



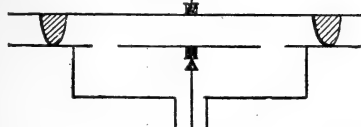
In making a measurement on the conductivity of the air with this instrument, the metal receiver was placed on top of the electro-scope as shown in fig. 1, and supported on ebonite blocks so as not to be in electrical connexion with the instrument proper. The receiver was then charged to any desired voltage, and the current through the air observed by the charge which was communicated in a given time to the electrode carried by the gold-leaf system.

The function of the compensator C, which consisted of a sliding tube-condenser, was to annul by its motion any charge

the system acquired through the conductivity of the air in the receiver; this motion being so regulated as to keep the potential of the gold-leaf always at zero, and thus minimise any tendency to promote a leak across the quartz insulation.

Thus for a determination of " q " it was necessary only to know the charge annulled in the time during which the compensator moved a standard distance.

Fig. 2.



For a determination of the charge corresponding to this standard distance the parallel plate condenser shown in fig. 2 was added in place of the electrode. The compensator tube was then charged to a known potential giving for the total motion a certain deflexion of the gold-leaf; and the voltage on the upper condenser plate was then so adjusted as to bring the gold-leaf back to its zero position. The charge could then be readily calculated from the dimensions of the parallel plate condenser and the voltage applied to it.

The corresponding values of voltage on compensator tube and charge annulled are given in Table II., and are illustrated by the curve in fig. 3 (Pl. V.).

These values were determined experimentally, and they showed that the capacity of the tube-condenser was, as its construction demanded, practically independent of the voltage applied to it.

TABLE II.

Voltage on Compensator = V.	Charge Annulled = E.	E/V.
	Electrostatic Units.	Charge (E.S.U.) per volt.
28.4	.1449	.00512
36.7	.1858	.00506
42.9	.2146	.00500
51.1	.2577	.00504
57.2	.2853	.00499
65.4	.3241	.00495
71.4	.3528	.00494
78.3	.3882	.00496
		Mean = .00501

From the following calculations it will be seen how the different constants of the instrument were used in making a determination of the value of " q " for the air contained in a certain lead receiver.

Reading of Compensator distance.	Time.	Remarks.
Scale Divisions (cms.).	min. sec.	
.254	1 02	Date of Experiment, Mar. 19th, 3 P.M.
.508	1 45	
.762	2 29	Locality. — Basement of Physics Laboratory.
1.016	3 15	
1.270	4 05	Potential applied to Cylin- der = +85 volts.
1.524	4 48	
1.778	5 33	Potential applied to Com- pensator = +13.0 volts.
2.032	6 17	
2.286	7 05	Vol. of Cylinder = 26700 ccs.
3.302	8 09	

From Table II. the charge annulled per volt on the compensator tube = .00501.

Therefore total charge annulled in 489 sec. was

$$.06513 \text{ e.s. units.}$$

Thus $.06513 = 3.4 \times 10^{-10} \times q \times 489 \times 26700$,

or $q = 14.67$, assuming the charge on an ion to be

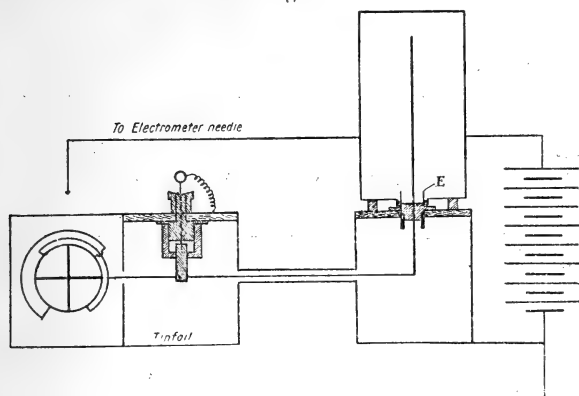
$$3.4 \times 10^{-10} \text{ E.S.U.'s.}$$

As a check on the values for " q " obtained for the ionization in the metallic receivers with this instrument, determinations were alternately made in the laboratory with the Wilson electroscope and with a sensitive Dolazelek electrometer. The method of using the latter is shown in fig. 4. With this arrangement it was possible, by simply removing the receiver, unscrewing the electrode, and slipping a metal cap over the earthed tube E, to allow for any charge acquired by the electrodes and the free quadrants through possible defective insulation of supports or by conduction through the air in any part of the apparatus, other than through that in the metal receiver.

The capacity of the instrument was found by measuring in each determination of the conductivity, the rate of motion of the needle both with an auxiliary condenser of known capacity attached to the electrode, and without.

To show how the determination of " q " with this instrument was arrived at, and to give an estimation of the magnitudes

Fig. 4.



involved, a measurement taken with the same cylinder and under the same conditions as obtained in the case of the determination with the Wilson instrument, is given in Table III. These values for " q ," as obtained with the two instruments, are seen to be practically the same, and so afford a conclusive test of the accuracy of the measurements.

TABLE III.

- (1) Date March 19, 4.00 P.M.
- (2) Lead Cylinder No. 1.
- (3) Locality.—Basement of Physics Laboratory.
- (4) Potential applied to needle +250 volts.
- (5) Rate of motion of needle without condenser=13.0 mm. per min.
- (6) Rate of motion of needle with condenser=+7.4 mm.
- (7) Rate of motion of needle due to defective insulation cylinder being removed and electrode capped =+1.0 mm.
 \therefore Rate due to conduction in air in cylinder alone
 =+12.0 mm.
- (8) Sensitiveness=832.2 mm. per volt.
- (9) Capacity of auxiliary condenser=128.04 cms.
- (10) Temp. 10° C., voltage on cylinder+85 volts.
- (11) Volume of cylinder=26700 c.cs.

From this we obtain

Capacity of electrometer 169.1 cms.

The calculation to determine " q " is then

$$\frac{169.1 \times 12}{832.2 \times 300} = q \times 3.4 \times 10^{-10} \times 60 \times 26700.$$

Whence $q = \underline{\underline{14.93}}$.

Before making the final measurements in these comparisons preliminary observations were made to ascertain what voltage it was necessary to apply to a receiver in order to obtain saturation currents. A set of the different voltages applied and the corresponding values of the currents obtained with a lead cylinder of comparatively high activity, but having dimensions the same as cylinder No. 1 mentioned above, are given in Table IV., and a curve to represent them in fig. 5 (Pl. V.). From the latter it may be seen that a potential of 60 volts gave a reading well over the knee of the curve, and as the voltage applied in all the determinations lay between 80 and 250 volts, it is clear that saturation currents were always obtained.

TABLE IV.

Voltage on receiver (positive).	" <i>q</i> " = No. of ions per c.c. per sec.
2	13.6
4	21.3
6	26.2
8	29.3
11.2	33.3
17.7	37.8
34.0	40.8
47.0	41.6
63.3	42.9
81.6	42.9
163.2	43.3
251.2	44.2

3. SOME SPECIAL OBSERVATIONS WITH LEAD CYLINDERS.

It will be remembered that Prof. McLennan, in his investigation of last year, drew the conclusion that ordinary commercial lead, in so far as he investigated it, contained in general an active impurity. Such being the case, one should expect a certain falling off in the course of time of the ionization in a lead receiver, due to the decay of the activity. It was decided then to determine again the conductivity of air in certain lead receivers used by him in June last in his investigation, to see whether any decrease from the former values could be noted. Lead cylinders described as numbers 1, 2, and 3 in Prof. McLennan's paper were used for this purpose, and the values of the ionization for each cylinder determined with both the electroscope and the electrometer. These numbers, which are given in Table V., show a considerable decrease in the value of "*q*" from the values for each receiver obtained by Prof. McLennan.

TABLE V.

March 6th.

q_0 = Number of ions per c.c. per sec. reduced to Zero centigrade.

Lead Cylinder.	By Electroscope.	By Electrometer.	Values obtained by Prof. McLennan.
	" q_0 ."	" q_0 ."	" q ."
No. 1	15.14	15.42	23
No. 2	135.8	139.6	160
No. 3	28.55	29.21	37

This result seemed somewhat surprising in view of the fact that Prof. McLennan had obtained the same numbers repeatedly during a period of over six months. As the experiments made by the writer with the three lead cylinders were carried out in a room in the new physical laboratory, and those by him in the older building, it was thought well to make a redetermination in the latter. On doing this, however, it was found that a slightly lower value even was obtained for the ionization in the measurements taken in this room, than in those in the new laboratory.

Although the experiments were made in a room supposed to be free or far removed from any active substances, it seems probable that there was in the old department, or in the rooms adjacent, some unobserved source of radiation present during Prof. McLennan's investigation which was absent during the measurements made by the writer, and this impurity was very probably removed when the old laboratory was vacated by the department of Physics and adapted to other purposes.

It might be pointed out that the lowest value for " q " hitherto recorded for the ionization in a lead cylinder, even with this possible additional influence, is that of 23 ions per c.c. per sec. given by Professor McLennan. The present value of 15 obtained with this cylinder is still lower, and would seem to indicate that we possessed in this lead receiver, one which contained little if any active impurity. It was therefore especially suitable, if proper screens could be found, for the investigation of any intrinsic activity associated with the metal lead itself.

4. PRELIMINARY OBSERVATIONS ON CONDUCTIVITY.

(a) *General Conditions of the Different Experiments.*

It has been noted by Prof. McLennan and other observers that when a metallic receiver has been thoroughly scoured with emery- or glass-paper, in order to remove any active coating which may have been deposited on its surface by exposure to the atmosphere, and after being washed with hydrochloric acid, ammonia, alcohol, and distilled water, then filled with freshly filtered air and allowed to stand, the ionization of the enclosed air gradually increases for some days, and ultimately reaches a steady value. Owing to this effect it was found necessary, in comparisons of the ionizing power of radiations existing in any two localities or under any two conditions, to make the observations with the ionization chamber under precisely the same conditions in both circumstances.

The different comparisons were made, therefore, either with a receiver freshly cleaned and freshly filled with filtered air immediately before taking the observations, or else with a receiver containing air which had been allowed to remain in it till the steady state had been reached.

Further, as it was impossible always to obtain observations at different times with the atmospheric conditions the same as to pressure and temperature, it was assumed that the ionization obtained in all localities would vary directly with the density of the air in the receiver, and in making any reductions which were necessary in order to reach values which were comparable, this assumption, which is amply warranted by the measurements of McLennan and Burton *, on the ionization of air at different pressures, has been adopted.

In this connexion it may be well to emphasize the extreme importance of taking every precaution, in making observations such as are described in the present paper, to secure absolute uniformity in the conditions of the measuring receivers. With the different receivers used in the present investigation, it was found that when the cylinders were thoroughly scoured and washed in the manner described above, the conductivity of freshly filtered air admitted into the chamber was always the same at any particular observing station, and thus by always working under these definite conditions it was possible to obtain very definite results.

Numerous investigators in this field of research have experienced considerable difficulty in arriving at concordant

* McLennan and Burton, Phys. Rev. 3. 1903.

results, but if the precautions mentioned are taken, it is possible to obtain a thoroughly reliable value for the conductivity of air enclosed in any metal receiver.

(b) *Daily Variation in the Conductivity of Atmospheric Air.*

Among other investigators, Wood and Campbell*, McKeon†, and Strong‡ have observed daily variations in the conductivity of air confined in metallic vessels; and inasmuch as it was not practicable to take observations in different localities at the same time of day in the present investigation, a preliminary set of observations was made throughout different days in a room in the Physics building, on the conductivity of the air confined in a lead receiver, in order to obtain evidence of the daily variation, and also, if such existed, to obtain an estimate of its magnitude.

TABLE VI.

Lead Cylinder 1, in steady state.—Voltage on Cylinder + 83.

Time.	q_0 =number of ions per c.c. per sec. reduced to 0° C.
Dec. 24th.	
10.23 A.M.	22.48
10.43	22.51
11.10	22.54
11.32	22.45
11.55	21.74
12.15 P.M.	22.73
12.40	21.96
1.00	22.94
Dec. 26th.	
9.35 A.M.	22.44
9.58	21.86
10.20	21.87
10.45	22.25
11.05	22.96
11.28	22.79
11.50	22.53
3.55 P.M.	22.65
4.15	22.54
4.40	22.94
	Mean <u>22.45</u>

Two sets of continuous readings taken in this way with the air in a lead cylinder in the steady state mentioned above, are given in Table VI., and from the values quoted it

* Wood and Campbell, *Phil. Mag.* Feb. 1907.

† McKeon, *Phys. Rev.* Nov. 1907.

‡ Strong, *Phys. Zeit.* Feb. 15th, 1908.

will be seen that there is no evidence of any appreciable regular variation in the conductivity. It is to be noted, too, that the extreme values obtained did not differ from the mean conductivity by more than 3 per cent. of the latter.

In the results which will be given later, it will be seen that variations amounting to as much as 50 per cent. were obtained in the conductivity, by a change in the observing station; and from the results obtained and given in Tables IX., X., and XI., it will be seen that any variation in conductivity due to daily changes in the value of the penetrating radiation from the earth which might have existed, were negligible in comparison with the variation in the conductivity due to a change in the point of observation.

(c) *Secondary Radiation from the Walls of a Room.*

In view of the existence of a penetrating radiation at the surface of the earth, having its origin either in the atmosphere or in the soil, and in view of the production by such radiations of secondary rays at the surface of substances traversed by them, it was thought advisable, before going on with the main part of the investigation, to see how far the influence of a secondary radiation excited in the walls of a room could be detected from those walls.

To obtain some information on this point, two plans suggested themselves: (1) to place the conductivity chamber at a selected distance from a wall, and to study the secondary rays excited at that wall by a quantity of radium placed at points on a circle with the chamber at its centre. (2) To study the variation in the ionization in a metal receiver with the radium at a fixed distance from the wall, and the chamber placed at points on a circle with the radium as centre. A set of measurements was made by following the first plan, but time has not permitted the carrying out of a series of observations with the second arrangement.

The electroscope provided with a zinc receiver in these measurements was placed at a distance of about one metre from the wall of a large room, and the radium enclosed in a lead box with walls 2 cms. in thickness was moved around the circumference of a circle, having as centre the cylinder and as radius the distance of the same from the wall.

The values for " n ," the number of ions per c.c. per sec., due to the primary radiations from the radium as well as the radiation emitted by the brick wall, together with the corresponding secondary rays excited in the metallic cylinder, are given in Table VII., and show a regular decrease in the ionization as the distance of the radium from the wall was

increased, with indications of a possible smaller maximum for a definite angle subtended at the wall by the line joining the radium with the electroscope. A curve representing the variation of "*n*" with the distance of the radium from the wall is given in fig. 6 (Pl. V.). From these results it is clear that for all positions of the radium a secondary radiation of considerable intensity was emitted by the brick wall under the excitation of the rays from the radium.

TABLE VII.

Cylinder of zinc.

Potential of cylinder = 250 volts, positive.

Radius of circle on which radium was placed about 1 metre.

Distance of radium from wall (cms.).	Ionization (arbitrary scale).
5.5 cms.	72.93
10.0	71.74
16.2	69.16
23.4	69.83
28.6	69.16
63.6	67.94
98.0	66.31
146.0	65.70
5.5	72.93

In order to obtain further information regarding this secondary radiation the instrument, provided with an aluminium cylinder, was moved to one of the corners of the room, and a similar set of observations made, the values obtained for "*n*" in this case being given in Table VIII. (p. 308) and illustrated by the curve in fig. 7 (Pl. V.).

Here again it is seen that the effect of the secondary radiation was quite marked. As the results show, the ionization steadily decreased to a minimum value, which corresponded approximately to the position in which the radium was on the line joining the electroscope to the corner of the room. After passing through this minimum value the ionization then steadily increased, and reached a maximum when the radium was slightly beyond a line drawn from the cylinder perpendicular to the far wall. After this the ionization fell away again as the distance of the radium from both walls was increased.

The maximum variation in the values of "*n*" for these

TABLE VIII.

Distance of aluminium receiver from far wall = 300 cms.

Distance of aluminium receiver from near wall = 105.5 cms.

Potential of receiver = 250 volts positive.

Radium placed on circle 100 cms. radius with receiver at centre.

Distance of radium from near wall.	Ionization (arbitrary scale).
5.5 cms.	51.86
10.9	50.89
20	50.68
41.4	49.95
55.1	49.13
76.4	49.80
110.9	50.63
153.7	50.00
197	48.08

experiments, it will be seen from Tables VII. and VIII., amounted to as much as 10 per cent. It seemed reasonable, therefore, to suppose that the presence of some such object as a brick wall might, when the penetrating rays from the earth impinged upon it, in the same way affect the natural ionization in any metallic cylinder; and care was therefore taken in the measurements made when determining the screening effect of the lake and of different soils, to place the electroscope and receiver as far away as possible from any building which might modify in some such way as that indicated, the ionization of the enclosed air.

5. SCREENING EXPERIMENTS.

Being assured from the foregoing experiments that changes in the ionization, due to daily variations, were inconsiderable in comparison with variations due to a change in position, three series of measurements were then undertaken with the object of investigating the screening effect of Lake Ontario, care being taken to choose positions of observation as far as possible from any artificial surroundings. The first and second series were made with lead cylinders after the ionization had reached the steady state, and the third with well-cleaned cylinders of lead, zinc, and aluminium containing freshly filtered air. A considerable decrease in the ionization when measured over the water was noticeable in every case;

a reduction of as high as 60 per cent. being recorded in the case of the freshly cleaned lead cylinder in the third series of measurements referred to above.

(1) *Measurements on board steamer 'Corona.'*

In the first set of measurements a series of observations was made in the laboratory on the conductivity of the air enclosed in a lead receiver which had not been recently cleaned, and the mean of these readings was found to give a value of 41.7 ions per c.c. per sec. for the conductivity. Measurements were then made on the same day, on board the s.s. 'Corona' during one of her voyages, and also at a number of points on land on the south side of Lake Ontario between Queenston Heights and Niagara Falls. These results are recorded in Table IX.

TABLE IX.

Comparison of Conductivity Experiments made on steamer 'Corona' with those made on Land.

" q_0 " = Number of ions generated per c.c. per sec. at 0° C.	Locality.
41.7 (mean value)	Physical Laboratory.
34.9 } 35.6 } 35.2 }	On board 'Corona.' Outward trip.
43.3	Queenston Heights.
42.4	Pavilion, Niagara Falls Park.
42.0	Tunnel of Ontario Power Co. 42 metres underground..
35.9 } 35.5 } 36.5 }	On board 'Corona.' Homeward trip.

Mean of results of land experiments = 42.3 ions per c.c. per sec.

Mean of results of steamer experiments = 35.6 ions per c.c. per sec.

Difference = 6.7 ions per c.c. per sec.

From the table it will be seen that the ionization at different points on the limestone soil of the Niagara District was practically constant. It will be seen, too, that the ionization obtained about 42 metres underground at the Falls was practically the same as that obtained at the surface on the limestone ridge. We see also from the figures, that the

screening action of the lake, minus any effect due to an intrinsic radiation from the boat itself, is represented by the value of 6.7 ions per c.c. per sec. with the particular lead cylinder used. From the figures which are given later in Table X., it will be seen that the screening effect of the lake in a cylinder of lead in the condition of that used in the above experiment, is represented by 9 ions per c.c. per sec., which shows that approximately 2.3 ions per c.c. per sec. must have been due to a radiation emitted by the steamer or by its contents.

(2) *Measurements made along the Water front of Toronto Bay.*

A second series of measurements was made with the lead cylinder referred to in the first portion of the paper as No. 1, after it had reached what has been referred to above as the steady condition, observations on the ionization being taken at different points along the water front of Toronto Bay.

The results given in Table X. show a total decrease of 9 ions per c.c. per sec. from the value of " q " obtained in the laboratory due to the screening action of the water, and they seem to indicate in addition that the ionization over sandbanks washed up by the waves was but little greater than that over water, even of a considerable depth.

TABLE X.
Dec. 28th.—Lead Cylinder 1, in steady state.

" q_0 " = Number of ions per c.c. per sec. (mean value).	Locality.
22.5	Laboratory of new Physics Building.
19.3	Under York St. bridge— "made land," 30 m. from water.
14.1	South end of west side of Eastern Gap, 3.40 m. from shore. Water 4.6 m. deep.
14.2	Sand-spit of Ward's Island, 110 m. from shore.
13.5	Toronto Canoe Clubhouse. Light pine structure on piles. Water 5.6 m. deep.

This result is probably due to the fact that any radioactive substances originally present in the sand have been washed away by the action of water.

In this connexion it is important to note that in making these determinations of the conductivity of air enclosed in lead receivers, it was frequently observed that in the measurements on the surface of the lake, with newly cleaned lead cylinders filled with freshly filtered air, the drop in conductivity observed was invariably about 50 per cent. less than the drop obtained with lead cylinders, which, after being well cleaned and filled with clean air, had been allowed to stand long enough to reach the steady state.

This difference in the drop in conductivity is well illustrated by the results obtained with the lead cylinder No. 1. With this cylinder in the steady state, as the numbers in Table X. show, the conductivity when measured in the laboratory corresponded to the generation of 22.5 ions per c.c. per second. With the same cylinder freshly cleaned and filled with well-filtered air the conductivity, as measured in the same room in the laboratory, always corresponded to the generation of approximately 15.3 ions per c.c. per second. In the measurements on the surface of the lake-water, however, the conductivity corresponded to the generation of 13.9 ions per c.c. per second when the cylinder was in the steady state; while, as will be seen from the results recorded in the next section, it corresponded to the generation of only 8.6 ions per c.c. per second when the cylinder had been freshly cleaned and filled with filtered air.

This difference in the values obtained for the drop in conductivity with the lead cylinder in the two conditions, can no doubt be traced to differences in the secondary radiation excited in the walls of the vessels by the penetrating radiation from the earth.

It is clear that the surface of the lead after being freshly cleaned must have gradually become covered with a deposit through oxidation and other causes, and it is reasonable to conclude that the presence of this deposit would produce such a modification in the intensity of the secondary radiation as to bring about the results described.

(3) *Measurements made in different localities and with different receivers freshly cleaned.*

The preliminary measurements just described made it abundantly evident that the lake water acted as a very efficient screen for the earth's radiation, a maximum decrease in the value of "*q*" of as much as 9 ions being recorded in

the last series of observations. A careful set of measurements was therefore undertaken, having for its object a determination of the relative decrease in the values of " q " over water, from those obtained in the laboratory, when freshly cleaned receivers of lead, zinc, and aluminium were used in turn as the containing vessels. In the case of the lead cylinder the conductivity was measured at a larger number of points, to determine if possible in what way the ionization was influenced by external conditions such as a change of soil.

The results, which are in many cases the mean of a number of observed values obtained on different occasions and differing but slightly from one another, are given in Table XI.

From the observations with the lead cylinder it will be seen that the ionization obtained over water of different depths was practically constant, the value for " q " being as before but slightly less than that obtained above a sandy soil. It is important to note also that the ionization on top of the University tower was the same as that found below on the campus, indicating that the atmosphere could not be the source of the penetrating radiation which gave the variations noted in the values of " q " for air confined in a closed metallic receiver.

From the values of " q " given in Table XI. for the three receivers, we obtain for the difference between the ionization in the laboratory and over the water the numbers 6.7, 7.4, and 6.0 for the receivers of lead, zinc, and aluminium respectively. These values then may be taken as giving a measure of the relative ionization in the three receivers due to that portion of the radiation from the earth which was cut off by the water, together with the secondary rays induced by this radiation in the different cylinders. That the actual numbers obtained for " q " at any observing station were not in the same ratio as these decreases, is a conclusive proof that the ionization measured at these stations was not due entirely to the radiation from the earth.

It might be well to call attention again to the fact that these decreases in the receivers of lead, zinc, and aluminium, of 6.7, 7.4, and 6.0, which are in the ratio of 1.1 to 1.23 to 1.00, must give a true measure of the total ionization in the three receivers which is due to a portion at least, if not the whole of the penetrating radiation from the earth. If, then, we could obtain cylinders of lead, zinc, and aluminium free from active impurities and possessing no intrinsic activity, we should expect the values for " q " at every point on the surface of the earth to be in this ratio.

Emphasis might also be laid upon the extremely low values

TABLE XI.

Observation Station. Number of Location Station.	Remarks.	Value of "q," the no. of ions generated per c.c. per sec.		
		Lead.	Zinc.	Aluminium.
1. The ice on Toronto Bay, 100 metres from the R. C. Yacht Club Wharf.	Water 10 metres deep. Ice, 30 cms. thick.	9		
2. The ice near Canoe Club Wharf, Toronto Bay.	Water 5.6 metres deep. Ice 30 cms. thick.	8.6	6.0	6.55
3. The ice on Grenadier Pond, an inlet of Lake Ontario. Station "a."	Water 2.5 metres deep. Ice 30 cms. thick.	9.2		
4. The ice on Grenadier Pond. Station "b."	Water 3.5 metres deep. Ice 30 cms. thick.	9.0		
5. At the shore of Grenadier Pond. Station "c."	9.1		
6. On a sand hill over- looking Grenadier Pond.	Height of hill about 50 metres. Dis- tance from shore 100 metres.	9.3		
7. University Lawn	Clay soil, sodded, frozen and covered with ice and snow 20 cms. deep.	11.2		
8. University Tower	26 metres high.	11.4		
9. In the open on the ground in the rear of the new Physical Laboratory.	Frozen clay, recently overturned, covered with 10 cms. of ice & snow.	13.2	11.1	10.4
10. A room in the basement of the new Physical Laboratory.	As the building is new it was supposed to be free from any radio- active contamination.	15.3	13.4	12.5

found for the ionization over the water when the cylinders were freshly cleaned and freshly filled with filtered air. These values of 8.6, 6, and 6.5 obtained for the receivers of lead, zinc, and aluminium are very much lower than those obtained by any other observer under any conditions, and

afford a conclusive proof of the efficacy of Lake Ontario water as a screen for the earth's penetrating radiation.

The experiments made at stations 5 and 6 are of special interest, as the conductivity obtained at these points was practically the same as that obtained in the experiments on the surface of the Lake which shows that the sand was entirely free from the radioactive substances which were probably present in the clays and rocks at other points of observation.

6. SECONDARY RAYS INDUCED BY THE γ RAYS FROM RADIUM..

To determine if this ratio of 1.1 to 1.23 to 1.0 for the three receivers of Pb, Zn, and Al would be found to hold also in the case of ionization due to the γ rays from radium, a series of measurements was undertaken with the electrometer used before.

A quantity of radium bromide encased in a lead box with walls 2 cms. thick, was used as the source of the penetrating rays, and the ionization measured in each receiver for different distances between it and the radium. The difference between the ionization before and after the radium was placed in position, was recorded as due to the primary rays from the radium together with the secondary rays excited by them in the receiver. From Table XII. it will be seen that " n ," the value of the ionization in each receiver due to the presence of the radium, varied inversely as the square of the distance d from the cylinder, the same variation being shown graphically in fig. 8 (Pl. V.). In this connexion it may be noted, however, that for distances less than 3 metres a much larger value for " n " was obtained than was demanded by the law of the inverse square.

TABLE XII.
Voltage on Cylinder : +250.

d = distance of Radium from Cylinder (metres).	Lead.		Zinc.		Aluminium.	
	n .	nd^2 .	n .	nd^2 .	n .	nd^2 .
6.0	223.5	6847	154.7	5569	135.4	4874
5.5	262.6	6933	181.9	5866
5.0	314.6	7022	229.6	5740	195.3	4882
4.5	381.9	7047	279.8	5665	246.3	4987
4.0	478.3	6899	359.7	5755	317.1	5074
3.5	7371	462.0	5668	421.7	5166
3.0	842.5	5206
	Mean	=6976	Mean	=5711	Mean	=4996

This variation from the law appeared at first sight somewhat difficult of explanation. On consideration, however, of the results obtained in the previous section, it at once seemed evident that the variation of " n " from the value for it demanded by the law of the inverse square was but another manifestation of the same secondary radiation excited by the presence of the radium in the neighbourhood of the brick wall, against which the electrometer was set up.

In Table XII. the value of the constant " nd^2 " has been calculated, giving for the receivers of Pb, Zn, and Al respectively the numbers 6976, 5711, and 4996, which are thus a measure of the ionization in the different cylinders due to the γ rays from radium *plus* the secondary rays induced by them in the enclosing metals. These numbers we see are in the ratio of 1.4 to 1.14 to 1.0 for the three metals Pb, Zn, and Al; whereas the ratios arrived at from the figures of Table XI. for the corresponding effects due to the earth's radiation were 1.1 to 1.23 to 1.00. The considerable difference between these ratios would thus seem to indicate a difference in penetrability between the γ rays from radium and the penetrating radiation from the earth. It is possible, however, that the discrepancy may have had an entirely different origin, and further measurements should be made to ascertain the cause of it before a satisfactory explanation can be offered.

7. EXPERIMENTS ON ABSORBING POWER OF WATER.

From the foregoing experiments with radium in conjunction with the effect noticed in 4 (c), we see that there was in each case some additional effect inside the cylinder which must be considered as due to the presence of the wall. From these experiments the idea presented itself that possibly the earth's penetrating radiation was the same at all points on its surface, and that the differences observed in the values for " q " for the land and water experiments were due not so much to differences in the absorbing power of the different soils, as to differences in a secondary radiation induced in the crust of the earth by this penetrating radiation.

To determine if any effect of this kind could be noted in the case of water for the penetrating rays from radium, the sample used in the investigation described in Section 4 was lowered under the ice on Grenadier Pond, and the ionization noted in a lead cylinder placed above it for different depths of the radium. In making these measurements the radium was hermetically sealed in a glass tube, which was

then enclosed in a tube of brass whose walls were 1 cm. in thickness.

The results (Table XIII.) showed that 2 metres of water completely cut off all effect from the radium, both primary and secondary, the same being illustrated by the curve in fig. 9 (Pl. V.).

TABLE XIII.

Lead Cylinder 1: 113 cms. above ice.

Depth of water over radium= d .	Number of ions due to radium=" n ."
0 metres.	4485
$1/2$	447.2
1	16.11
2	.69
3.65 at bottom.	.62

This result, surprising as it at first sight appeared, is exactly what one would expect from a consideration of the values of Table XI., which show that practically the same value for the ionization was obtained over water of depths ranging from 2.5 to 10 metres. Owing also to this fact that complete absorption of the γ rays from radium took place, it is clear that over the water of Lake Ontario at least there is no appreciable secondary effect due to the earth's penetrating rays such as has been shown to be emitted by a brick wall under bombardment by the γ rays from radium.

Since in addition we know from an observation made by Prof. McLennan that the waters of Lake Ontario contain no appreciable radioactive emanation, the conclusion is forced upon us that in the case of the experiments described above with the cylinders of lead, zinc, and aluminium, the water of the lake acted as a perfect screen for the earth's penetrating radiation, and the values of " q " recorded in Table XI. for Observation Station No. 2 may be taken to represent the conductivities impressed upon the air, either by intrinsic radiations arising from the metals of the receivers themselves, or else by radiations from active impurities still remaining in them.

Taking in the case of the lead cylinder the value of 8.6 ions per c.c. per sec. as due to intrinsic activity, we have a means then of calculating the ionization in this cylinder due to the soil alone in any position on the surface of the earth.

From the value of 11.1 ions obtained with the lead cylinder over clay (given in Table XI.), a calculation shows that 2.6 ions per c.c. per sec. in such a cylinder is due to the soil alone. If now we assume, as seems justifiable from the experiments of Professor McLennan, *loc. cit.*, on the conductivity of air enclosed in lead receivers, that the ratio of the ionization due to secondary rays in a lead cylinder of the dimensions of those used in this investigation, is twice that due to the primary, it follows that approximately 0.9 ions would be generated in free air over a clay soil by the earth's penetrating radiation in the locality referred to above.

On the basis of Strutt's determination of the radium content of the rocks and soils, Strong* has recently deduced a value of 0.8 ions as an upper limit for the ionization in free air due to the penetrating rays from radioactive substances in the soil; and this number it will be seen is in good agreement with the experimental result obtained in this investigation.

In passing it might be noted that this calculation throws some light on the results obtained (Table VI.) for the regular daily variation, which it will be remembered was found to be inappreciable and which is exactly what should be expected if only some two or three of the ions generated in the cylinder per c.c. per sec. were due to a penetrating radiation from the earth.

8. SUMMARY OF RESULTS.

The results obtained in the preceding investigation may be summed up as follows:—

- (1) No evidence of a regular daily variation was noted.
- (2) It has been shown that there is a penetrating secondary radiation set up by penetrating rays such as those from radium, in the brick wall of a room.

- (3) Proofs have been adduced to show that the water of Lake Ontario acts as a perfect screen both for the earth's radiation and, if a sufficient depth be taken, for the γ rays from radium. On this account and owing to the fact that the water of Lake Ontario contains no active impurity, it has been possible to determine what portion of the ionization in

* Strong, *Phys. Zeit.* ix. pp. 117-119 (Feb. 15, 1908).

the receivers used in this investigation was due to residual active impurities and to intrinsic activity in the metals of the receivers.

(4) Based on this fact, a determination has been made of the ionization in free air due to radioactive impurities in a clay soil, and this value 0.9 ions per c.c. per sec. has been found to be in close agreement with a value deduced by Strong from Strutt's determination of the radium content of the earth.

(5) The ratio of the ionization in cylinders of lead, zinc, and aluminium due to the radiations from the earth has been determined, and has been found different from the ratio for the ionization due to the gamma rays from radium; a result which needs confirmation but which points to a difference in the penetrability of the two radiations.

(6) The values obtained in the open for the ionization in well cleaned receivers of lead, zinc, and aluminium are lower than any hitherto recorded, the numbers 8.6, 6.0, and 6.5 respectively being obtained over the water of Lake Ontario.

Considered as a whole, the experiments described above are interesting from the light which they throw on the question of the radioactivity of metals and substances generally. The values obtained for " q " for the three cylinders at Station 2, Table XI., differ from each other but little. They are moreover of the order of magnitude of effects which might easily be accounted for by active impurities in the metals, since differences as large as these values of " q " may easily be obtained with cylinders made from different samples of almost any metal selected at random. Considering also the difference in the atomic weights of the three substances, aluminium, zinc, and lead, and having in mind that radioactivity is a property associated with atomic structure, it would seem that, if these metals could be obtained entirely free from active impurities, and the conductivity of air contained in vessels made from them studied, it would be found, if the observations were carried out under conditions or in places where no ionization was possible from penetrating radiations arising from external sources, to drop to a very low value if it did not entirely vanish.

In conclusion, I desire to thank Professor McLennan, both for his valuable suggestions and for his assistance at all times throughout the investigation.

XXIV. *Note on the Theory of the Greenhouse.**By Professor R. W. Wood*.*

THERE appears to be a widespread belief that the comparatively high temperature produced within a closed space covered with glass, and exposed to solar radiation, results from a transformation of wave-length, that is, that the heat waves from the sun, which are able to penetrate the glass, fall upon the walls of the enclosure and raise its temperature: the heat energy is re-emitted by the walls in the form of much longer waves, which are unable to penetrate the glass, the greenhouse acting as a radiation trap.

I have always felt some doubt as to whether this action played any very large part in the elevation of temperature. It appeared much more probable that the part played by the glass was the prevention of the escape of the warm air heated by the ground within the enclosure. If we open the doors of a greenhouse on a cold and windy day, the trapping of radiation appears to lose much of its efficacy. As a matter of fact I am of the opinion that a greenhouse made of a glass transparent to waves of every possible length would show a temperature nearly, if not quite, as high as that observed in a glass house. The transparent screen allows the solar radiation to warm the ground, and the ground in turn warms the air, but only the limited amount within the enclosure. In the "open," the ground is continually brought into contact with cold air by convection currents.

To test the matter I constructed two enclosures of dead black cardboard, one covered with a glass plate, the other with a plate of rock-salt of equal thickness. The bulb of a thermometer was inserted in each enclosure and the whole packed in cotton, with the exception of the transparent plates which were exposed. When exposed to sunlight the temperature rose gradually to $65^{\circ}\text{C}.$, the enclosure covered with the salt plate keeping a little ahead of the other, owing to the fact that it transmitted the longer waves from the sun, which were stopped by the glass. In order to eliminate this action the sunlight was first passed through a glass plate.

There was now scarcely a difference of one degree between the temperatures of the two enclosures. The maximum temperature reached was about $55^{\circ}\text{C}.$ From what we know about the distribution of energy in the spectrum of the radiation emitted by a body at 55° , it is clear that the rock-salt plate is capable of transmitting practically all of it, while the glass plate stops it entirely. This shows us that

* Communicated by the Author.

the loss of temperature of the ground by radiation is very small in comparison to the loss by convection, in other words that we gain very little from the circumstance that the radiation is trapped.

Is it therefore necessary to pay much attention to trapped radiation in deducing the temperature of a planet as affected by its atmosphere? The solar rays penetrate the atmosphere, warm the ground which in turn warms the atmosphere by contact and by convection currents. The heat received is thus stored up in the atmosphere, remaining there on account of the very low radiating power of a gas. It seems to me very doubtful if the atmosphere is warmed to any great extent by absorbing the radiation from the ground, even under the most favourable conditions.

I do not pretend to have gone very deeply into the matter, and publish this note merely to draw attention to the fact that trapped radiation appears to play but a very small part in the actual cases with which we are familiar.

XXV. *Molecular Diameters.* By WILLIAM SUTHERLAND*.

GREATER absolute precision and better mutual consistency were introduced into the measurement of molecular diameters by the kinetic theory of gases, when Jeans applied electrical data to give the number of molecules of a gas in a cm.^3 under normal conditions (Phil. Mag. [6] viii. 1904, p. 692). But in his calculations he took no account of the effect of cohesive forces in the viscosities, conductivities, diffusivities, and collisional virials of gases which he used in the calculation of molecular diameters. Now in "The Viscosity of Gases and Molecular Force" (Phil. Mag. [5] xxxvi. 1893, p. 507) it was shown that if $2a$ is the diameter of a molecule, and C a parameter proportional to the mutual potential energy of two molecules in contact, T denoting absolute temperature, the molecules behave as if devoid of attractive force, but enlarged so that $(2a)^2$ is replaced by $(2a^2)^2(1 + C/T)$. Thus the temperature law of the viscosity of a natural gas is $\eta \propto T^{3/2}/(1 + C/T)$, instead of the law $\eta \propto T^{3/2}$ which holds for the ideal perfect gas. On this account the quantities given by Jeans as the diameters of molecules $2a$, are in reality $2a(1 + C/T)^{1/2}$. By means of the values of C and with $T=273$ it is easy to obtain the true values of $2a$. In the Landolt-Börnstein-Meyerhoffer *Tabellen* values of C for many of the gases in the list of Jeans are

* Communicated by the Author.

given, and also values of viscosity in other cases, enabling C to be calculated. In the table below I have given the mean value of C for each gas from the *Tabellen*, and in brackets those values calculated from data given in the *Tabellen*. Strictly the same sort of correction ought to be applied separately to each of the four methods used by Jeans for finding $2a$, but it is practically as useful at the present stage to apply the correction to his mean values obtained from the results of the four methods. These are given in the second column of the following table under the heading $2a(1+C/T)^{\frac{1}{2}} \times 10^8$, the last column giving $2a \times 10^8$. Water

Gas.	$2a(1+C/T)^{\frac{1}{2}} \times 10^8$.	C.	$2a \times 10^8$.
	cms.		cms.
Hydrogen	2.03	72	1.81
Helium	1.81	76	1.60
Carbon monoxide...	2.86	(156)	2.28
Ethylene	3.81	249	2.75
Nitrogen	2.91	110	2.46
Air	2.84	115	2.38
Nitric oxide	2.82	(195)	2.15
Oxygen	2.73	128	2.25
Argon	2.79	160	2.22
Carbon dioxide	3.36	258	2.41
Nitrous oxide	3.52	(167)	2.77
Chlorine.....	4.11	(199)	3.13
Water-vapour	3.39		
Ethyl chloride	4.68		

vapour and ethyl chloride at ordinary temperatures are too far from the condition of a typical gas as regards the conditions of collision to permit of a satisfactory estimation of C without special experiments and discussion.

Melbourne, September 1, 1908.

XXVI. *On the Radio-active Products present in the Atmosphere.* By W. WILSON, M.Sc., Graduate and Hatfield Scholar of the University of Manchester*.

SINCE the experiments of Elster and Geitel† in 1901, various experimenters working at different parts of the earth's surface have found that a wire suspended in the air and maintained at a high negative potential becomes radio-active. Bumstead‡ showed that the activity thus obtained

* Communicated by Prof E. Rutherford, F.R.S.

† Elster & Geitel, *Phys. Zeit.* ii. (1901).

‡ Bumstead, *Am. Jour. Sci.* 1904.

decays as if the disintegration products, both of the emanations of radium and of thorium, were present. He found that for a twelve hours' exposure the activity due to the thorium was about 15 per cent. of the whole. Blanc *, working in Rome, made a series of experiments to determine the ratio of the activities due to the two deposits, and found that for a long exposure (3 days) the activity due to the thorium deposit was 60 per cent. to 70 per cent. of the whole activity obtained.

Since it is of some importance to know the relative amounts of these substances at different parts of the earth's surface, the following experiments have been made at Manchester during the past year.

A copper wire 36 feet long, maintained at a negative potential of from 8000–12,000 volts by means of a Wimshurst machine, was suspended from an insulated support out of an upper window of the University Physical Laboratory. The potential was measured by means of a Kelvin Static Voltmeter. After the wire had been exposed to the atmosphere for a sufficient time (3 to 8 hours) it was removed and coiled round a square brass cage, which was then placed in an electroscope by means of which the activity could be measured from time to time. The natural leak of the instrument was determined before and after the experiment by taking a number of readings with a "clean" wire coiled round the cage. The wire when drawn in was always covered with soot particles.

It was found that the activity of the wire fell off with the time as if products of radium and thorium were present. The readings were continued for about 48 hours.

The whole decay curve of the thorium deposit could be deduced from the later readings, since the activity due to the radium deposit has practically disappeared after four hours. The decay curve of the radium deposit could then be obtained by subtraction.

The ratio which the activity of the thorium deposit would bear to that due to the radium, if the wire were exposed till equilibrium was obtained, was calculated for each experiment, and the mean value found was 1.64, or 62 per cent. of the initial activity is due to the deposit from thorium. The extreme values obtained were 59 per cent. and 69 per cent., but most were within 2 or 3 per cent. of the mean.

In a recent paper Dadourian † has calculated the relative amounts of radium and thorium present in the atmosphere near the earth's surface, on the assumption that the active

* Blanc, Phil. Mag. May 1907; *Phys. Zeit.* May 1908.

† Dadourian, *Le Radium*, April 1908.

deposits obtained are due to the direct disintegration of the emanations in the vicinity of the wire. He finds that there is 30,000 to 50,000 as much radium as thorium emanation in the air of New Haven, Conn., U.S.A., and from Blanc's results 20,000 to 30,000 at Rome.

We can calculate this ratio from our results as follows :—

When the emanation and active deposit are in equilibrium, an equal number of atoms of each product break up per second. Now the initial activity of the thorium deposit is due to α particles from ThB and ThC. The number of each breaking up per second is λN , where N is the number of atoms of emanation necessary to keep ThB and C on the wire in equilibrium, and λ its radioactive constant.

The initial ionization is then given by $\lambda N(R_1 + R_2)n$, where R_1 and R_2 are the ranges of the α particles from ThB and ThC, and n is the number of ions produced by an α particle per cm. of its path (supposed to be the same for all). Similarly, if λ' , N' , R_1' , R_2' are the corresponding values for the radium products, the ionization due to the radium deposit is $\lambda' N'(R_1' + R_2')n$.

The ratio $\frac{\lambda N(R_1 + R_2)n}{\lambda' N'(R_1' + R_2')n}$ has been shown to have the value 1.64, which gives $\frac{N'}{N} = 3700$ on substituting the known values $R_1 = 5.0$, $R_2 = 8.6$, $R_1' = 4.8$, $R_2' = 7.0$, $\lambda = \frac{1}{87}$, $\lambda' = \frac{1}{463,000}$.

In the same manner we find from the results given by Blanc and Dadourian the ratios 2600 to 4000 at Rome, and 9000 to 18,000 at New Haven. We see, then, that the radium and thorium emanations exist in the atmosphere in about the same proportion in this country as in Rome, while in America there is relatively less thorium.

The discrepancy between these results and those of Dadourian may be explained as follows :—

In equation (9) p. 105 of his paper he makes use of the formula

$$I = vR_3M_3,$$

where v is the number of ions produced by an α particle per cm. of its path, R_3 the range of the α particle, and M_3 the number of atoms of RaC present.

This equation should read $I = \lambda_3 vR_3M_3$, that is the ionization is proportional to $\lambda_3 M_3$ the number of atoms breaking up per second. If we substitute this in Dadourian's equation we obtain values corresponding to those given above. The

number obtained, however, is only the virtual ratio, the actual number being somewhat larger, since, as Blanc has shown, the active deposit present, especially at some distance from the ground, is not produced *in situ* but is carried away from the surface and remains suspended in the air for a considerable time before it is removed.

We can, however, form some idea of the amount of thorium present in the earth's crust from the foregoing results.

Since radium is a disintegration product of uranium the number of atoms of the radium emanation breaking up per second is equal to the number of atoms of uranium breaking up per second, the two being in equilibrium.

Similarly, the numbers of atoms of thorium and the corresponding emanation breaking up per second are equal.

Now, we have seen that there is per c.c. of the atmosphere near the earth's surface, 3700 times as much radium emanation as thorium emanation. On account of its longer life the radium emanation extends upwards into the atmosphere to a much greater extent than does that of thorium; and thus there is present in the whole of the atmosphere 3700 K times as much radium as thorium emanation, where K can be deduced from the coefficients of diffusion and the periods of the two emanations.

For the same reason, however, it can penetrate into the atmosphere from a depth in the ground equal to K times that from which the thorium emanation can emerge, and so comes from a volume K times as great.

The ratio then of the amounts of the emanations of radium and thorium which are in equilibrium with the uranium and thorium per c.c. of the earth's crust is $\frac{3700 K}{K} = 3700$.

We see from this that, taking the radioactive constants of the thorium and radium emanations as $\frac{1}{87}$ and $\frac{1}{463,000}$ respectively, there must be per c.c. of the earth's crust 1.42 times as many thorium as radium atoms breaking up per second.

Now, Bragg* has shown that the uranium atoms break down five times as quickly as the thorium atoms, which points to there being about 7 times as much thorium as uranium present per c.c. of the earth's crust. Uranium and thorium then exist in the earth in amounts of the same order of magnitude; and in calculations bearing on the heat of the

* Bragg, Phil. Mag. June 1906.

earth the action of the thorium family as well as that of the uranium family should be taken into consideration.

Strutt found that the average amount of radium present in the surface rocks was 5×10^{-12} gram per c.c. Taking the life of thorium as 10^9 years and that of radium as 2×10^3 years, we see from our previous values that there must be about 7×10^5 as much thorium as radium, *i. e.* there must be about 10^{-6} gram per c.c. This amount is of course still very small, only amounting to about 10^{-5} per cent. of the whole mass.

S. J. Allen, working in Pittsburg, Conn., has lately published a paper on "Radioactivity in a Smoke-laden Atmosphere"*. He finds the same type of results for negatively charged wires and obtains no activity on positively charged ones. With wires, however, suspended in the atmosphere without charge, he obtains about 25 per cent. of the activity found on a negatively charged one.

Similar experiments were made in the present case during smoky fogs, and although the wires when drawn in were always covered by a thick layer of dirt, no activity was observed either on a positively charged wire, or on one which had been suspended in the atmosphere for six weeks without charge.

I wish to thank Prof. Rutherford for suggesting the experiments and for his valuable help and encouragement from time to time.

Physical Laboratory,
The University, Manchester.
Sept. 1908.

XXVII. *The Transverse Oscillations of a Canal of Circular Section.* By ROBERT A. HOUSTOUN, M.A., Ph.D., D.Sc. †

THE problem of the transverse oscillations of water contained in a canal of circular section has not yet been solved, though Lord Rayleigh has made an approximate determination of the frequency of the slowest mode by a general method for the case, where the surface is at the level of the axis ‡. The object of this note is to show how the same case may be solved by the substitution of a series.

Take the origin at the centre of the circle. Let the axis of x be horizontal and the axis of y be measured positively

* Phys. Rev. Dec. 1907.

† Communicated by Prof. A. Gray, F.R.S.

‡ Lamb's Hydrodynamics, 3rd edition, § 256.

upwards. Let a be the radius of the section. Then ϕ , the velocity potential, must satisfy the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at all points within the semicircle, the condition

$$\frac{\partial \phi}{\partial n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

at all points on the semicircle and

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at all points on the surface.

The problem cannot be solved by transforming conformally the case of triangular section solved by Kirchhoff *, because when the triangle is transformed into the semicircle, the surface condition no longer holds.

Suppose that the time-factor is $e^{i\sigma t}$. Then the surface condition becomes

$$\sigma^2 \phi = g \frac{\partial \phi}{\partial y}.$$

Let ψ be the stream function. Then, omitting the time factor, we may assume

$$\phi + i\psi = A_0 + A_1(x + iy) + A_2(x + iy)^2 \dots\dots$$

If we write the constants A_n in the form $P_n + iQ_n$, we obtain

$$\phi = P_0 + \sum_1^\infty r^n (P_n \cos n\theta - Q_n \sin n\theta),$$

$$\psi = Q_0 + \sum_1^\infty r^n (Q_n \cos n\theta + P_n \sin n\theta).$$

This satisfies (1).

Expressing the surface condition in polars we obtain

$$\sigma^2 \phi = \frac{g}{r} \frac{\partial \phi}{\partial \theta} \quad \text{for } \theta = 0.$$

$$\text{and} \quad \sigma^2 \phi = -\frac{g}{r} \frac{\partial \phi}{\partial \theta} \quad \text{for } \theta = \pi.$$

Substituting the expression for ϕ , we find that the surface condition is satisfied, provided that

$$nQ_n = -\sigma^2 P_{n-1}/g. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* *Ibid.* § 255.

Using (4) we find that

$$\psi = Q_0 + \sum_1^{\infty} r^n (P_n \sin n\theta - \frac{\sigma^2}{g} \frac{P_{n-1}}{n} \cos n\theta).$$

In order to satisfy (2), ψ must be constant for $r=a$ and $\pi < \theta < 2\pi$. We may, without loss of generality, assume that $\psi - Q_0 = 1$. Then writing $a^n P_n = S_n$ and $\sigma^2 a/g = x$, we have

$$1 = S_1 \sin \theta + S_2 \sin 2\theta + \dots$$

$$-x(S_0 \cos \theta + \frac{S_1}{2} \cos 2\theta + \frac{S_2}{3} \cos 3\theta + \dots). \quad (5)$$

If the constants x , S_0 , S_1 , S_2 , &c. be chosen so that this equation holds for all values of θ between π and 2π , then ϕ satisfies all the prescribed conditions.

Since the series on the right of equation (5) must be symmetrical about OY, $S_0 = S_2 = S_4 = \dots = 0$. The calculation of the other constants is extremely laborious, and we shall retain only the first three. To solve for x , S_1 , S_2 , and S_3 , we require four equations. Since the series is symmetrical about OY, it is necessary to equate it to unity only for the range $\pi < \theta < \frac{3\pi}{2}$. Hence, putting $\theta = \pi$, $\pi + \frac{\pi}{6}$, $\pi + \frac{\pi}{3}$, and $\frac{3\pi}{2}$ in succession, we obtain from (5)

$$\begin{aligned} 1 &= -x \left(\frac{S_1}{2} + \frac{S_3}{4} + \frac{S_5}{6} \right), \\ 1 &= -\frac{S_1}{2} - S_3 - \frac{S_5}{2} - x \left(\frac{S_1}{4} - \frac{S_3}{8} - \frac{S_5}{6} \right), \\ 1 &= -\frac{S_1 \sqrt{3}}{2} + \frac{S_5 \sqrt{3}}{2} - x \left(-\frac{S_1}{4} - \frac{S_3}{8} + \frac{S_5}{6} \right), \\ 1 &= -S_1 + S_3 - S_5 - x \left(-\frac{S_1}{2} + \frac{S_3}{4} - \frac{S_5}{6} \right). \quad (6) \end{aligned}$$

Eliminating S_1 , S_3 , S_5 ,

$$\left| \begin{array}{cccc} 1, & \frac{x}{2}, & \frac{x}{4}, & \frac{x}{6} \\ 1, & \frac{1}{2} + \frac{x}{4}, & 1 - \frac{x}{8}, & \frac{1}{2} - \frac{x}{6} \\ 1, & \frac{\sqrt{3}}{2} - \frac{x}{4}, & -\frac{x}{8}, & -\frac{\sqrt{3}}{2} + \frac{x}{6} \\ 1, & 1 - \frac{x}{2}, & -1 + \frac{x}{4}, & 1 - \frac{x}{6} \end{array} \right| = 0. \quad (7)$$

The roots of this equation should give the frequencies of different modes. It has one real root, $x=1.345$, and two complex ones. From $x=1.345$ we obtain $\sigma = \sqrt{g/a} 1.160$. Lord Rayleigh finds for the gravest mode, $\sigma = \sqrt{g/a} 1.1644$.

If we substitute $x=1.345$ in equations (6) we find

$$S_1 = -1.764,$$

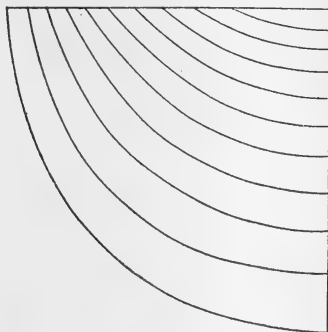
$$S_3 = .5895,$$

$$S_5 = -.0538.$$

Hence

$$\begin{aligned} \psi &= \left(\frac{r}{a}\right) S_1 \sin \theta + \left(\frac{r}{a}\right)^3 S_3 \sin 3\theta + \left(\frac{r}{a}\right)^5 S_5 \sin 5\theta - x \left(\left(\frac{r}{a}\right)^2 \frac{S_1}{2} \cos 2\theta \right. \\ &\quad \left. + \left(\frac{r}{a}\right)^4 \frac{S_3}{4} \cos 4\theta + \left(\frac{r}{a}\right)^6 \frac{S_5}{6} \cos 6\theta \right) \\ &= -\left(\frac{r}{a}\right) 1.764 \sin \theta + \left(\frac{r}{a}\right)^2 1.187 \cos 2\theta + \left(\frac{r}{a}\right)^3 .5895 \sin 3\theta \\ &\quad - \left(\frac{r}{a}\right)^4 .1982 \cos 4\theta - \left(\frac{r}{a}\right)^5 .0538 \sin 5\theta + \left(\frac{r}{a}\right)^6 .0121 \cos 6\theta. \end{aligned}$$

The accompanying figure represents the stream-lines calculated by the above formula. Since they are symmetrical about OY only one quadrant is given. Hence we see that with three constants we obtain a satisfactory representation of the slowest mode.



If we take more constants we obtain an equation of higher degree for x , and it seems that its other real roots may give the higher modes. I have, however, not been able to discover any way of discussing the general equation for x , and to calculate particular cases would be exceedingly laborious.

XXVIII. *On the Coefficients in certain Series of
Bessel's Functions.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN your December number, pp. 947–948, Mr. P. W. Bridgman refers to “an apparent oversight in the usual method of determining the coefficients in certain series of Bessel’s functions,” which he illustrates by reference to a paper by me read to the Cambridge Philosophical Society in 1887 and printed in their Transactions, vol. xiv. Owing probably to lack of familiarity with recent pure mathematical terminology, I am uncertain whether I fully understand Mr. Bridgman’s conclusion.

He seems to admit the accuracy of the differential equation

$$\frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} + \frac{d^2v}{dz^2} = 0,$$

and of the solution I gave—novel I think at the time— $v = \Sigma E \sinh kz J_1(kr)$ in my notation. What I understand him to question is the applicability of this solution to the torsional problem I supposed it to apply to, viz. the case where the shearing stress $\widehat{z\theta} \equiv n dv/dz$ is given over the end $z=l$ of a right circular cylinder of radius a (the end $z=0$ being fixed) as an arbitrary function $f(r)$ of the axial distance. By taking $J_2(ka)=0$ I was able to satisfy the condition $\widehat{r\theta}=0$ over $r=a$, and “following the usual methods”—i. e. making use of the fact that $\int_0^a r J_1(kr) J_1(k'r) dr$ vanishes if k and k' are two different roots of $J_2(ka)=0$ —I supposed the expansion of $f(r)$ in J_1 ’s possible over the end $z=l$, thus determining the constants E . In this I believed myself to follow the example of the ordinary text-books.

Mr. Bridgman says my solution is clearly in error, for when applied to the ordinary case $f(r)=r$ the determining equation makes the constants E all zero, and “gives the absurd result that a cylinder experiences no torsion under a torsional force varying as the distance from the axis.” If, however, the E ’s all vanish, then the stresses vanish as well as the strains; thus the result on the physical side is simply that a cylinder under no stress is under no strain, a result which to me presents no absurdity. The real fact is that it never occurred to me that my new solution should include the simple ordinary case of the text-books $v=rz$. For this

$d^2v/dz^2 = 0$, and so the equation in r is not of the Bessel's type at all. This case I supposed a distinct one by itself and always treated it as such (*cf.* Phil. Mag. Feb. 1886, p. 86, a paper which, though published earlier, was written subsequently to the one under discussion). It will I think be more satisfactory to take the example which I actually applied the solution to, viz. $f(r) = Tr^3$, with T constant. For this I found

$$v = 8(T/n) \Sigma [\{J_1(kr) \sinh kz\} / \{k^4 J_0(ka) \cosh kl\}],$$

the corresponding stresses being

$$r\theta = -8T \Sigma [\{J_2(kr) \sinh kz\} / \{k^3 J_0(ka) \cosh kl\}],$$

$$z\theta = 8T \Sigma [\{J_1(kr) \cosh kz\} / \{k^3 J_0(ka) \cosh kl\}],$$

the former vanishing over $r=a$ in virtue of $J_2(ka) = 0$.

If I follow Mr. Bridgman, his point is that Tr^3 by itself cannot be expanded in a Bessel series, but that $Tr^3 + Br$, B being a constant determined by a simple integration which Mr. Bridgman gives, is expressible in terms of the exact Bessel's series which I supposed to represent Tr^3 alone. If this should prove to be the case, then all that would be necessary to make my solution complete would be to add to the value for v the single term $-Brz/n$, answering to a torsional force of the simplest type. The same would be true for the most general form of $f(r)$, the value of B varying with the function. If I rightly follow Mr. Bridgman, the theorem he quotes is due to Dini, and amounts to accepting r as a debased but necessary form of $J_1(kr)$ —or in the general case r^i as a form of $J_i(kr)$.—There is, Mr. Bridgman says, a misprint in Dini's work and a misrepresentation of it in the only text-book it appears in, thus some more definite confirmation appears desirable before finally accepting it.

Dec. 19.—Since writing the above I have been able to consult my friend Mr. Arthur Berry, and through him Dr. E. W. Hobson, F.R.S.—who is a recognized authority on Bessel Functions—on the point in pure mathematics raised by Mr. Bridgman. These gentlemen substantiate the accuracy of Mr. Bridgman's interpretation of Dini's result. This means that in the problem treated by me in expanding any function $f(r)$ representing torsional forces, in addition to a series of what are ordinarily recognized as Bessel functions with arbitrary constants, there must be a term $A_0 r$, of which

the constant is determined exactly as if the term represented a true Bessel function through the relation

$$A_0 \int_0^a r^2 \cdot r dr \equiv \frac{1}{4} A_0 a^4 = \int_0^a r f(r) r dr.$$

For instance, using my original notation, in the case of a torsional force Tr^i over $z=l$, the term additional to ordinary Bessel terms is

$$v = 4(i+3)^{-1} r z T a^{i-1} / n.$$

With this addition and mode of interpretation, the solution of 1887 includes the ordinary text-book case for which $i=1$, and so possesses a greater generality than I originally supposed. The additional term, as explained above, is always of the ordinary text-book type, but it is important unless i is large. I am thus much indebted to Mr. Bridgman for raising the point, and to Mr. Berry and Dr. Hobson for their assistance. Considering the date of the solution it seems curious that the point has not been raised before.

C. CHREE.

XXIX. *Notices respecting New Books.*

A Study of Splashes. By Professor A. M. WORTHINGTON, F.R.S.
With 197 Illustrations from Instantaneous Photographs.
Longmans, Green & Co. 1908. Price 6s. 6d. net.

THIS enticing volume is the outcome of an experimental study lasting over fourteen years. Yet even so, its publication was at the last delayed in order to be able to include additional and unexpected information relating to cases in which the height of fall is increased above a certain critical value. It is impossible in a short note to do more than call attention to the extreme beauty of many of the photographs reproduced here; and to express delight at the very lucid and ingenious explanations which accompany the reproductions. What is to be desired greatly (and what we believe Professor Worthington himself desires greatly) is that some hydrodynamician may see his way to tackle the problems presented and express them in mathematical language. The present publication will do much to spread a knowledge of the phenomena themselves.

XXX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 228.]

December 2nd, 1908.—Prof. W. J. Sollas, Sc.D., LL.D., F.R.S.,
President, in the Chair.

THE following communication was read:—

‘The Geological Interpretation of the Earth-Movements associated with the Californian Earthquake of April 18th, 1906.’ By Richard Dixon Oldham, F.G.S.

At the time of the San Francisco earthquake movement took place along a fault, known as the San Andreas Fault, which can be traced for a distance of about 200 miles. A remeasurement of the primary triangulation in the region shaken by the earthquake revealed considerable displacements, increasing in amount as the fault is neared, and of such nature that places to the east of the fault were shifted southwards while those to the west of it were shifted northwards. The author points out that the extent and peculiar distribution of these displacements negative the supposition that the fault was the cause—it must rather be regarded as a consequence of, or an incident in, the earthquake, this word being used to denote the disturbance in its entirety.

He also considers that the displacements cannot be explained in a satisfactory manner on the supposition that they are the result of strains affecting the crust of the earth as a whole, but may be explained by the difference in character and behaviour of the materials composing the greater part of it, where pressures are great enough to produce the phenomena of solid flow, and of those in the outer skin, where the pressures are not great enough to produce any material difference in the behaviour of rocks from that which we associate with solidity, as experienced at the surface of the earth. The surface-displacements constituting the earthquake, as ordinarily understood, arise from disturbances in the outer skin; but in great earthquakes, like the one dealt with in the paper, these may be the result of more deep-seated disturbances affecting the whole crust of the earth. A distinction is drawn between these two forms of disturbance, and the term *bathyseism* is proposed for the deep-seated disturbance: the wave-motion which impresses itself on distant seismographs and constitutes the *teleseism* or world-shaking earthquake being the product of the *bathyseism*.

The deep-seated cause, or *bathyseism*, of the San Francisco earthquake is regarded as the result of a widespread strain, of the nature of a shear, such as might have been produced by displacements approximately parallel to the general direction of the coast-line, and by forces which must have been very different from those concerned in the formation of the San Andreas Fault. This fault cannot, consequently, be regarded as the cause of the earthquake, nor the earthquake as an incident in the growth of the fault.

FIG. 1 A.

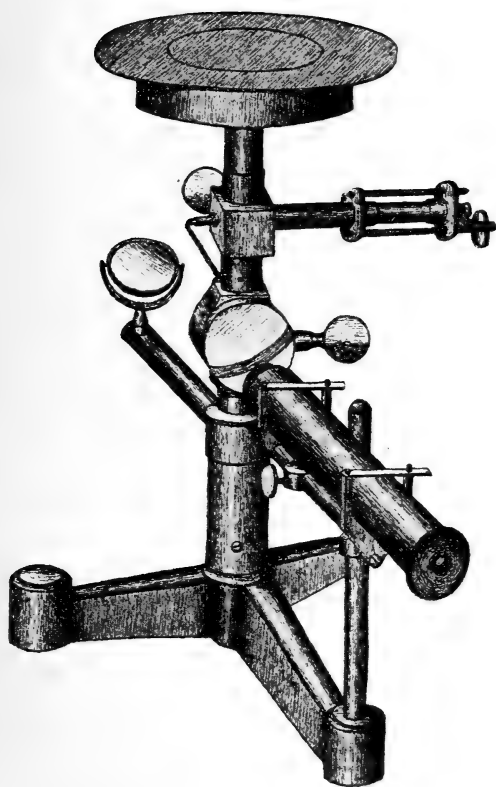




FIG. 6.

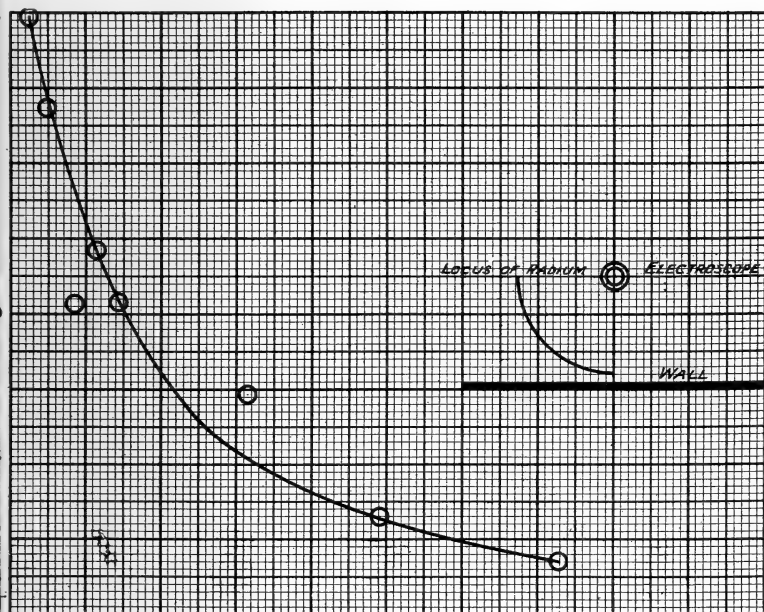


FIG. 3.

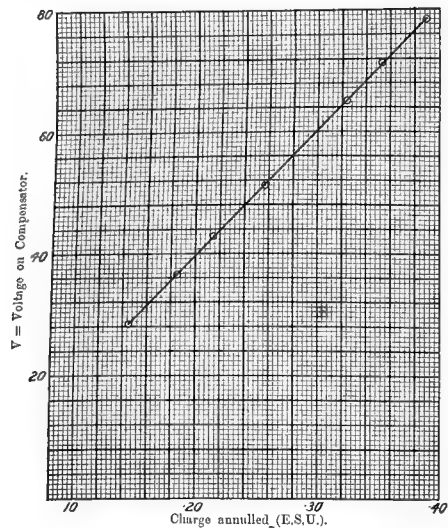


FIG. 5.

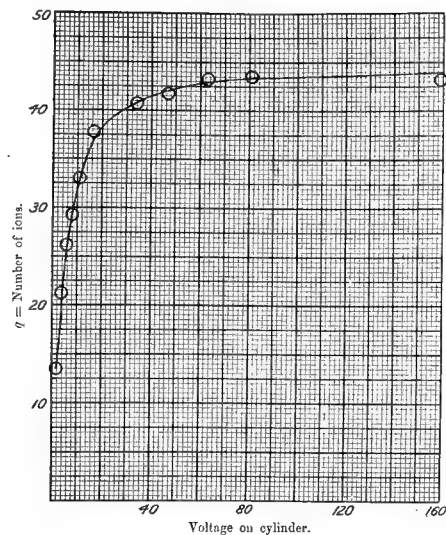


FIG. 6.

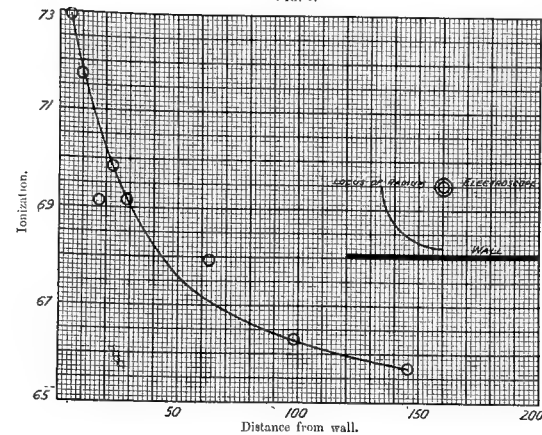


FIG. 7.

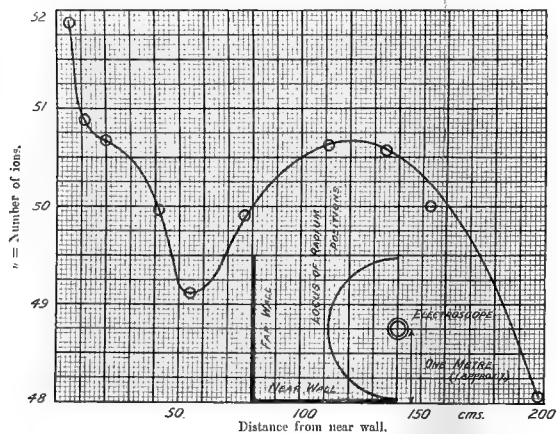


FIG. 8.

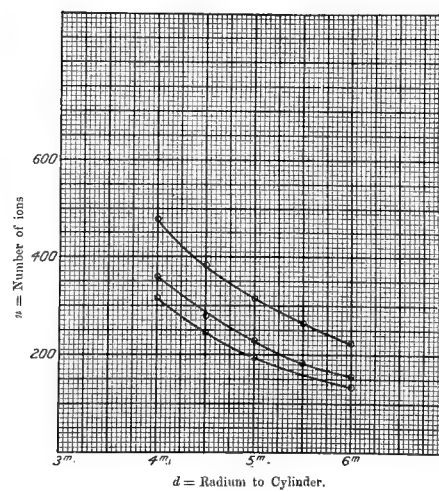
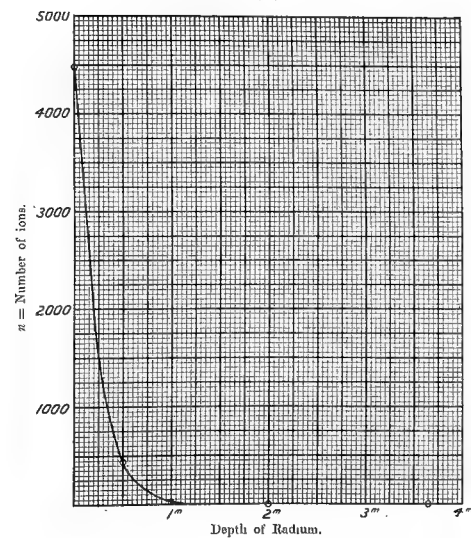
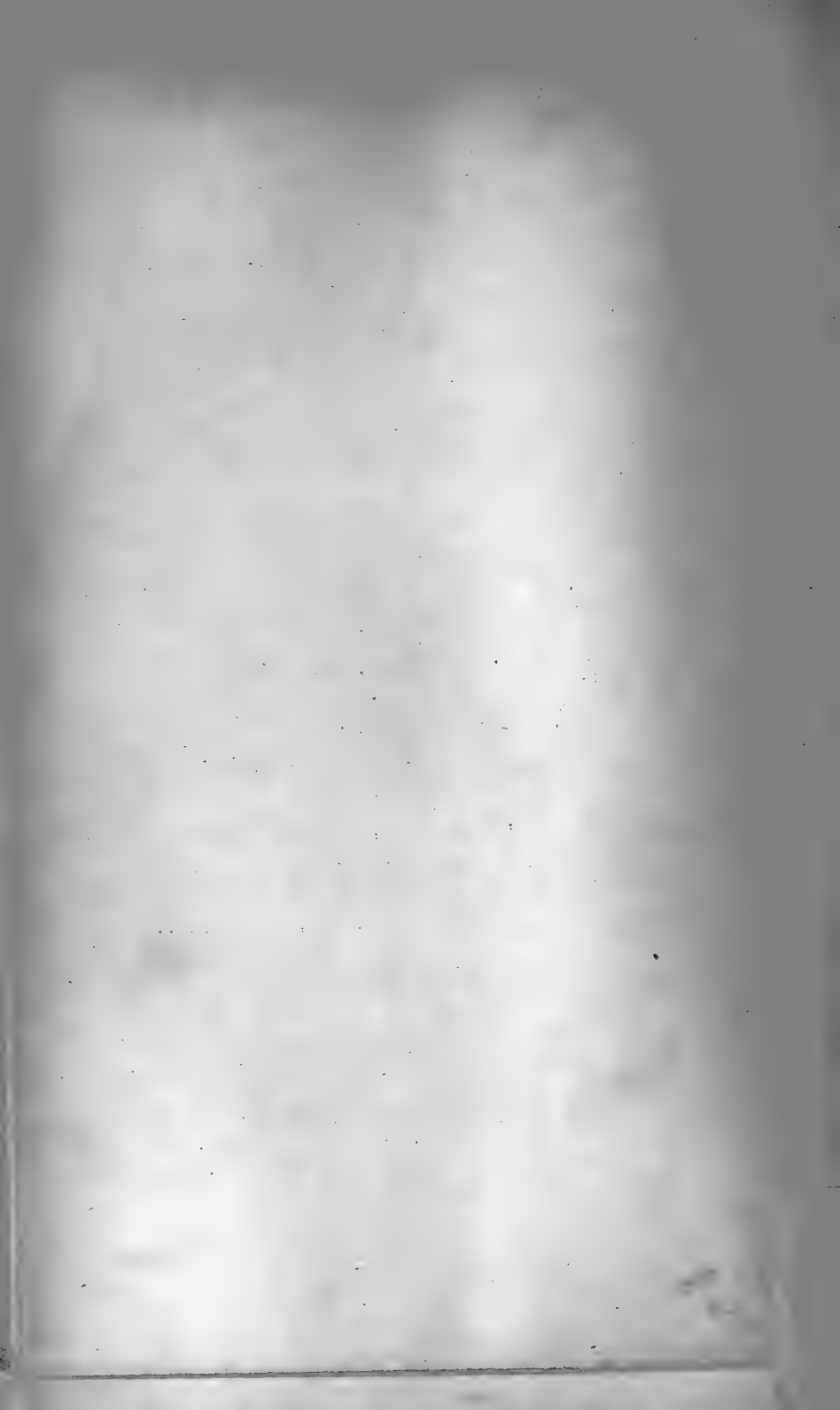


FIG. 9.





INDEXED

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MARCH 1909.

XXXI. *On the Theory of Surface Forces.*—IV. *On very thin Liquid Films and very small Liquid Drops.* By G. BAKKER*.

§ 1. *At a fixed temperature the structure of a plane capillary layer in contact with its vapour is fully determined.*

ON the supposition that the capillary layer is not under the influence of an external force, such as that of gravitation, magnetism, electricity, or the adhesion of the walls of the vessel, we can demonstrate in the following way, that at a fixed temperature the plane capillary layer of a given fluid in contact with its vapour is always the same thing. That is to say, the plane capillary layer which limits a large bulk of liquid has exactly the same structure as the capillary layer which limits for instance a black spot in a very thin liquid film of the same body.

For a point of the plane capillary layer the hydrostatic pressure in a direction perpendicular to the surface is, as I have demonstrated †, equal to the vapour-pressure (p_1). If we thus consider with Young ‡ the hydrostatic pressure as the difference of the thermic pressure θ (repulsive force of Young) and the cohesion (force of cohesion of Young) and, if we denote the cohesion in the mentioned direction by S_1 , we have

$$p_1 = \theta - S_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Communicated by the Author.

† "On the Theory of Surface Forces, II.," Phil. Mag. October 1907, p 515.

‡ Thomas Young, "On the Cohesion of Fluids," Phil. Trans. 1805. Phil. Mag. S. 6. Vol. 17. No. 99. March 1909. 2 A

If we consider further, for a point of the capillary layer, the thermic pressure as independent of the direction (Stefan, Fuchs, Rayleigh, van der Waals), and only as a function of the density, the cohesion S_1 must be in consequence of the equation (1) also a function of the density, because p_1 is a constant. By this I mean the following:—

If one considers at a fixed temperature, firstly, a plane capillary layer which limits a *large bulk of liquid*, and, secondly, a set of capillary layers which limit very thin liquid films (black spots), the cohesions S_1 have the same value in the corresponding points of all the considered capillary layers. (Two points are called corresponding points when, in these points, the value of the density is the same.)

The same property can also be demonstrated for the cohesion in the direction *parallel* to the surface of the capillary layer.

When, namely, V denotes the potential of the forces of cohesion and ρ the density in a point of the capillary layer, we have

$$d\theta = -\rho dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

According to Rayleigh *, we have further :

$$V = -2a\rho - \frac{c_2}{1.2} \frac{d^2\rho}{dh^2} - \frac{c_4}{1.2.3.4} \frac{d^4\rho}{dh^4} - \dots$$

and for the cohesion S_2 *parallel* to the surface of the capillary layer I have given † :

$$S_2 = a\rho^2 + \frac{c_2}{1.2} \rho \frac{d^2\rho}{dh^2} + \frac{c_4}{1.2.3.4} \rho \frac{d^4\rho}{dh^4} + \dots$$

Therefore

$$\frac{1}{2} V\rho = -S_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we thus consider θ as a function of the density, the relation (2) gives likewise this property for the potential V , and the relation (3) demands the same for S_2 . Hence it follows that S_2 must have the same value in the corresponding points of the considered capillary layers, *when only for the vapour, which touches these capillary layers, θ , ρ , and V have the same values.*

If p_2 denotes the pressure for a point of the capillary

* Phil. Mag. Feb. 1892, p. 210.

† Zeitschr. f. phys. Chem. xlviii. p. 12, and Koninkl. Akad. van Wetenschappen Amsterdam, Proceedings, Dec. 20, 1899, pp. 260 & 261.

layer in a direction parallel to their surface, we have, further,

$$p_2 = \theta - S_2,$$

and we see thus that in the corresponding points of the capillary layers, which are touched by the same vapour, p_1 , p_2 , and V have the same values. Now, we consider the capillary layer as a continuous series of states between two fixed densities. Each state (phase) is determined by its density ρ and the hydrostatic pressures p_1 and p_2 . This density ρ and the hydrostatic pressures p_1 and p_2 having the same values in the corresponding points of the considered capillary layers, we see that a set of films touched by the saturated vapour must be limited by congruent capillary layers or parts of congruent capillary layers. The capillary layer which limits a large bulk of liquid is, of course, complete. For the two capillary layers, however, which limit a black spot, we must examine the question further, and we shall find that the capillary layers in this case must be also complete. Indeed, if we could have a black spot, which could consist of two *incomplete* capillary layers, which touch each other in their plane of maximum density, this plane should divide the black spot in two congruent layers of smaller thickness, and because this plane should be a plane of symmetry the intensity of the force of cohesion must be null in this plane. In my theory, the departure from the law of Pascal being proportional to the square of the intensity of the force of cohesion *, we must have for each point of the plane of symmetry :

$$p_1 = p_2 ;$$

and, further,

$$p = \frac{p_1 + p_2}{2} = p_1.$$

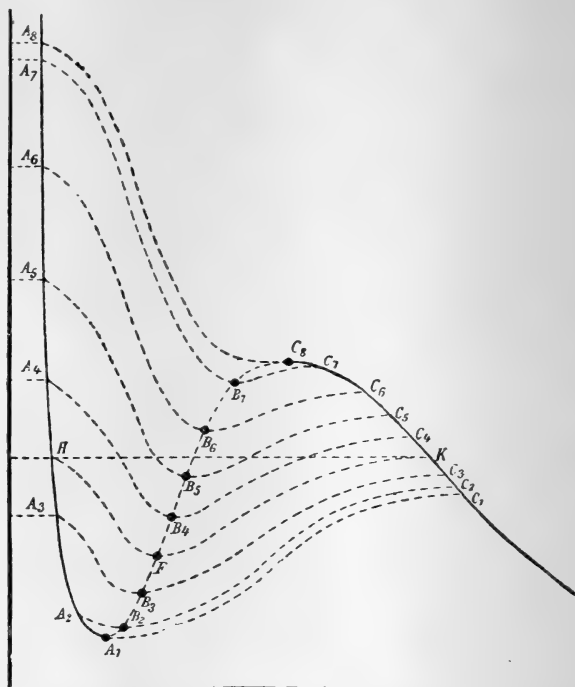
Now, for every *plane* capillary layer (complete or incomplete) the hydrostatic pressure in a direction perpendicular to its surface must be a constant †, and therefore the pressure p must have in the considered plane the same value as the vapour-pressure. In fig. 1 the curve HFK gives for a complete plane capillary layer, for each point, the relation between the pressure p and the reciprocal value of the density. Now, if we could have a black spot, which might consist of two *incomplete* capillary layers, which touch each other in their plane of maximum density, the relation between p and $\frac{1}{\rho}$ for each of the two symmetric incomplete layers, out

* Phil. Mag. Oct. 1907, p. 523, Formula (22).

† L. c. p. 515.

of which the film (black spot) would consist, would be given by a part of the curve HFK (fig. 1). The pressure p in the plane of symmetry having the same value as the vapour-

Fig. 1.



pressure, the relation between p and $\frac{1}{\rho}$ for the whole film would be given by a curve such as in fig. 2 (p. 339), where p and C_4 have the same ordinate. (By this consideration the letter C_1 of fig. 2 must be changed into K.) Because, however, the values of the ordinates of the curve HFK between the points H and K are always smaller than the vapour-pressure, the curves between p and K (C_4) cannot be incomplete. Hence we conclude :

A set of films touched by the saturated vapour must be limited by capillary layers, which are complete and congruent with the capillary layer that limits a great bulk of liquid.

The minimum value of the thickness of a film must therefore be greater than twice that of the capillary layer. Indeed, if we consider a film as consisting of two complete capillary layers which touch each other in their plane of

maximum density, the potential V in this plane could not have the same value as the potential in the corresponding points of the capillary layer which limits a large bulk of liquid. Hence follows: The film must always consist of a layer of liquid, limited by two *complete* capillary layers, and the layer of liquid must have a minimum thickness, which has the same value as the radius of the sphere of action. The thickness of the capillary layer and the sphere of action being of the same order of greatness, *one can take for the thickness of the complete capillary layer approximately the third part of the minimum value of a liquid film.*

Hitherto we have supposed that the capillary layer was in contact with *saturated* vapour. Let us now consider the possibility of the existence of a *plane* capillary layer in contact with vapour which has *not* the ordinary tension.

Using the potential function

$$-f \frac{e^{-\frac{r}{\lambda}}}{r}$$

for the forces between the elements of volume, the cohesion S_2 in a direction normal to the lines of force is* :

$$S_2 = \frac{1}{8\pi f} \left\{ \left(\frac{dV}{dh} \right)^2 + \frac{V^2}{\lambda^2} \right\}, \quad (4a)$$

and the cohesion S_1 in the direction of the lines of force :

$$S_1 = -\frac{1}{8\pi f} \left\{ \left(\frac{dV}{dh} \right)^2 - \frac{V^2}{\lambda^2} \right\}. \quad (4b)$$

Hence

$$p = \frac{p_1 + p_2}{2} = \theta - \frac{S_1 + S_2}{2} = \theta - \frac{V^2}{4a}.$$

Differentiating this equation we find

$$\frac{dp}{dv} = \frac{d\theta}{dv} - \frac{V}{2a} \frac{dV}{dv}$$

We have, further, μ_1 denoting the thermodynamical potential of the homogeneous phase of the vapour :

$$V + 2\alpha p = \mu_1 - \mu^\dagger. \quad (5)$$

Hence

$$\frac{dp}{dv} = \frac{d\theta}{dv} \frac{\mu_1 - \mu}{2\alpha p} \quad (6)$$

* Phil. Mag. Dec. 1906, p 560.

† Phil. Mag. Oct. 1907, p. 516, equation 4.

Whereas the equations (4) and (5) are true likewise for a curved capillary layer, the equation (6) may be used equally well for a curved as for a plane capillary layer. That means: if a plane capillary layer could exist in stable equilibrium with vapour not having the ordinary pressure, the relation between p and ρ would be the same as for a curved capillary layer in contact with the same vapour*.

If, for instance, in fig. 1, C_3 denotes the pressure and the value of $\frac{1}{\rho}$ for the vapour, which touches a concave surface of liquid (consider a spherical bubble of vapour in the interior of a liquid), I have found for the relation between p and $\frac{1}{\rho}$ a curve such as $C_3B_3A_3$. Likewise the relation $p=f\left(\frac{1}{\rho}\right)$ is presented for instance by the curve $C_6B_6A_6$, when the state of vapour, which touches now a spherical drop of liquid, is given by the point C_6 &c.

If a plane complete capillary layer exists in contact with the vapour, the state of which is given by the point C_4 , the relation between p and $\frac{1}{\rho}$ is, likewise as for a curved capillary layer, given by the curve $A_4B_4C_4$. Now, for the homogeneous phases which limit a complete capillary layer, the potential of the forces of cohesion is given by the formula of Gauss-van der Waals:

$$V = -2a\rho.$$

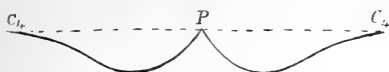
The equation (5) gives, therefore, $\mu_1 - \mu = 0$; that means: the thermodynamical potentials in the two homogeneous phases which limit the complete capillary layer must have the same value. The states of these two homogeneous phases are thus given by the points C_4 and A_4 . We have, however, still a second condition for the stability of our plane capillary layer: the pressure in the points A_4 and C_4 must have the same value. The latter condition being contrary to the first, we see that the plane capillary layer might not be complete.

If we consider the possibility of an incomplete plane capillary layer, the relation between p and $\frac{1}{\rho}$ would be given by a part of the curve $C_4B_4A_4$; and for a film which could exist of two incomplete capillary layers which touch each

* The supposition is of course that the thermic pressure θ , being a pure function of the density, has the same value for the two corresponding points in the two considered capillary layers.

other in their plane of maximum density the relation between p and $\frac{1}{\rho}$ would be given by the curve of fig. 2.

Fig. 2.



In this manner we should have for the point P two different values of $\frac{dp}{dv}$, and because this is impossible we conclude that our incomplete plane capillary layer could not exist.

If we do not wish to use the potential function

$$-f \frac{e^{-\frac{r}{\lambda}}}{r}$$

for the forces between the elements of volume, we may use for our consideration the equations

$$p_2 = \theta - S_2$$

$$\text{and } S_2 = -\frac{1}{2} V \rho. \quad (\text{See above.})$$

In the same manner as above, we can now demonstrate that the curve

$$p_2 = f\left(\frac{1}{\rho}\right)^*$$

is fully determined by the state of the vapour, from which again follows the impossibility of a film consisting of two incomplete capillary layers.

In a general manner we find thus that, for a fixed temperature, the plane capillary layer is fully determined and must be limited on one side by the saturated vapour (of ordinary pressure) and on the other side by the "ordinary" homogeneous liquid phase.

§ 2. Thickness of the plane capillary layer and surface-tension of very thin liquid films.

A case wherein a set of liquid films of different thickness in the same space is in equilibrium with the vapour and with each other, we have in the experiment of E. S. Johannott, Jun. In a first research about thin liquid films, Johannott has found† that the black spots in thin liquid films may

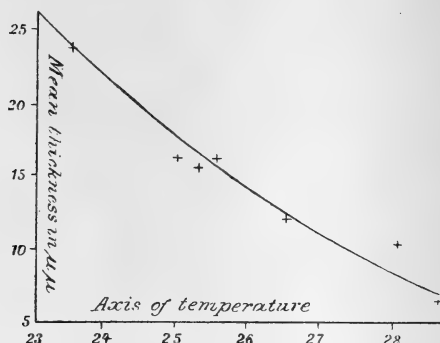
* p_2 is the pressure in a direction parallel to the surfaces of the capillary layer.

† Phil. Mag. xlvii. (1899) p. 501.

consist of different parts. For a first set of films he has found, according to the experiments of Reinold and Rücker, a thickness between 40 and 12 $\mu\mu$. If the atmosphere of the space wherein the liquid films were formed was not saturated, he obtained a film of about 6 $\mu\mu$.

In a second investigation* Johannott has continued his research by the aid of a Michelson's interferometer. In order to obtain a thickness which could be measured with sufficient exactness, he produced, in a long box of brass, 221 films of soap-water parallel behind each other. The relation between the temperature and the average thickness of the films may be represented by fig. 3.

Fig. 3.



By careful examination with the microscope, Johannott could often distinguish in the black spots five parts of different thickness. The minimum value of the thickness of the films was circa 6 $\mu\mu$. *Although the vapour in the box of Johannott was mixed with air, I believe that we may thus safely adopt also in the case wherein the films are only in contact with saturated vapour, 6 $\mu\mu$ as the order of magnitude of the minimum thickness of the plane capillary layer.* Now I have demonstrated above that the two capillary layers which limit a liquid film cannot touch one another, and that the thickness of the plane capillary layer is circa the third part of the minimum value of a liquid film. We find thus for the thickness of the plane capillary layer circa 2 $\mu\mu$. The fact that the two capillary layers which limit a liquid film cannot touch one another, and that therefore each liquid film must consist of a layer of liquid limited by two complete

* *L. c.* xi. (1906) pp. 746-753.

capillary layers, may be also expressed in the following manner:—

The metastable phases of the theoretical isotherm of Thomson-van der Waals cannot be in equilibrium unless this set of phases of different densities is limited on both their sides by a homogeneous phase.

Johonnott has also measured the surface-tensions of the films and he has found that *the surface-tension was independent of the thickness of the films*. This fact follows immediately from my considerations above. Indeed, when the films always consist of a liquid film limited by two complete capillary layers, the tension of the film must be always twice that of the capillary tension of Laplace, and for a fixed temperature the surface-tension in the film must be independent of their thickness.

In this periodical* I have endeavoured to explain the apparition of the black spots in very thin liquid films as the vanishing of the phases of the theoretical isotherms of James Thomson, which would be unstable if they were isolated. From my considerations above, however, it follows that the capillary layer must be always complete, and the apparition of the black spots cannot be explained otherwise than as the consequence of a temporary and local diminution of the surface-tension in the considered points, in consequence of the temporary and local increase of the evaporation in these points.

If H denotes the surface-tension of Laplace, ρ_1 and ρ_2 respectively the densities of the liquid and saturated vapour, the ascending of a liquid in a capillary tube which is wetted by the liquid is proportional to the expression

$$\frac{H}{\rho_1 - \rho_2}.$$

Now, this expression being proportional to $T_k - T$ †, where T_k denotes the critical temperature, while T denotes the (absolute) temperature of observation, we have

$$\frac{H}{\rho_1 - \rho_2} = \kappa \left(1 - \frac{T}{T_k} \right),$$

or, putting $\frac{T}{T_k} = \vartheta$,

$$\frac{H}{\rho_1 - \rho_2} = \kappa(1 - \vartheta). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

* Phil. Mag. Oct. 1907, p. 511.

† J. E. Verschaffelt, Comm. from the Lab. of Phys. Leiden, Nr. 28, p. 15, and E. C. de Vries, Inaugural-dissertation and *Arch. Néerl. Science*, xxviii. (1894) pp. 210-219.

It is very probable that this relation exists up to the critical temperature. Indeed, the theory of capillarity of van der Waals gives in the immediate neighbourhood of the critical temperature

$$H = H_0(1-\vartheta)^{\frac{3}{2}}$$

$$\text{and } \rho_1 - \rho_2 = 2\beta\rho_k(1-\vartheta)^{\frac{1}{2}},$$

where H_0 and β are constants. Hence

$$\frac{H}{\rho_1 - \rho_2} = \kappa(1-\vartheta).$$

Generally I put, therefore,

$$H = \kappa(1-\vartheta)(\rho_1 - \rho_2), \quad . \quad . \quad . \quad . \quad (8)$$

κ being a constant.

In his paper, "Remarques sur le Théorème des états correspondants" (*Annales de Toulouse*, v.), Mathias gives empirical formulæ for the densities of the liquid and the saturated vapour. For the density of the vapour he gives:

$$\rho_2 = A(1-\vartheta - 1.124 \sqrt{1-\vartheta} + 0.579^2),$$

where A is a constant.

For CO_2 , for instance,

$$\rho_2 = 1.295(1-\vartheta - 1.13 \sqrt{1-\vartheta} + 0.579^2).$$

For CO_2 in the liquid state, he gives, further,

$$\rho_1 = 1.064(\vartheta - 0.569 + 1.655 \sqrt{1-\vartheta}).$$

The difference of the two densities is therefore given by the formula

$$\rho_1 - \rho_2 = \alpha\vartheta + \beta \sqrt{1-\vartheta} - \gamma,$$

where α , β , and γ are positive constants.

For CO_2 we have:

$$\alpha = 2.36 \quad \text{and} \quad \gamma = 2.32,$$

but in reality α and γ must have the same value, ρ_1 and ρ_2 having the same value at the critical temperature.

I put, therefore,

$$\rho_1 - \rho_2 = \alpha(\vartheta - 1) + \beta \sqrt{1-\vartheta}.$$

Substituting this expression in the formula (8), we find

$$H = \kappa(1-\vartheta)^{\frac{3}{2}}(\beta - \sqrt{1-\vartheta}). \quad . \quad . \quad . \quad . \quad (9)$$

In the immediate neighbourhood of the critical temperature $\sqrt{1-\mathfrak{S}}$ may be neglected with respect to β and we get the formula of van der Waals :

$$H = \kappa(1-\mathfrak{S})^{\frac{3}{2}},$$

which is only true in the neighbourhood of the critical temperature, while my formula is general.

For ether I have found :

$$H = 145(1-\mathfrak{S})^{\frac{3}{2}} - 119(1-\mathfrak{S})^2.$$

	For $t=20^{\circ}$.	150°.	190°.
Formula	H = 16·49	3·12	0·15
Observed.....	H = 16·49	2·98	0·16

Observation.—Mathias has found for $\rho_1 - \rho_2$:

$$\rho_1 + \rho_2 - 2\rho_k = \alpha(1-\mathfrak{S}),$$

where α denotes a constant.

We have thus :

$$H = \kappa(\rho_1 - \rho_2)(\rho_1 + \rho_2 - 2\rho_k). \quad . \quad . \quad . \quad (10)$$

In the neighbourhood of the melting-point the density of the vapour may be neglected with respect to the density of the liquid, and for many substances the density ρ_0 at the melting-point is about three times the critical density. For instance, Mathias has calculated * :—

Substance.	ρ_k .	One-third of the density at t° .	t° .
C_6H_6	0·3038	0·3000	0°
C_6H_5F	0·3543	0·3491	-0°·75
C_6H_5Cl	0·3665	0·3635	35°
C_6H_5Br	0·4860	0·4836	53°
C_6H_5I	0·5843	0·5808	77°·8
CH_3OH	0·2775	0·274	0°
C_2H_5OH	0·2786	0·269	0°
C_3H_7OH	0·2777	0·2733	0°
$(C_2H_5)_2O$	0·2631	0·2453	0°
CH_3COOH	0·3516	0·3510	16°·3
CCl_4	0·5557	0·5440	0°
$SnCl_4$	0·7414	0·7465	14°·9
SO_2	0·5200	0·5043	-30°

* E. Mathias, *Le Point Critique des Corps Purs*, p. 10.

H_0 being the value of H at the melting-point, formula (10) gives :

$$H_0 = 3\kappa\rho_k^2,$$

Hence we find for the constant of Laplace :

$$\frac{H}{H_0} = \frac{1}{3} \frac{\rho_1^2 - \rho_2^2}{\rho_k^2} - \frac{2}{3} \frac{\rho_1 - \rho_2}{\rho_k}.$$

In this periodical* I have given for the difference of the hydrostatic pressure p_1 , *perpendicular* to the surface of the capillary layer, and the average value \bar{p} of the pressure p_2 *parallel* to the surface of the capillary layer :

$$p_1 - \bar{p} = \frac{25\pi f}{32a^2} \left\{ \frac{H}{\rho_1 - \rho_2} \right\}^2, \quad . \quad . \quad . \quad (11)$$

where f denotes the constant of the used potential function

$$-f \frac{e^{-\frac{r}{\lambda}}}{r},$$

while a is the coefficient of the expression $a\rho^2$ of Laplace for the so-called molecular pressure or the function a in the equation of state of van der Waals. Although a is a function of the temperature, I have, as a first approximation, considered a and f as constants; κ being a new constant, we had

$$p_1 - \bar{p} = \kappa \left\{ \frac{H}{\rho_1 - \rho_2} \right\}^2;$$

or, the expression $\frac{H}{\rho - \rho_2}$ being proportional to $1 - \mathfrak{S}$,

$$p_1 - \bar{p} = \kappa(1 - \mathfrak{S})^2. \quad . \quad . \quad . \quad (12)$$

If the pressure is expressed in atmospheres, I have found for ether, $\kappa = 238$. When h denotes the thickness of the plane capillary layer, we have, further †,

$$h = \frac{H}{\rho_1 - \bar{p}}. \quad . \quad . \quad . \quad (13)$$

Substituting respectively the formula (9) for H and the formula (12) for $p_1 - \bar{p}$ in (13), we find :

$$h = \frac{\alpha}{\sqrt{1 - \mathfrak{S}}} - \beta,$$

where α and β are constants.

* Phil. Mag. Oct. 1907, p. 521.

† Ibid. p. 522.

For ether I find :

$$h = \left\{ \frac{6.014}{\sqrt{1-\vartheta}} - 4.93 \right\} \mu\mu. \quad . \quad . \quad . \quad (14)$$

At the temperature $T = \frac{1}{2}T_k$, or $\vartheta = \frac{1}{2}$, or $t = -39^{\circ}.3$ Celsius, we find thus for the thickness of the plane capillary layer of ether :

$$h = 3.57 \mu\mu.$$

If water was "conformable" with ether in the sense of van der Waals, the proportion between the thickness of the capillary layers of water and ether at corresponding temperatures would be equal to the expression :

$$\sqrt[3]{\frac{T_k}{p_k}},$$

and we should have :

$$\frac{h_{\text{ether}}}{h_{\text{water}}} = \frac{236}{151} = \text{circa } 1.5.$$

For water at the temperature $T = \frac{1}{2}T_k$ or 46° Celsius, we should therefore calculate :

$$h = \frac{3.6}{1.5} = 2.4 \mu\mu.$$

The thickness of the capillary layer, calculated by the aid of the formula (14), increases with the temperature. For a temperature of 15° Celsius, we must therefore take a value which is a little smaller than 2.4. *Such a value would be therefore in perfect accordance with the researches of Johannott, which give (see above) a value of the order of greatness of $2\mu\mu$.*

§ 3. The surface-tension and the radius of small drops.

In the same manner as I have demonstrated above for a *plane* capillary layer, we can show that a spherical drop is also fully determined by the pressure of the vapour which surrounds it. Now, we have seen further that the metastable phases of the Thomson-van der Waals Isotherm might not be in equilibrium in the *plane* capillary layer if the capillary layer was not limited at *both* sides by a homogeneous phase (of liquid and vapour), so that every plane liquid film always consists of a *liquid* layer limited by two *complete* capillary layers. In the same manner as we have observed that the interior liquid layer of a plane liquid film of which the thickness has its minimum value has a thickness of the order of the radius of the sphere of action, and in the

same manner as we have found for the order of greatness of the thickness of a liquid film of minimal thickness circa three times the thickness of a complete plane capillary layer, we conclude :

when at a fixed temperature the size of a spherical drop of liquid has its minimal value, the radius of its spherical capillary layer has a value of the order of greatness of the thickness of a capillary layer.

For the value of the "radius"* of a spherical capillary layer Kelvin has given the formula :

$$R = \frac{2H}{p_{\text{liq.}} - p_{\text{vap.}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where H denotes the departure from the law of Pascal for the spheri-capillary layer, and $p_{\text{liq.}}$ and $p_{\text{vap.}}$ respectively the pressures in the corresponding liquid and vapour phases which surround the capillary layer. For the case when the drop has its minimal size I have calculated by the aid of the formula (15) the value of R, and I have found in the following manner a value, which really is of the order of the thickness of the capillary layer. The state of the liquid phase in the interior of the "liquid" drop (that is to say, the homogeneous part) when the drop has its minimal size and the state of the homogeneous phase of the vapour which surrounds the drop, are respectively given by the points A_8 and C_8 of the fig. 1 above. This figure has the following meaning † :—

Every pair of points of the isotherm, for which the thermodynamical potential has the same value (as A_8 and C_8 , A_7 and C_7 , &c.), corresponds above the rectilinear part HK of the empiric isotherm to a spherical drop of liquid, such, that the state in the interior of the drop and the state of the vapour which surrounds it, are determined in a singular manner by the situation of this pair of points. In the same manner, every pair of points below the rectilinear part HK of the empiric isotherm (A_3 and C_3 , A_2 and C_2 , &c.), for which the thermodynamical potential has the same value, corresponds to a spherical bubble of vapour. If we now construct the curves, such as $A_6B_6C_6$, $A_5B_5C_5$, &c., which present the relation between

half the sum $p = \frac{p_1 + p_2}{2}$ of the maximum and minimum

* By this radius I mean a value between the values of the radii of the spheres which limit the spherical capillary layer internally and externally. The effective radius would be therefore about $\frac{3}{2}$ times the thickness of the capillary layer.

† Phil. Mag. April 1908, p. 430.

pressure and the reciprocal value of the density for every point of the capillary layers which envelop the spherical drops and the spherical vapour-bubbles, the minima of these curves present exactly the unstable part $A_1FB_6C_8$ of the theoretical isotherm of James Thomson.

In the case that the liquid drop has its minimal size the pressures $p_{\text{liq.}}$ and $p_{\text{vap.}}$ have their maximum value, and are thus given by the ordinates of the points A_8 and C_8 , while the abscissas of these points denote respectively the reciprocal values of the densities of the liquid and the vapour which surround the spherical capillary layer. Since we do not seek an exact value but only the order of greatness of R , we may use for the homogeneous phase the equation of state of van der Waals. The equality of the thermodynamical potentials in the points A_8 and C_8 gives approximately :

$$2(p_{\text{liq.}} - p_1)v_1 = (p_{\text{vap.}} - p_1)(v_2 + v_{\text{max.}}) \quad . \quad . \quad (16)$$

$p_{\text{liq.}}$ and $p_{\text{vap.}}$ have the known significations, v_2 and v_1 are respectively the abscissas of K and H , and denote therefore respectively the ordinary vapour- and liquid-volume. Further, $v_{\text{max.}}$ is the abscissa of C_8 , while the abscissas of A_8 and H are considered to be equal.

We shall at first make the calculation for ether at the temperature for which the isotherm touches the axis of volume; that is for the temperature $T = 0.844T_k^*$. This temperature is for ether, $121^\circ.5$ Celsius. The equation of state of van der Waals gives for the sum of the three densities which correspond to a definite pressure :

$$\rho_1 + \rho_2 + \rho_3 = 3\rho_k \quad . \quad . \quad . \quad (17)$$

For the ordinary vapour-pressure p_1 we have :

$$v_1 = \frac{1}{\rho_1} = 1.7 \text{ cm.}^3, \quad v_2 = \frac{1}{\rho_2} = 27.3, \text{ and } v_k = 3.8 \text{ cm.}^3$$

Hence, $v_3 = \frac{1}{\rho_3} = 6.12$. The ordinary vapour-pressure p_1 of ether at $121^\circ.5$ Celsius is $7896.2 \text{ mm.} = 10.528 \times 10^6 \text{ dyne per cm.}^2$

The rule of Maxwell-Clausius gives further as a first approximation, $p_{\text{min.}}$ being null :

$$p_1(v_3 - v_1) = (p_{\text{vap.}} - p_1)(v_2 - v_3) \dagger.$$

* J. D. van der Waals, *Kontinuität*, p. 105 (1899).

† We seek only the order of greatness of R and make no pretence to give an exact value.

Substituting the values of p_1 , v_1 , v_2 , and v_3 , we find :

$$p_{\text{vap.}} = 12.725 \times 10^6 \text{ dyne per cm.}^2$$

To find a value for $v_{\text{max.}}$ in the equation (16), we use the equation (17) for the case that the pressure is denoted by the ordinate of the point C_8 . That gives :

$$\frac{1}{v_1} + \frac{2}{v_{\text{max.}}} = \frac{3}{v_k}.$$

Hence

$$v_{\text{max.}} = 10 \text{ cm.}^3$$

The equation (16) gives therefore :

$$2(p_{\text{liq.}} - 10.528 \times 10^6) \times 1.7 = 2.197 \times 10^6(27.3 + 10),$$

and we find

$$p_{\text{liq.}} = 34.63 \times 10^6 \text{ dyne per cm.}^2 ;$$

and further,

$$p_{\text{liq.}} - p_{\text{vap.}} = (34.63 - 12.72)10^6 = 21.91 \times 10^6.$$

The equation of Kelvin,

$$R = \frac{2H}{p_{\text{liq.}} - p_{\text{vap.}}}$$

gives thus :

$$R = \frac{H}{10.95} \times 10^{-6} \text{ cm.} \quad . \quad . \quad . \quad . \quad (18)$$

The H in this equation is the departure from the law of Pascal in a *spherical* capillary layer and is different from the constant of Laplace, which corresponds to a plane capillary layer. I shall, however, demonstrate that the value of H in the equation (18) is of the same order of greatness as the constant of Laplace. Indeed I have already observed that a spherical drop has a liquid nucleus, which has a diameter of the order of greatness of the thickness of a plane capillary layer. The parts of the spherical capillary layer, which are situated on opposite sides of the liquid nucleus, can also not very much influence each other. That is to say, as a first approximation we might consider our spherical capillary layer as a part of a *plane* film of minimal thickness that is rolled up.

Also in the following way we see that the departure from the law of Pascal in a *spherical* capillary layer, which limits a drop of *minimal size*, has a value of the same order of greatness as for a *plane* capillary layer. We have namely for the capillary constant of Laplace and *likewise more generally for*

the departure from the law of Pascal in a curved capillary layer* :

$$H = - \frac{\lambda}{\sqrt{a}} \int_{\rho_1}^{\rho_2} \left\{ \sqrt{\theta_1 - \theta + \rho_1 \int_{\rho_1}^{\rho} \frac{d\theta}{\rho}} + \frac{1}{4a} \left[\int_{\rho_1}^{\rho} \frac{d\theta}{\rho} \right]^2 \cdot \frac{1}{\rho} \frac{d\theta}{d\rho} \right\} d\rho, \quad (19)$$

where λ and a are constants, while θ denotes the thermic pressure in the considered point of the capillary layer. For a *plane* capillary layer ρ_1 and ρ_2 are respectively the reciprocal values of the abscissæ of the points H and K in fig. 1, and therefore the *ordinary* densities of the liquid- and vapour-phases. In our case, however, where the capillary layer is curved, the limits of the integral in the formula (19) are the densities of the points A_8 and C_8 instead of those of the points H and K. Now the thermic pressure, which is considered in the capillary layer as a pure function of the density (Stefan, Fuchs, Rayleigh, van der Waals), increases with the density. The contribution to the terms of the considered integral is therefore the most important in the neighbourhood of the points H and A_8 , while the contribution corresponding to the points between C_8 and K is much less important.

The difference of the abscissæ of the points A_8 and H being very small, we see thus that the *order of greatness* of H given by the expression (19) is the same for a *curved* capillary layer as for a *plane* capillary layer. For the considered temperature $T = 0.844T_k$ or 121.5 Celsius the capillary constant of Laplace is for ether : 5.17 dynes per cm. The equation (18) gives thus :

$$R = \frac{5.17}{10.95} \times 10^{-6} = \text{about } 5 \times 10^{-7} \text{ cm.} = 5\mu\mu.$$

R being a mean value between the two spheres which limit the spherical capillary layer, we find for the radius of the sphere which limits the drops a value between 5 and about $8\mu\mu$, while my formula (14) for the thickness of the *plane* capillary layer of ether gives at 121.5 Celsius, $h = \text{about } 10\mu\mu$. We find thus actually for the radius of a liquid drop of ether which has its minimal size a value of the same order of greatness as for the thickness of a plane capillary layer at the same temperature.

Secondly, we make the corresponding calculation for a liquid drop of ether at the temperature of 0° Celsius or

$$\vartheta = \frac{T}{T_k} = 0.585.$$

* Phil. Mag. Dec. 1906, p. 566. The demonstration of the formula for *curved* capillary layer is quite the same as that at the *locus citatus*.

The equation, which gives the equality of the thermodynamical potentials respectively in the interior of the drop and in the vapour which surrounds the drop, was :

$$(v_1 + v_1')(p_{\text{liq.}} - p_1) = (p_{\text{vap.}} - p_1)(v_2 + v_{\text{max.}}). \quad (20)$$

Now we have : $v_1 = 1.36 \text{ cm.}^3$ and $v_2 = 1273.4$. The value of $v_{\text{max.}}$ can be calculated by

$$\frac{1}{v_1} + \frac{2}{v_{\text{max.}}} = \frac{3}{v_k},$$

where $v_k = 3.8$. That gives $v_{\text{max.}} = 37 \text{ cm.}^3$

Further, we have for the ordinary vapour-pressure of ether at 0° Celsius :

$$p_1 = \frac{184.4}{760} \times 1.0133 \times 10^6 = 2.458 \times 10^6 \text{ dyne per cm.}^2$$

The equation (20) gives therefore :

$$2 \times 1.3(p_{\text{liq.}} - 2.458 \times 10^5) = (p_{\text{vap.}} - 2.458 \times 10^5) 1310.4$$

or

$$1310.4 p_{\text{vap.}} - 2.6 p_{\text{liq.}} = 3217.768 \times 10^5. \quad (20 a)$$

The rule of Maxwell-Clausius gives again :

$$(p_1 - p_{\text{min.}})(v_3 - v_1) = (p_{\text{vap.}} - p_1)(v_2 - v_3), \quad (21)$$

where $p_{\text{min.}}$ denotes the ordinate of the point A_1 in fig. 1. This point lies in our case far below the axis of volume. To find the absolute value of the ordinate of the point A_1 , I consider the part of the figure between the axis of volume and the part of the isotherm below this axis, firstly as the sum of two parabolic segments, with a common tangent, and secondly as the value of the integral :

$$\int p dv.$$

Using the equation of state of van der Waals, the segments, cut from the axis of volume by the isotherm, are given by the equation

$$\frac{RT}{v-b} = \frac{a}{v^2}.$$

For $T = 0.585 T_k$ I have calculated for these segments :

$$1.49 v_k \quad \text{and} \quad 0.326 v_k.$$

In this way we find for the absolute value of $p_{\text{min.}}$:

$$p_{\text{min.}} = 4.45 p_k.$$

For ether we have :

$$p_k = 35.6 \text{ atm. (S. Young)} = 4.5092 \times 10^6 \text{ dyne per cm.}^2$$

The equation (21) gives by the substitutions of the calculated values :

$$p_{\text{vap.}} = 3.178 \times 10^5 \text{ dyne per cm.}^2$$

Further, the equation (20 a) gives :

$$p_{\text{liq.}} = 36.411 \times 10^6 \text{ dyne per cm.}^2$$

By the aid of the formula of Kelvin, we calculate thus finally :

$$R = \frac{H}{p_{\text{liq.}} - p_{\text{vap.}}} = \frac{2 \times 19.3}{36.411 - 0.3178} \times 10^{-6} = 1.01 \times 10^{-6} \text{ cm.} = 10.1 \mu\mu.$$

We find therefore again a value of the order of greatness of the thickness of the *plane* capillary layer at the same temperature. If water was conformable in the sense of van der Waals with ether, 0° Celsius for ether would correspond to 100° Celsius for water, and the thickness of their capillary layers at the considered temperatures would be proportional to the values of the expression :

$$\sqrt[3]{\frac{T_k}{p_k}}.$$

That is about 1.5. At 100° Celsius the value of the radius R in the formula of Kelvin for a drop of water of *minimal size* would be thus about 6 or 7 microns. R being a mean value between the radii of the two spheres which envelop the spherical capillary layer of the drop, I put for the effective radius of a water-drop of *minimal size* at 100° Celsius :

$$R = 10 \mu\mu.$$

In the same manner as we have determined the order of greatness of the radius of a liquid drop, which has its minimal size, we can treat a spherical bubble of minimal size in the interior of a mass of liquid. We shall make the calculation again for ether at the temperature $T = 0.844T_k$ or $t = 121.5$ Celsius. The pressures $p_{\text{liq.}}$ and $p_{\text{vap.}}$ are in this case given by the ordinates of the points A_1 and C_1 of fig. 1. The equality of the thermodynamical potentials gives here in the same manner as above :

$$(v_1 + v_1')(p_1 - p_{\text{liq.}}) = (v_2 + v_2')(p_1 - p_{\text{vap.}}), \quad . \quad (20)$$

where v_1 and v_2 have the same meaning as above, while v_1' and v_2' are respectively the abscissæ of the points A_1 and C_1

of fig. 1. For our case the axis of volume is a tangent to the isotherm, and therefore

$$p_{\text{liq.}} = 0.$$

Further, I put approximately,

$$p_1 v_2 = p_{\text{vap.}} \cdot v_2'$$

and $v_1' = 2b = \frac{2}{3}v_k$ (van der Waals).

The equation (20) gives therefore

$$\frac{p_1}{p_{\text{vap.}}} - \frac{p_{\text{vap.}}}{p_1} = \frac{3v_1 + 2v_k}{3v_2}.$$

Hence,

$$\frac{p_1}{p_{\text{vap.}}} = 1.048.$$

Now $p_1 = 7896.2 \text{ mm.} \cdot 10.528 \times 10^6 \text{ dyne per cm.}^2$

The equation of Kelvin gives therefore for the order of greatness of the radius of the bubble of vapour of *minimal size*:

$$R = \frac{2H}{p_{\text{vap.}} - p_{\text{liq.}}} = \frac{2H}{p_{\text{vap.}}} = \frac{2 \times 5.17}{10.528} \times 1.048 \times 10^{-6} \\ = \text{about } 10^{-6} \text{ cm.} = 10 \mu\mu.$$

We find thus a value of the order of greatness of the thickness of the *plane* capillary layer at the considered temperature.

OBSERVATION I.

If we calculate the energy required to convert a film of *measurable* thickness at a constant temperature in a film of *minimal* thickness in the manner, given by Kelvin in his popular lecture "The Size of Atoms," we shall find for a water-film of 1 gramme, which has a minimal thickness of $6 \mu\mu$ (Johannott) an amount of work which is equivalent to almost 9 minor cal. *When we go on to stretch the film*, the state of the film becomes labile, and, if our considerations and calculations above are exact, the most complete crumbling would be attained when the film was converted into spherical drops with a diameter of about $6 \mu\mu$.

For the surface of a film of $6 \mu\mu$ (per gramme) we find further

$$\frac{2}{6 \times 10^{-7}} = \frac{10^7}{3} *.$$

* Because I calculate *the order of greatness* and not the *exact* value of the surface, I have taken for the mean density of the film unity.

The thickness of the capillary layer, which limits the spherical drop, being of the order of greatness of $2 \mu\mu$, and its homogeneous part having a radius of about $1 \mu\mu$, we take for the radius of the sphere, about which we distribute the surface energy, a value of $2 \mu\mu$. The collective surface of the spherical drops of minimal size becomes therefore :

$$\frac{4\pi \cdot 2^2 \cdot 10^{-14}}{\frac{4}{3}\pi \cdot 2^3 \cdot 10^{-21}} = \frac{3}{2} \times 10^7,$$

or 4.5 times the value which we find for the surface of the plane film. *The order of greatness of the value of the departure from the law of Pascal of the spherical capillary layer, which limits a drop of minimal size, as we have demonstrated above, being the same as the capillary tension in a plane capillary layer (constant of Laplace), we find thus for the energy required to crumble most completely a mass of water of 1 gramme a value which has the order of greatness of $4.5 \times 9 =$ about 40 minor cal.*

The finest haze of water at ordinary temperature is therefore still far remote from the vapour state.

In his calculation of the smallest limit of the thickness of water-films, Kelvin firstly supposes that the contractile force of a thin plane film is independent of its thickness. We have seen that the theory as well as the experiment of Johonnott gives really the same result. *Supposing secondly that the film remains stabile*, Kelvin finds for its minimal thickness a value of about $0.1 \mu\mu$, and this would give for the thickness of the capillary layer a value of about $0.03 \mu\mu$. If, however, my considerations above are exact, the minimum of the thickness of the plane capillary layer of water at ordinary temperatures would be about $2 \mu\mu$ and *the second supposition of Kelvin may not be admitted.*

OBSERVATION II.

By the aid of the numbers given by Loschmidt, we find at 100° Celsius for the number of the molecules in a cubic millieron of water : 32. Now I have found above for the order of greatness of the radius of a drop of water, which has its minimal size at 100° Celsius : $10 \mu\mu$. Hence follows easily by calculation that the order of greatness of the number of molecules in the considered water-drop of minimal size is about 80,000. Accumulations of 100 or 200 molecules for instance cannot therefore form a stabile water-drop in the vapour of water at 100° Celsius. *Only when a kernel of sufficient size is formed, is its field of force strong enough to prevent*

"*crumbling into molecules.*" If the temperature is higher the considered kernel must be larger, the velocity of the molecules being greater. This consideration is also in accord with my theory. Indeed, I have found that for a fixed temperature the radius of a liquid drop of *minimal size* is of the same order of greatness as that of the thickness of the *plane* capillary layer at the same temperature. Now my formula for the thickness h of the plane capillary layer becomes for instance for ether

$$h = \left\{ \frac{6 \cdot 014}{\sqrt{1-\mathfrak{S}}} - 4 \cdot 03 \right\} \mu\mu,$$

where $\mathfrak{S} = \frac{T}{T_k}$ and gives for $T = 0 \cdot 585 T_k$ or 0° Celsius :

$h = 4 \cdot 5 \mu\mu$, while we have found for the R in the formula of Kelvin, $R = 10 \mu\mu$. At the temperature $T = 0 \cdot 99 T_k$, we shall find : $h = 55 \mu\mu$. R being of the same order of greatness as h , we shall find about $100 \mu\mu$ for the order of greatness of R at the temperature : $T = 0 \cdot 99 T_k$. The minimal size of a liquid drop of ether at $T = 0 \cdot 99 T_k$ is therefore about the thousandfold of the value at 0° Celsius. Further, the critical density being about a third of the liquid density at 0° Celsius, a liquid drop of ether, *which has its minimal size* at $T = 0 \cdot 99 T_k$, must contain a number of molecules which is the 300-fold of this at 0° Celsius. If therefore in the immediate neighbourhood of the critical temperature condensation takes place, *it begins with relatively large "drops"* *. In fig. 1 the points A_8 and C_8 give for a drop of minimal size respectively the state in the interior of the drop, and in the vapour which surrounds it. In the immediate neighbourhood of the critical temperature, the parts C_8K and A_8H of the curve in fig. 1 being small, the state (A_8, C_8) for a drop on one side and the state given by (H, K) on the other side are not very different. Hence follows that a small variation of the temperature † or of the pressure converts the state (H, K) into the state such as (A_8, C_8) .

In connexion with my observation on the *relatively large* size of the drops which have their minimal size in the immediate neighbourhood of the critical temperature, we have therefore two conditions for the formation of mist. When we make a corresponding observation for the pair of points A_1C_1 in fig. 1, which gives the state of a bubble of *minimal*

* These "drops" are, however, ultramicroscopic, and form therefore firstly an *invisible* haze, which nevertheless by accumulations of several "drops" quickly passes into mist.

† In this case we must consider the complete set of isotherms.

size of vapour in the liquid, we see that we may expect in the immediate neighbourhood of the critical temperature in the proof-tube for critical researches, formation of mist and rain in the vapour and small vapour bubbles in the liquid. As every one knows, these phenomena are observed. Teichner for instance observes: "It is remarkable that this rain breaks out in both phases: in the vapour phase small drops of fluid fall down, and in the liquid small bubbles of gas ascend" *.

XXXII. *The Kinetic Energy of the Positive Ions emitted by Hot Platinum.* By F. C. BROWN, *Experimental Science Fellow, Princeton University* †.

THE object of the present paper is to set forth an investigation into the magnitude and the mode of distribution of the kinetic energy of the positive ions emitted by hot platinum.

The method of investigation was the same as that used by Richardson and Brown ‡ for the negative ions. It consists in measuring the rate of charging up of a plate opposite a parallel plate containing a strip of the hot metal, both plates being initially at zero potential. The ions from the hot metal cause the opposite plate to acquire a positive potential tending to stop further charging up. From the rate at which the current varies with the potential, both the mean value of the kinetic energy and the way in which it is distributed among the different particles can be determined. For a detailed description of both the theory and the practice of the method, reference must be made to the previous paper. As is there pointed out, the method only yields information about that portion of the kinetic energy of the ions which depends on their velocity components normal to the emitting surface.

It was shown (*loc. cit.*) that whatever the law of distribution of energy may be, the current between the plates (supposed infinite) will be given by

$$i = c \frac{dV}{dt} = ne \int_{\sqrt{\frac{2e}{m}V}}^{\infty} F(u_0) du_0 \int_{-\infty}^{\infty} f(v_0) dv_0 \int_{-\infty}^{\infty} f(w_0) dw_0, \quad (1)$$

where c is the capacity of the upper plate and its connexions,

* G. Teichner, *Ann. d. Phys.* xiii. p. 597 (1904). Compare also: H. Kamerlingh Onnes and G. H. Fabius, Repetition of de Heen's and Teichner's experiments on the critical state: Communications from the Physical Laboratory at the University of Leiden, Nr. 98, 1907.

† Communicated by Professor Richardson.

‡ Phil. Mag. Sept. 1908, pp. 355-366.

V is the potential at time t , n is the number of ions emitted by the lower plate per second, and u_0 , v_0 , and w_0 are the velocity components at the time of emission.

In the particular case where the distribution of velocity among the ions follows Maxwell's laws, this equation reduces to

$$\log_e \frac{i}{i_0} = -\frac{ve}{R\theta} V, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where i is the current against a potential V , i_0 is the current at zero potential, ve is the quantity of electricity required to liberate half a cubic centimetre of hydrogen in a water voltameter at zero degrees centigrade and 760 millimetres pressure, θ is the absolute temperature, and R is the constant in the equation $pv=R\theta$ taken for unit volume of gas under standard conditions.

We shall now describe the results of experiments which show that this relation is fulfilled within the limits of experimental error.

Experimental Results.

As the determination of the kinetic energy was based upon the rate of charging up of an insulated plate by the positive ions, it was quite necessary that there should be no negative ions emitted to the upper plate. The positive ionization appears greatest in platinum when it is first heated. After a few hours' heating in a high vacuum it disappears almost completely. For one platinum strip the magnitude of the positive ionization decayed with the time of heating about as shown in the accompanying table.

TABLE I.

Date.	Hours heated.	Positive current, amperes.	Temperature-resistance, scale.	Pressure, millim.	Negative potential on Upper plate.
Dec. 2	0	5×10^{-9}	7800	·01	40
Dec. 14	$4\frac{1}{2}$	1.3×10^{-9}	7800	·006	400
„	$4\frac{1}{2}$	6×10^{-10}	7800	·006	0
Dec. 18	9	8×10^{-11}	7800	·030	} [Not greater than 10 volts.]
Dec. 26	13	3×10^{-11}	7500	·009	
Dec. 28	18	10^{-12}	7400	760·000	
Dec. 29	21	3×10^{-13}	7500	760·000	

A part of the irregularity in the decay of the positive ionization may have been due to oxygen leaking into the vacuum system. It may also have been due to negative mixed with the positive ionization. As Professor Richardson has already pointed out, the negative ionization increases enormously with the temperature, and also does not begin to appear in new platinum until a higher temperature is reached than is necessary to obtain positive ions. Therefore the observations for determining the kinetic energy of the positive ions were taken before the platinum strip had been heated as much as two hours; in fact, the observations were taken just as soon as the hot platinum could be adjusted to zero potential and to a temperature satisfactorily constant; also the temperature was kept considerably lower than the corresponding temperatures in connexion with the negative ions. The temperature of the platinum strip ranged between 1067 and 1293 degrees on the absolute scale, whereas the corresponding temperatures used in obtaining the negative ions were between 1473 and 1840 degrees absolute.

The highest potential to which the upper plate was charged by the positive ions was about 0.5 volt. For some time there was some question as to whether there was not at that potential an equilibrium point between the positive and the negative ionization. As is well known*, the negative ionization increases very rapidly with an increase in the potential until the saturation-current is approached. In the above case, there would be few or no negative ions reaching the upper plate at zero potential; but as the plate charged up, the number of negative ions would increase and the number of positive ions would fall away, until the number of negative ions became equal to the number of positive; there would then be no further change in the potential of the upper plate. Obviously the theory that has been deduced could not apply to the experimental results if such conditions prevailed. That such conditions did not prevail was settled conclusively by the following method:—Before and after taking the observations, a positive potential of 2 volts was placed on the upper plate, the hot platinum remaining at zero potential, when it was found that there was no negative leak. Clearly, then, there could be no negative ionization when the upper plate was charged to 0.5 volt or less. The fact that the pressure in the vacuum was frequently high, tended to keep the negative ionization small.

Otherwise than has just been stated, the procedure in the manipulation was like that followed by Richardson and

* Cf. O. W. Richardson, *Phil. Trans. Series A*, vol. xx. p. 520.

Brown as described in the paper already referred to. Three series of observations were taken with different strips of platinum. The width of the platinum strip was least at the middle, where it was hottest.

One of the series of observations may be seen in Table II.

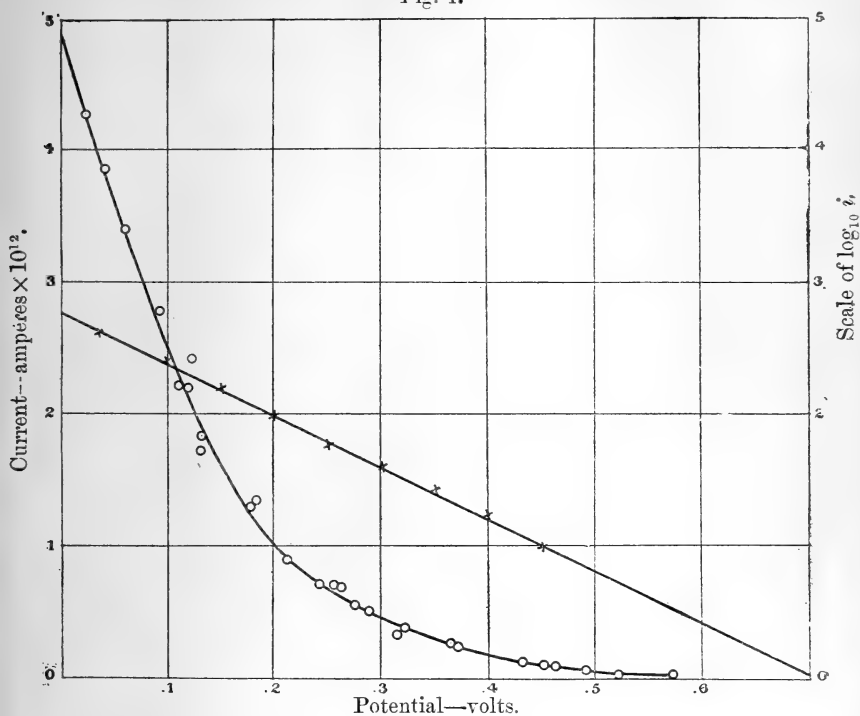
TABLE II.

Capacity, microfarads.	Pressure, millim.	Time.	Electrometer Read- ings (millims.).			Increment of po- tential, scale- divisions.	Interval of time, seconds.	Current, amperes × 10 ⁻⁴ .	Mean Potential, volts.
			Heating current :						
			Direct.	Re- versed.	Mean.				
·0001	·009	10 sec.	27	27					
			28	28					
			30	30	28.3	28.3	10	242.8	.125
		20 sec.	34	33					
			34	37					
			37	34	34.8	6.5	10	57.7	.274
		30 sec.	39.5	40					
			39	40					
			38	38					
		10 sec.	40	41					
			42	39	39.6	4.8	10	39.3	.320
		10 sec.	26	27	26.5	26.5	10	227	.120
		20 sec.	34	34	34	7.5	10	68.6	.262
		30 sec.	38	38	38	4	10	34.3	.314
		1 min.	47	47	47	9	30	25.7	.374
		1.5 min.	51.8	51.8	51.8	4.8	30	13.7	.430
		2 min.	55	55	55.0	3.2	30	9.2	.460
		3 min.	59.4	59	59.2	4.2	60	6.3	.497
		4 min.	60.2	61.8	61	1.8	60	2.8	.520
·001	·0085	10 sec.	5	—	5	5	10	429	.022
		20 sec.	9	9	9	4	10	343	.061
		30 sec.	12	12	12	3	10	280	.097
		1 min.	18	18.4	18.2	6.2	30	174	.130
		1.5 min.	22.6	23	22.8	4.6	30	135	.177
		2 min.	26.1	27	26.5	3.7	30	102	.210
		3 min.	31.1	32	31.5	5	60	71.5	.240
		4 min.	35	36	35.5	4	60	55.8	.286

Resistance 6750; temperature 1293° absolute scale.

The data for the curves shown in fig. 1 were obtained from this table. The agreement with equation (2) is at least as good as the expected error warrants. The curve showing

Fig. 1.



the relation between the logarithm of the current and the potential is almost a perfect straight line, and the experimental value of the gas constant comes to be

$$R = 3.5 \times 10^3.$$

A summary of the results of these three series of observations may be seen in Table III.

TABLE III.

Series.	Pressure, millim.	Absolute Temperature.	Largest current.	R.
1	28.0	1193	2×10	4.0×10^3
2	15.0	1067	1.7×10^{-12}	3.5×10^3
3	0.009	1293	5×10^{-12}	3.5×10^3
			Mean	3.6×10^3

The value of the gas constant calculated for the same number of particles as those for which R is given in the table is 3.7×10^3 . The agreement is quite satisfactory. The table also seems to show that the mean kinetic energy is independent, or almost independent, of the air-pressure surrounding the hot platinum. The fact that the temperatures were so low and so near the chief calibration temperature, that of the melting-point of potassium sulphate, may account for the close agreement in the value of R . The temperature should be accurate to within 0.5 per cent. The range of temperature is about twice as great as is the range of R when expressed as percentage variation. This would lead us to expect that the theoretical formula (2) would hold not only for the temperatures given but also for any temperatures obtainable. To test this experimentally is quite impossible, because of the appearance of the negative electrons at higher temperatures than those given in Table III.

The value of the mean energy of the ions in terms of the gas constant R may also be calculated from the data in Table II. by the use of equation (13) in the paper by Richardson and Brown (*loc. cit.*). That equation reduces to

$$R = \frac{ve \cdot t \cdot i_0 i}{\theta c (i_0 - i)}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where t is the time elapsed since the upper plate began charging up, i_0 is the current at zero potential and i the current after time t , c is the capacity of the upper plate, and θ is the absolute temperature.

Two determinations from one of the sets of data gave 4.3×10^3 and 4.8×10^3 as the value of R . We should expect to get the most accurate and consistent calculated values of R from equation (2). By drawing curves through the points representing the value of the current, and also through those representing the logarithm of the current for different potentials, the errors of single observations are largely eliminated. In addition, equation (3) requires an accurate knowledge of the magnitude of i_0 , which it is not necessary to know in the use of equation (2).

Discussion of Results.

The close agreement between the value of the kinetic energy and that required for thermal equilibrium, would appear to militate against the somewhat prevalent idea that the positive ions arise from chemical action. In that case, we should expect the kinetic energy of the emitted ions to be

increased by the heat of reaction. The small value of e/m recently obtained for the positive ions (384 for platinum) by Professor Richardson, indicates that they are not positive electrons. The balance of evidence appears to be in favour of the view that they are atoms or molecules of some foreign substance emitted from the metal by a process similar to evaporation and involving small thermal changes. The matter can hardly be regarded as definitely settled.

The author takes great pleasure in acknowledging his indebtedness to Professor Richardson for his interest and advice throughout the course of the investigation.

XXXIII. *A Note on the Electron Theory of the Carbon Arc.*

By J. A. POLLOCK, *Professor of Physics in the University of Sydney* *.

1. **INTRODUCTION.**—A general description of the phenomena of the carbon arc, from the point of view of the electron theory, is given by Professor J. J. Thomson in his book on the Conduction of Electricity through Gases †; here it is proposed to discuss merely the questions arising from a consideration of the results of a research on the Resistance and Electromotive Forces of the Electric Arc by Mr. Duddell ‡. This investigation, carried out with characteristic ingenuity and skill, yielded the important discovery of both a back and forward electromotive force in the arc, and the measurements obtained finally solve the old problem of the nature of the distribution of the potential between the carbons.

2. *Experimental Results.*—In the case of a direct current, 9.91 ampere arc, 6 millimetres long, between solid Conradty Norris carbons, 11 millimetres in diameter, Mr. Duddell gives the following figures for the arc potential-difference :—At the surface of the anode a back electromotive force of 16.7 volts, followed by a fall of potential of 16 volts within an extremely small distance from the anode surface, then a fall of 25 volts through the vapour column, a fall of 11.7 volts within a small distance from the cathode, and a forward electromotive force of 6.1 volts at the cathode surface.

3. *The Electromotive Forces in the Arc.*—When an ion of mass m moves freely between two points differing in potential

* Communicated by the Author. Read before the Electrical Association of New South Wales.

† See also Stark, *Ann. der Physik*, xii. p. 673 (1903).

‡ Duddell, *Phil. Trans. A*, 203, p. 305 (1904).

by V units, it acquires a velocity which may be found from the expression for the work done, viz. :—

$$\frac{1}{2}mv^2 = Ve,$$

where v is the final velocity, and e the ionic charge. If then at any section of a circuit, energy is communicated to the ions so that they are projected from that section with a velocity v , there is at the section what is electrically equivalent to an electromotive force which is given by the expression

$$V = \frac{1}{2} \frac{m}{e} v^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

v being now the velocity of projection.

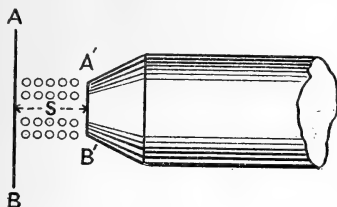
Considering that electrons and positive ions are projected from the surface of glowing carbon, with velocities depending on the temperature, we should therefore expect to find in a circuit containing a junction of heated carbon and a gas, what may be called a surface potential-difference at the outer layer of the carbon. This surface electromotive force is, in any instance, the difference between that due to the projection of electrons, and that arising from the emission of positive ions. As it is likely that the value of $mv^2/2e$ is far greater in the case of electrons than in that of positive ions, it is only necessary, for purposes of general description, to consider the electromotive force due to the projection of the former.

If the hot carbon is the negative electrode, so that the electrons are projected in the direction in which negative ions are being carried by the field, the surface potential-difference is a forward electromotive force helping the flow of the current, as discovered by Mr. Duddell at the cathode of the arc. If, on the other hand, the hot carbon is the positive electrode, so that the electrons are projected in a direction opposite to that in which negative ions are being carried by the field, the surface potential-difference is a back electromotive force, opposing the flow of the current, as at the arc anode.

The theory thus accounts in a simple way for the existence and direction of the electromotive forces as found by experiment. With the values of the forces, previously given in section 2, knowing that e/m for electrons is equal to 1.86×10^7 electromagnetic units, by substitution in equation 1, we arrive at the result that at the anode of the arc, electrons are projected with a velocity of 2.5×10^8 , and at the cathode with a velocity of 1.5×10^8 , centimetres per second.

4. *The Anode Fall of Potential.*—Let $A'B'$, in figure 1, represent the trace of the anode surface; from this surface

Fig. 1.



electrons are projected in great numbers, owing to the high temperature of the crater, and with a considerable initial velocity. These electrons collide with molecules of the gas at somewhat various distances from the anode surface; the average range may be represented by a length S , and the collisions may be considered, for the purposes of general description, to take place about a plane parallel to the anode surface at a distance S from it, whose trace is given by the line AB in the figure. The electrons travel through the distance S with negatively accelerated motion, as they are moving in opposition to the force in the field, and the outcome of their collisions with molecules is the production, at AB , of molecular negative ions. As all the ions so made travel to the anode, but, on account of their greater mass, with a movement so much slower than that of the electrons which created them, negative ions accumulate between $A'B'$ and AB . Further, as the projected electrons move with negatively accelerated motion towards AB , while the molecular ions travel with a positive acceleration from it, the accumulation will be at a maximum about that section.

The number of positive ions emitted from the anode surface is small compared with the number of electrons projected, the electrification to the right of AB is therefore negative, and its density far greater than that in the vapour column to the left of that plane.

By Poisson's equation it is known that this accumulation of negative ions near the anode surface is associated with a steep potential gradient, so that the well-known anode fall of potential is thus accounted for.

The negative ions accumulate until the anode fall of potential is sufficient to bring the projected ions to rest at the end of their average range S . The fall cannot be greater than this, otherwise some of the electrons would be brought to rest before collision with molecules, in which case few

molecular ions would be formed, and the fall of potential would decrease as the density of the electrification diminished. From this point of view one would expect the anode fall of potential to be equal to the surface difference of potential responsible for the velocity of projection.

In the case considered, the back electromotive force at the anode is 16.7 volts. This is the value of the discontinuity of the potential due to any heat effects which have an influence on the passage of the electrons through the surface layer, including the part represented by the energy with which the electrons emerge into the gas. According to the above argument the latter is equal to the anode fall of potential, which in this case is 16 volts. The difference between this value and that of the back electromotive force being but 0.7 volt, seems to justify the assumption underlying the whole discussion, that in the expression for the surface discontinuity of potential the term representing the translational energy of the projected electrons is predominant.

5. *The Cathode Fall of Potential.*—Let figure 1 now represent the negative carbon; from the surface A'B' electrons are projected with a velocity of 1.5×10^8 centimetres per second, due to a surface difference of potential of 6.1 volts. The velocity of the electrons increases after projection, owing to the electric force in the field, and assuming that the cathode fall of potential of 11.7 volts occurs in the distance S, corresponding to their average range, the electrons collide with the molecules of the gas at AB, with a velocity of 2.6×10^8 centimetres per second, equal to that acquired by the free fall through 17.8 volts.

In the hot gas collisions under such circumstances result in ionization, so that both positive and negative ions are produced at the section AB, the negatives moving to the left in the figure, while the positives travel slowly to the cathode.

Those electrons which collide with molecules, after a flight less than the average range, may not have sufficient energy to ionize, in which case some negative molecular ions are produced between AB and the cathode surface, to be lost, perhaps, by recombination with incoming positives. As ionization certainly takes place about AB, there is between this plane and the cathode a maximum density of positive electrification; this, by Poisson's equation, means a steep potential gradient which corresponds to the well-known cathode fall of potential.

It may be considered that in the development of the arc

the magnitude of the cathode fall of potential reaches a limit when ionization by collision, at the end of the average range of projection from the cathode, is fully established. In this case the value found by Mr. Duddell may be taken as a constant, characteristic of the carbon arc.

6. *The Vapour Column.*—From the ease with which the arc is displaced, by even small magnetic forces, one is led to consider that a large proportion of the current through the vapour column is due to electrons moving with high velocity. As the electric force in the column, about 4 volts per millimetre in the case under consideration, is utterly inadequate to guide unaltered a swarm of electrons through molecules and positive ions, the mode of conduction in the vapour column would seem to be different from that usually considered in the case of gases.

Possibly, with a critical value of the cathode fall, the collision of the projected corpuscles with molecules of the highly heated gas, may start a progressive exchange of electrons among the oriented atoms, throughout the vapour column, somewhat like that suggested by Professor J. J. Thomson* for the case of conduction through metals.

In such a case it may be considered that the electrons would be handed on, as it were, from atom to atom with unimpaired energy, and would thus reach the anode surface with the critical velocity of the corpuscles starting the exchange, together with that acquired during the flight through the anode fall of potential, or with a speed of 3.6×10^8 centimetres per second.

In view of Professor Townsend's† discovery of the expulsion of two electrons from an atom in ionization by collision, the above suggestion, as to the mode of conduction through the vapour column, would lead to the consideration of an ever increasing volume of ionization towards the anode, thus accounting for the shape of the arc, and an area of crater proportional to the arc length. Without modification the idea would, however, be inconsistent with the fact of a zero density of electrification throughout the vapour column, as the positive ions made near the anode in moving towards the negative electrode would not meet an equally dense negative stream.

7. *Summary.*—The application, to the case of the arc, of the idea of the projection of electrons from the surface of

* Thomson, "The Corpuscular Theory of Matter" (1907).

† Townsend, Proc. R. S., A. 80, March 1908.

incandescent carbon, leads to the following general description of the distribution of the potential between the carbons :—

1. A forward electromotive force at the cathode surface.
2. A back electromotive force at the surface of the anode depending on the temperature of the crater.
3. An anode fall of potential equal to the back electromotive force at the anode.
4. A cathode fall of potential, such that the result of its addition to 1 is a critical value.

The description agrees with the facts as discovered by Mr. Duddell.

As the magnitudes of the above quantities are independent of the current and arc length, the sum of 2, 3, and 4, minus 1, represents what has been called the back electromotive force of the arc, in the case considered 38·3 volts.

With the figures given, the velocities of the electrons are as follows :—

At projection from anode, $2\cdot5 \times 10^8$ centimetres per second.

At projection from cathode, $1\cdot5 \times 10^8$ centimetres per second.

At collision at end of cathode fall of potential $2\cdot6 \times 10^8$ centimetres per second.

At collision with anode surface, possibly $3\cdot6 \times 10^8$ centimetres per second.

The Physical Laboratory,
The University of Sydney, July 3rd, 1908.

XXXIV. *The Discharge of Electricity from Glowing Carbon.*

By J. A. POLLOCK, *D.Sc.*, *Professor of Physics in the University of Sydney*, and A. B. B. RANCLAUD, *B.Sc.**

[Plate VI.]

1. *Introduction.*

THE experiments, of which a description is here given, were undertaken in connexion with an investigation of the phenomena associated with the relighting of the carbon arc. For the object in view, an arrangement was required in which carbon rods should be situated somewhat as they are in an arc-lamp, and in which the temperature of one of the rods could be readily controlled. The plan adopted was as follows :—A cylinder of carbon, 4·5 centimetres long and 0·5 centimetre in diameter, was electrically heated, and a

* Communicated by the Authors. Read before the Royal Society of New South Wales.

circuit arranged to include an air-gap, of a few millimetres, between the middle point of the heated cylinder and the end of a comparatively cool carbon rod. The currents in the circuit have been measured for different temperatures of the carbon cylinder, and for various voltages across the air-gap up to the point at which an arc forms, with a view to finding the conditions under which the change from the non-luminous to the luminous discharge occurs in the case of hot carbon in air at natural pressure. The conditions for the similar change in connexion with a hot lime cathode in air at low pressure have been investigated by Professor J. J. Thomson*.

2. Experimental Detail.

The apparatus is shown in plan and elevation in fig. 1 (Pl. VI.), C, the hot carbon cylinder, is held by large blocks of carbon A & B (not shown in elevation), through which the heating current passes. The cool carbon rod D is fixed in a holder E which can be moved in the direction of its length by turning the nut F, the latter being prevented from longitudinal movement by the stop G. The holder E is attached to a cross piece provided with an adjustable stop H, the cross piece working on pivots so that the cool carbon rod D can be very quickly placed in position before the heated cylinder C or readily removed.

A scheme of the connexions is shown in fig. 2, where A and B are storage-cell batteries, G a galvanometer, and P a potentiometer; by this arrangement any difference of potential desired could be set up between H and C, allowance being made, of course, for the potential gradient in the hot rod due to the heating current. A fine wire fuse was used to protect the galvanometer from excessive currents.

When the cold carbon is close to the hot one its temperature rises, and the current varies with time. To secure uniformity measures have been taken in the following way:—with the cool carbon removed, the hot cylinder was brought to its full temperature; then by tilting the holder E, fig. 1, the cool carbon was put quickly into position at the required distance from the cylinder; in 6 to 10 seconds after this the galvanometer deflexion became fairly steady, and all observations have been taken within those limits of time.

The heated carbon cylinder was a piece of the carbon rods supplied by Messrs. Siemens Brothers for use with their Lilliput arc-lamps, while the cooler carbon was a portion of

* Thomson, 'The Conduction of Electricity through Gases,' 2nd ed. p. 477 (1906); see also Horton, Phil. Trans. A. 207. p. 149 (1907).

Conradty arc-lamp carbon, turned down at the end as shown in the figure.

The temperature of the heated carbon was found with the aid of a Holborn-Kurlbaum optical pyrometer, calibrated at the Reichsanstalt. The temperatures are given as observed, they are therefore in all cases the equivalent black body values.

To get an estimate of the temperature of the cool carbon when currents were measured, the junction of a thermoelectric thermometer of platinum and platinum-iridium wires, 0.5 millimetre in diameter, was placed in the position of the end of the carbon rod, the rod being temporarily removed. The temperature given by the thermometer, under any circumstances as to distance of the junction from the heated cylinder and length of time in position close to it, is considered to be that of the end of the carbon under similar conditions.

3. *Flow of Negative Electricity from the Hot Carbon.*

The general form of the relation, without reference to exact scale, between potential-difference and current for temperatures of the hot rod in the neighbourhood of 1800°C. , is shown in fig. 3, where ordinates represent currents, and abscissæ potentials of the hot relatively to that of the cool carbon.

For the flow of positive electricity from the hot carbon it is sufficient to say that any increase, above the value for zero current, in the potential of the hot carbon, gives a much smaller current than a decrease of the same amount. This smaller current, however, under the circumstances of our experiments, is not the definite measure of the stream of positive ions from the hot carbon; it is the resultant of such a stream and one of negative ions from the cooler electrode.

The current of negative electricity from the hot carbon depends on the temperature of the carbon, on the potential-difference between the electrodes, and on the distance separating them. At low voltages the current is of the order of a milliampere and is not accompanied by any luminosity. As the potential-difference increases a critical value, depending on the temperature and on the distance between the carbons, is reached at which an arc forms, and the current jumps instantly from milliamperes to amperes; the critical points are shown at X in the figures.

To exhibit the form of the connexion between potential-difference and current, for the non-luminous régime when

the hot carbon is negative with respect to the cool one, the relations between them, for distances separating the carbons of 1, 2, and 3 millimetres, are shown in figs. 4, 5, and 6. The final portions have been drawn vertical, the abscissæ of the vertical parts being the values of the voltages at which arcing occurred. The numbers attached to the curves are the values of the temperatures of the hot and cool carbons respectively.

4. Critical Values.

The phenomenon of an abrupt change from a non-luminous to a luminous discharge is described by Professor Thomson* for the case of a gas at low pressure. A similar sudden change takes place in air at normal pressure, and under the circumstances of our experiments, the alteration, from the one régime of current flow to the other, can be brought about by a change in the potential-difference between the carbons of about a volt.

The currents just before arcing occurs can be deduced by graphical extrapolation, as shown in the above diagrams; the approximate values so found are collected in Table I.

TABLE I.

Distance between Carbons.	Temperature.		Critical Voltage for arcing.	Current just before arcing.
	Hot Carbon.	Cool Carbon.		
1 mm.	1800° C.	1360° C.	volts.	milliamperes.
"	1680	1240	43·5	2·37
"	1550	1100	55·0	2·50
"	1415	920	68·5	2·66
"			83·0	2·82
2 mm.	1800	1120	53·0	1·85
"	1680	980	66·0	1·95
"	1550	860	81·0	2·07
"	1415	650	97·0	2·20
3 mm.	2060	1070	39·5	0·97
"	1800	880	70·0	1·12
"	1680	750	85·0	1·20

The relation, as given by our observations, between the temperature of the hot carbon and the critical value of the potential-difference between the electrodes necessary to start arcing, is shown in fig. 7 for distances of 1, 2, and 3 millimetres between the carbons.

* Thomson, *loc. cit.*

All the observations given in this paper, with the exception of those in the following table, refer to the case where the hot and cool carbons are in the same horizontal plane. The potential-difference necessary to start arcing, however, depends very considerably on the position of the hot relatively to the cool carbon, and we give in Table II. measurements of this quantity with the hot carbon vertically below, and with it vertically above, the cool positive; in the lower part of the table are given the values of the critical voltage with the hot carbon vertically below the cool one, and with it in the same horizontal plane as the other electrode.

TABLE II.

Distance between Carbons.	Temperature.		Critical voltage for arcing.		Difference.
	Hot carbon.	Cool carbon.	Hot carbon below.	Hot carbon above.	
3 mm.	1930° C.	1420° C.	volts. 56·0	volts.	volts.
"	"	346	...	74·0	18·0
"	2110	1510	39·5		
"	"	650	...	54·5	15·0
"	2230	1550	26·5		
"	"	860	...	39·0	12·5
"	2460	1620	12·5		
"	"	1240	...	20·5	8·0
1 mm.	1930	1580	36·0		
"	"	1280	...	44·0	8·0
2 mm.	"	1530	42·0		
"	"	840	...	52·0	10·0
3 mm.	"	1420	56·0		
"	"	346	...	74·0	18·0
				Carbons in same horizontal plane.	
1 mm.	1930	1580	36·0		
"	"	1500	...	42·0	6·0
2 mm.	"	1530	42·0		
"	"	1200	...	46·0	4·0
3 mm.	"	1420	56·0		
"	"	1000	...	59·5	3·5
"	2230	1550	26·5		
"	"	1240	...	29·5	3·0

It is seen from the table that the potential-difference

required for arcing is less when the hot carbon is vertically below the cool one than when it is vertically above, or than when the two carbons are in the same horizontal plane.

5. *Distribution of Potential between the Carbons.*

It is well known that from the surface of hot carbon, electrons and positive ions are projected. The rate of emission of the latter is small, and its consideration may be neglected in the discussion of the present experiments when the hot carbon is negatively electrified. The electrons collide with molecules of the surrounding gas, at somewhat various distances from the hot carbon surface, the average of which may be called the average range of projection. The collisions result in the creation of molecular negative ions, which, if they are not removed, accumulate near the surface of the carbon, thus establishing a potential gradient through the average range of projection and a consequent movement of the ions towards the hot surface. Such a distribution of potential can be realized with the apparatus previously described by applying between the carbons the potential-difference requisite, at any temperature, to make the current in the circuit zero, the hot carbon, in this case, being positive relatively to the cool one. This distribution may be represented, from a merely illustrative point of view, by the curve 1 in fig. 8, where V is the potential and x the distance from the cool carbon, the hot carbon rise of potential ab being supposed to take place through a distance comparable with the average range of projection of the electrons.

If the hot carbon is made slightly less positive than the value required for zero current, ac in curve 1 will slope a little upwards; those few molecular negative ions formed near the surface of the hot carbon beyond the point a will be drawn by the field to the cool rod, though the great majority will still go to the hot carbon. A further stage, with the hot carbon negative with respect to the cool one, is illustrated by curve 2. As the potential-difference between the carbons increases, the hot carbon rise of potential ab becomes less and less, and the proportion of the negative ions carried to the cool carbon becomes greater. If the potential-difference rose to such a value that the hot carbon potential rise disappeared, the current would be saturated; in our experiments an arc forms long before this stage is even approached.

The deduction of an analytical expression for the relation between potential-difference and current, in the case under consideration, presents difficulties; it involves a statement of the circumstances of the appearance of the molecular

negative ions near the hot carbon, and requires a knowledge of the relation between the mobility of the ions and distance from the hot surface, which depends on the temperature gradient existing between the carbons.

6. *Change from the Non-luminous to the Arc Discharge.*

In the carbon arc there is a very characteristic cathode fall of potential which indicates an accumulation of positive ions near the cathode surface. In view of the fact that, in the experiments under discussion, the arc forms long before the current is saturated, while there is yet a potential *rise* at the cathode, and therefore no possibility of ionization by collision near the hot surface, one must look to the anode for the origin of the change from the non-luminous to the arc discharge. As a confirmation of this view we have noticed that the change to the luminous régime of current flow is always heralded by the appearance of a small white-hot spot somewhere on the face of the positive carbon. After the advent of the spot, the development to the full arc takes place too rapidly for its phases to be followed by the eye, although if the circuit is immediately opened, the formation of the luminous discharge may be prevented.

If I is the current of negative ions, E the potential gradient at the anode surface, and λ the length of the last free run of the ions, at the end of which they collide with the anode, the energy reaching the anode surface per second is $IE\lambda$; it is here suggested that for the arc to form, the potential-difference between the carbons must reach a value necessary to make the magnitude of $IE\lambda$ sufficient to raise a portion of the anode surface to such a temperature that positive ions are somewhat freely emitted. These ions, in travelling to the cathode, first annul the hot carbon rise of potential, and then by accumulating near the cathode surface create a cathode fall of potential. With a cathode fall the positive ions bombard the cathode surface with considerable energy, thereby raising its temperature. As a result the electrons are projected in greatly augmented numbers, and with enhanced velocity, their speed being still further increased during the free flight in the now reversed field. When the velocity of the electrons, at the end of their average range, reaches the value of 2.6×10^8 centimetres per second, it may be considered, as suggested by one of us*, that the arc discharge is fully established.

In our experiments, the currents when arcing commences

* Pollock, *suprà*, p. 361.

have been estimated, but we have been unable to deduce the magnitude of the potential gradient at the anode surface. The value of this latter factor will include a part due to the projection of electrons from the surface of the anode, which will be greater the higher the anode temperature.

In the above description it is considered that the development of the arc from the non-luminous discharge takes place in two stages. The first one commences with the somewhat copious emission of positive ions from the anode surface, when the energy of its bombardment by negative ions reaches a critical value; the second one begins when the velocity, at collision with gaseous molecules, of electrons projected from the cathode, commences to increase on account of the presence of positive ions near the cathode surface; it is completed when this velocity is such as to start a mode of conduction through the vapour column, characteristic of the fully developed arc, in which, perhaps, electrons are handed on from atom to atom through the column, as suggested in the paper just mentioned.

The view we have taken of the origin of the change from the non-luminous to the arc discharge receives some support, we think, from the observations of the differences in the voltages required for arcing given in Table II. When the hot carbon is vertically below the cool one, the flow of molecular ions is helped by the convection current of hot gas, whereas in the reverse position the flow is opposed by the current: the value of $IE\lambda$ may thus, for the same potential-difference, be different in the two cases. In addition, the temperature of the cool carbon when below the hot one is much lower than when above it; the critical value of $IE\lambda$ may, therefore, have to be higher for arcing to start in the former case than in the latter. If the change to the luminous discharge is considered to originate near the cathode surface, the observed differences in the arcing voltages, due to alteration in the relative positions of the hot and cool carbons, would be difficult to explain.

7. Summary.

The flow of negative electricity from hot carbon, in a circuit containing an air-gap, up to three millimetres in length, between a hot and a cool carbon rod, has been investigated for temperatures of the hot rod from 1100°C. to 1800°C. , and for various voltages up to the point at which an arc forms between the carbons, the experiments being made in air at natural pressure.

If I is the current of negative ions, E the potential gradient

at the anode surface, and λ the length of the last free run of the ions at the end of which they reach the anode surface, it is suggested that arcing commences when $IE\lambda$ attains a value sufficient to raise a part of the anode surface to such a temperature that positive ions are somewhat freely emitted.

The Physical Laboratory,
The University of Sydney.
Aug. 26th, 1908.

XXXV. *The Absorption of the Radioactive Emanations by Charcoal.* By R. W. BOYLE, M.Sc.*

[Plate VII.]

THE experiments of Sir James Dewar in 1903 drew attention to the property possessed by the charcoal of coconut of absorbing gases in a remarkable degree. Since then this property has been investigated by several experimenters†, and has been applied by Sir William Ramsay in separating the inert gases helium and neon from the air, and measuring the quantities in which they exist in the atmosphere. The results in the papers mentioned below may be summarized as follows:—

- (1) The absorption of gases by coconut charcoal is greatly increased by lowering the temperature of the charcoal.
- (2) The charcoal has a greater "affinity" for some gases than for others, there being something of the nature of a selective absorption. In consequence of this property, it is possible to separate from one another, in some degree, the constituents of a mixed gas.
- (3) Of the inert gases of the argon family, argon is absorbed about in the same degree as ordinary gases, but helium and neon in a much less degree, and helium less than neon.
- (4) The absorption is also influenced by the pressure of the gas in contact with the charcoal, the pressure-concentration-curves varying in a regular manner with the temperature.

* Communicated by Prof. E. Rutherford.

† "The Absorption and Thermal Evolution of Gases occluded in Charcoal of Low Temperature." Dewar, Proc. Roy. Soc. vol. lxxiv. p. 122.

"The Separation of the most volatile Gases from Air without Liquefaction." Dewar, Proc. Roy. Soc. vol. lxxiv. p. 122.

"The Determination of the Amount of Neon and Helium in Atmospheric Air." Ramsay, Proc. Roy. Soc. May 24th, 1905.

"The Law of Distribution in the Case in which one of the Phases possesses Mechanical Rigidity." "Absorption and Occlusion." Travers, Proc. Roy. Soc. July 21st, 1905.

It was pointed out by Professor Rutherford, in a letter to 'Nature,' October 6th, 1906, that this property of coconut charcoal also holds for the radioactive emanations; and in his letter are accounts of experiments demonstrating the fact. Professor Rutherford found that a slow current of air charged with the emanations of radium, thorium, or actinium is deprived of some or all of its emanation in the passage through a tube filled with coconut charcoal, and for this purpose the charcoal need only be at ordinary temperature. The complete withdrawal of the emanation only takes place when the air is passed very slowly through the charcoal; if the speed is increased, the amount of emanation emerging is increased also. In another experiment, Rutherford found that if a tube containing less than a gramme of the charcoal is open to a vessel containing the emanation from several milligrammes of radium bromide, in the course of time the emanation is absorbed by the charcoal. If some powdered willemite is mixed with the charcoal, the gradual absorption is shown by the increasing brilliancy of phosphorescence of the willemite. It is not necessary to heat or exhaust the charcoal previously, but when either is done the emanation is more rapidly absorbed. The charcoal retains the emanation at ordinary temperature, but the greater part of it is expelled by heating to a low red heat.

The phenomenon of the absorption of emanation is not only of much interest in itself, but also in the fact that it may be of use in determining directly the amount of emanation existing in any particular gas. Mr. Eve, of this laboratory, is now utilizing the method to determine the amount of radium emanation existing in the free atmosphere.

It may be noted that, since the emanation must be mixed with some gas, there is special interest in studying the information contained in the papers of Dewar and Ramsay concerning the absorption of the inert gases of the argon family which exist in the atmosphere. The cases are analogous in that the emanations are believed to be chemically inert gases, and are mixed with the containing gas in extremely small proportions; but there is the distinctive difference that the emanations decrease in amount with time, while the gases of the argon family do not.

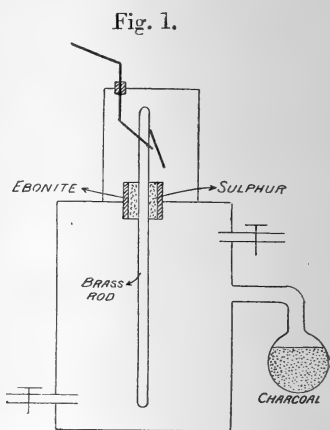
It is important to determine the laws which govern the absorption of the radioactive emanations, to know how far it depends on the speed of the containing gas through the absorbent, the temperature of the absorbent, and other conditions. An investigation of these points is the main object of this paper.

In consequence of the delicacy of electrical tests for determining the presence of minute quantities of emanations from radioactive substances, it is possible to carry out experiments with quantities of emanation far too small for examination by chemical or other known physical methods, yet the measurements are capable of a high degree of accuracy.

Radium Emanation.

A short account may be given of a few preliminary experiments on the diffusion of radium emanation from vessels containing the emanation mixed with air to vessels containing coconut charcoal. The apparatus used (fig. 1)

was an emanation electroscope, which consists of two chambers, one above the other. The lower chamber is an air-tight cylindrical vessel of brass, of about 1 litre cubical capacity, serving as an emanation reservoir, while the upper is a vessel covering the gold-leaf system which can be observed through a mica window by a microscope with a scale in the eyepiece. The gold leaf is at the upper end of a brass rod which extends down into the lower chamber and is well insulated. The lower chamber is fitted with tubes and stopcocks, through which air charged with emanation can be admitted or withdrawn at will by means of a pump. Connected with the emanation reservoir by a short tube of large cross-sectional area was a glass bulb containing a few grammes of coconut charcoal in granulated form. The outside of the electroscope was earthed, and the ionization currents measured by the rate of movement of the gold leaf. The ionization current, or activity, at any time is proportional to the amount of emanation present in the emanation reservoir.



When radium emanation was admitted into and retained in the electroscope, it was found that the decay of activity did not follow an exponential law, for while at the first the activity fell to half-value in 1.0 days, in the course of five or six days the rate was half-value in 1.6 or 1.7 days.

When the emanation reservoir was refilled with emanation and the charcoal vessel removed, so that the emanation could

diffuse to the open air, the decay of activity did follow an exponential law, the time for diminution to half-value being 1.0 day. This result may be expected, for it means that the amount of emanation diffusing to the open air per second is proportional to the amount of emanation in the reservoir. P. Curie and Danne* performed experiments of this kind with capillary tubes as the communication from the emanation reservoir to the outside air, and found that the loss of emanation from the reservoir was exponential.

It should be noticed that, in these experiments, if the emanation reservoir were sealed, so that there could be no diffusion to charcoal or to open air, the activity would fall to half-value in 3.75 days.

From the figures above, the rate of diffusion of the emanation from the reservoir to the charcoal was at first the same as the rate of diffusion to the open air, viz., half-value in 1.0 day; but as time went on, this rate gradually became less. A probable explanation of this fact is that the layer of charcoal first reached by the emanation acts as a good absorbent until it becomes saturated, very little emanation passing to the under layers until the top layer reaches this condition. In consequence, it will take time for the emanation to diffuse from the upper to the lower layers of charcoal, with the result of a gradual lessening of the amount of emanation absorbed from the reservoir.

Another experiment was made with the emanation diffusing to a very thin layer of charcoal, and the results obtained support this explanation. In this case the activity diminished at first at the rate of half-value in 1.3 day, but kept gradually changing until, after twenty-four hours, the rate was half-value in 3.2 days. At this stage there could be very little absorption, since the natural decay of the emanation itself is half-value in 3.7 days. We may say, then, that it took about a day to saturate the thin layer of charcoal.

In the experiments just described, the charcoal has been contained in a vessel outside the emanation reservoir; when the charcoal is in the reservoir itself, the absorption takes place more rapidly, as the time for the emanation to diffuse through the connecting tubes is eliminated. In an experiment where a tray containing charcoal was placed in the bottom of the emanation reservoir, on admitting some emanation the absorption took place so quickly that there was not the usual rise of activity due to the active deposit, but, on the contrary, an immediate, rapid fall. With only a thin layer of charcoal—a sprinkling covering the bottom of the tray—there is a

* Curie & Danne, *Comptes Rendus*, cxxxvi. p. 1314 (1903).

slight rise of activity due to the active deposit, but not nearly in the same degree as if the charcoal had been absent.

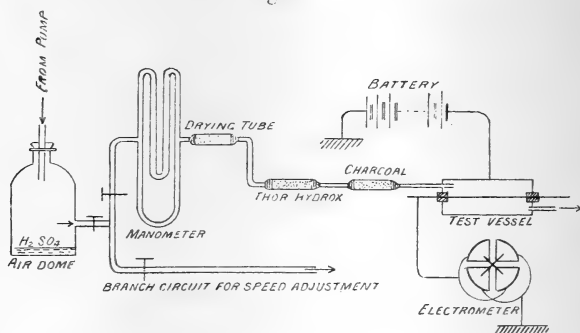
The above experiments were all performed in a constant-temperature room, so that all effects on the diffusion by changes of temperature and draughts of air were avoided.

More experiments, along the same lines as the above, are being carried on.

Thorium Emanation.

In the case of thorium emanation, owing to its short period of decay—to half-value in 54 seconds,—it is necessary to use a flow method of experiment. In other words, the emanation must be mixed with some gas, and conducted through tubes containing absorbing, or non-absorbing, material into a testing vessel, where it ionizes the air, and the ionization currents can be measured by means of an electrometer. The apparatus used for the experiments is represented by the accompanying diagram (fig. 2).

Fig. 2.



An air-pump supplies a current of air through a tube of thorium hydroxide, where the emanation mixes with the air and thence passes through tubes containing absorbing or non-absorbing material, as the case may be, into a testing-vessel of the ordinary cylindrical type. An accurately calibrated manometer in the circuit measures the velocity of the air-current. The testing vessel is connected to one pole of a battery of E.M.F. sufficient for complete saturation, the other pole of the battery being earthed. The central rod of the testing vessel is connected with a suitable electrometer, and a condenser of adjustable capacity is placed in parallel with it. It should be noticed that, with this arrangement, absorption of the emanation takes place when the charcoal has absorbed its full amount of the atmospheric gases.

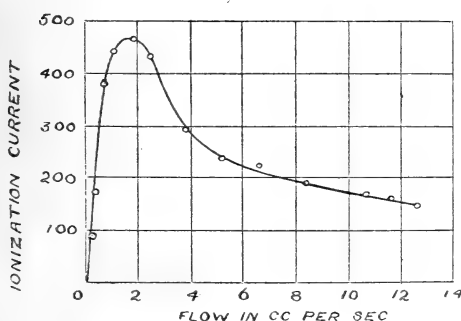
The weight of thorium hydroxide used was 37 grms.

Using this apparatus when there is no absorbing material between the thorium hydroxide and the testing-vessel, and taking observations of the ionization current with varying speed of the air-current, we get a curve such as is shown in fig. 3 (Table I.), which has ionization (i) for ordinates, and speeds of the air-current in c.c. per sec. (q) for abscissæ. We shall hereafter call this the i - q curve.

TABLE I. (Fig. 3.)
Emanation directly into testing vessel.

Flow in cc./sec.	Ionization Current in arbitrary units.
0.30	91
0.37	174
0.76	380
1.12	441
1.86	464
2.52	433
3.80	291
5.18	238
6.60	224
8.40	192
10.7	169
12.6	149

Fig. 3.



The shape of this curve can be deduced from theoretical considerations by the following:—

The thorium hydroxide, in radioactive equilibrium, gives off free atoms of emanation at a constant rate, and these are

all removed by the air-current. In consequence, there will be the same number of emanation atoms leaving the thorium-tube per second for any speed of the air-current.

The ionization current in the testing vessel is proportional to the number of emanation atoms breaking up per second in the testing-vessel; and this number is λ times the number of emanation atoms present, λ being the coefficient of decay of the emanation.

Let N = the number of free emanation atoms given off by the thorium hydroxide per sec.,

q = the flow of the air-current in c.c. per sec.,

T = the time for the air to pass from the thorium hydroxide to the testing-vessel.

Then, with q c.c. of air entering the testing-vessel are associated $Ne^{-\lambda T}$ atoms of emanation. If we denote the number of emanation atoms per c.c. of air at the entrance of the testing-vessel by n_0 , then

$$n_0 = \frac{Ne^{-\lambda T}}{q}.$$

Let V be the total free volume of the conducting tubes between the thorium hydroxide tube and the testing-vessel; then

$$T = \frac{V}{q},$$

$$\text{and } \therefore n_0 = \frac{Ne^{-\lambda T}}{q} = \frac{Ne^{-\frac{\lambda V}{q}}}{q}.$$

To obtain the total number of emanation atoms, P , present in the testing-vessel, consider an element of volume dw , with a volume w between the entrance and the element, as shown by fig. 4. $\frac{w}{q}$ is the time for the air to traverse the volume w , and therefore, if n = the number of emanation atoms per c.c. of air within the element,

$$n = n_0 e^{-\frac{\lambda w}{q}}.$$

The total number of emanation atoms in the element dw

$$= ndw$$

$$= n_0 e^{-\frac{\lambda w}{q}} dw;$$

$$\therefore P = \int_0^W n_0 e^{-\frac{\lambda w}{q}} dw,$$

where W is the whole volume of the testing-vessel.

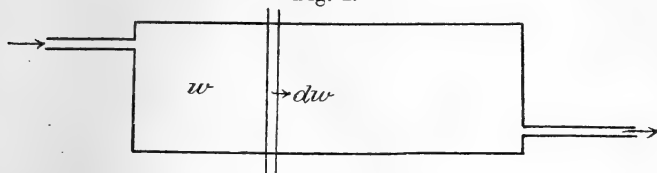
Integrating, we get

$$P = \frac{qn_0}{\lambda} [1 - e^{-\frac{\lambda W}{q}}],$$

and substituting for n_0 ,

$$P = \frac{N}{\lambda} e^{-\frac{\lambda V}{q}} [1 - e^{-\frac{\lambda W}{q}}].$$

Fig. 4.



The ionization current is proportional to the number of emanation-atoms breaking up per second, which is λ times the number present, *i. e.* λP . Hence, if i is the ionization current,

$$i = K\lambda P,$$

where K is a constant ; and therefore

$$i = KN e^{-\frac{\lambda V}{q}} [1 - e^{-\frac{\lambda W}{q}}]. \quad . \quad . \quad . \quad (1)$$

If we plot a curve with ionization currents as ordinates and flow of air in c.c. per second as abscissæ, we can see from (1) what the shape of the curve should be.

Denoting $-\lambda V$ as a , and $-\lambda W$ as b , we have

$$i = KN e^{\frac{a}{q}} [1 - e^{\frac{b}{q}}],$$

from which we can at once see that $i=0$ when $q=0$ or ∞ . Differentiating, we get

$$\frac{di}{dq} = KN e^{\frac{a}{q}} [(a+b)e^{\frac{b}{q}} - a],$$

from which it follows that i is maximum when

$$q = \frac{a}{\log \frac{a}{a+b}},$$

or when

$$q = \frac{\lambda V}{\log \frac{V+W}{V}}. \quad . \quad . \quad . \quad (2)$$

The ionization, then, must rise from zero to a maximum, and fall off to nothing as the speed of the air-current increases to infinity.

It can be seen that the curve of fig. 3 conforms to these conditions.

The relation deduced in (1) shows us that the ionization-current depends not only on q , the speed of the air-current, and on W , the volume of the testing-vessel, but also on V , the free volume of the space through which the emanation has to pass before it reaches the testing-vessel. When V is increased, i is decreased; and, following from the condition for a maximum (2), q must be increased to give the maximum ionization-current. Hence it follows, that in experimenting to compare the absorption by different substances, or by different quantities of the same substance, the free volume between the thorium hydroxide and the testing-vessel should be constant. In the experiments described later, where it was desired to compare the absorption, care was taken to have this condition fulfilled.

When an absorbing material is inserted between the thorium hydroxide and the testing-vessel, there is less emanation available for ionization and the i - q curve is much altered *. In the case of charcoal, in granulated form, it has been found that the amount of absorption depends on

- (a) The nature of the charcoal,
- (b) The speed of the air-current through the absorbent,
- (c) The amount of charcoal surface exposed to the emanation,
- (d) The temperature of the charcoal.

In the experiments on these points, the results of which are given later, the i - q curves in the cases of non-absorption were obtained by substituting for the tube containing the charcoal a similar tube containing a non-absorbing sand of the same volume and of the same size grain as the charcoal used. This precaution was necessary in order to have the same volume between the thorium hydroxide and the testing-vessel in the case of non-absorption as in the case of absorption. In all experiments with a given sample of sand or charcoal, an attempt was made to have the grains of uniform size by sifting through a set of sieves and collecting separately the residues caught in the meshes of each sieve.

- (a) *Nature of the absorbent.*

Fig. 5 (Table II.) shows the i - q curves, taken under the same experimental conditions, for the cases of absorption by ordinary wood, animal, and coconut charcoal, all of the

* The Effect of Temperature and of Velocity of Gas-Current on the Absorption of Radioactive Emanations by Charcoal." R. W. Boyle, Trans. Royal Soc. Canada, May 1907.

TABLE II. (Fig. 5, Pl. VII.)
Multiply currents by 6.16×10^{-13} to reduce to amperes.

Non-Absorption.		Wood Charcoal.		Animal Charcoal.		Coconut Charcoal.	
Flow in cc./sec.	Ionization Current.	Flow in cc./sec.	Ionization Current.	Flow in cc./sec.	Ionization Current.	Flow in cc./sec.	Ionization Current.
Nat. leak at beginning.		Nat. leak at beginning.		Nat. leak at beginning.		Nat. leak at beginning.	
0.40	3.10	0.53	0.81	0.37	1.46	1.07	1.63
0.49	31.6	1.07	32.7	0.60	3.48	1.48	6.24
0.63	33.9	1.52	113.9	0.97	26.8	1.89	16.9
0.86	90.8	1.95	141.0	1.34	73.0	3.07	30.1
1.26	123.6	2.37	147.0	1.67	88.4	3.47	42.8
1.62	161.1	2.84	136.4	2.39	97.1	4.15	46.1
2.34	169.4	3.62	123.2	2.48	98.0	4.91	45.7
3.14	154.5	4.76	101.7	3.23	86.9	6.09	44.4
3.77	130.8	5.95	83.6	3.80	76.1	8.77	43.7
4.78	110.5	6.80	73.2	4.33	71.1	Nat. leak at end.	42.0
5.51	91.5	7.07	70.7	5.40	54.8		2.26
7.10	84.1	8.40	62.9	6.70	32.5		
8.66	77.0	Nat. leak at end.	1.84	7.86	43.9		
	67.3			8.68	39.4		
	4.07			Nat. leak at end.	36.2		
					3.42		

same volume and same size grain. The wood charcoal weighed 0.77 gm., animal 2.25 gms., and coconut 2.17 gms.

The curves show that coconut charcoal absorbs the most, and ordinary wood charcoal the least of the three, and that the variation in absorptive power is very marked. In fact, two different samples of the same kind of charcoal can show appreciably different absorbent powers. I have had samples of coconut charcoal differing considerably in this respect, the better absorbers being softer, less gritty, and less dense than the others.

(b) *Speed of the air-current.*

An inspection of the curves mentioned above, or of any of those following, will show that the amount of absorption depends on the speed of the air-current which conducts the emanation through the absorbent material.

The absorbent must have time to absorb, and at quick speeds it does not get as good a chance.

The following facts, taken from fig. 5 and tabulated, illustrate very well (a) and (b). Here the percentage absorption for any given speed is represented by

$$100 \times \frac{\text{Difference between non-absorption and absorption ordinates}}{\text{Non-absorption ordinate}}.$$

The table shows that the percentage absorption is increased as the speed of the air-current is decreased.

TABLE III.

Absorbent.	Speed of Air-Current.	Ionization Current, Absorption.	Ionization Current, Non-Absorption.	Per cent. Absorption.
WOOD CHARCOAL.	0.7	55	93	40.9
	1.0	94	138	31.9
	1.5	139	164	14.8
	4.5	86	92	6.5
	8.0	63	665	5.3
ANIMAL CHARCOAL.	0.5	11	55	80.0
	0.7	31.5	93	66.2
	1.0	62	138	55.0
	2.0	98.5	164	39.9
COCONUT CHARCOAL.	1.5	16	164	90.3
	2.0	28.5	164	82.6
	4.5	43	92	53.3
	8.0	40	66.5	39.8

(c) *The surface exposed to the emanation.*

It is natural to suppose that the amount of absorption will in some way depend on the amount of the charcoal absorbing surface—that, in fact, the absorption will be increased if the surface be increased.

To prove the point experiments were performed, the results of which are shown graphically in fig. 6. To save space, the tables showing the actual numbers are omitted. Each division of the ordinates corresponds to 6.17×10^{-13} ampere.

For a particle of any material the ratio of its surface to its mass becomes greater when the size of the particle is reduced; hence one lot of charcoal, equal in mass to a second lot, will have a greater or less surface than the second according as the size of its grains is smaller or larger than the size of the grains of the second.

In this particular experiment three equal masses (2.17 gms.) of the *same* sample of coconut charcoal were used; but the size of the grain was different in each case. The *i-q* curves (fig. 6) were taken for all three. I. is the curve for the grains which were just caught in a sieve of ten meshes to the inch, II. for twenty meshes to the inch, and III. for thirty. It can be seen that the larger the grain the less is the absorption, or, in other words, the smaller the surface the less the absorption.

The experiment shows that it is not the mass of the absorbent on which the amount of absorption depends, but rather the amount of surface which the mass exposes to the emanation.

By increasing the quantity of charcoal of a given size grain the amount of absorbing surface is also increased, and consequently there is increased absorption. This is shown by the curves in fig. 7, which show the experimental *i-q* curves for 1.18, 2.17, and 3.35 gms. of the same sample of coconut charcoal of the same size grain (just caught in a sieve thirty meshes to the inch).

It must be noticed that the greater the quantity of charcoal—or, for that matter, the greater the power of absorbing by any condition—the greater must be the speed of the air-current to get any effect at all in the testing-vessel. Thus, there is practically complete absorption for 1.18 gms. of charcoal up to a speed of 0.5 c.c. per second, up to 0.8 c.c. per second for 2.17 gms., and up to 1 c.c. per second for 3.35 gms.

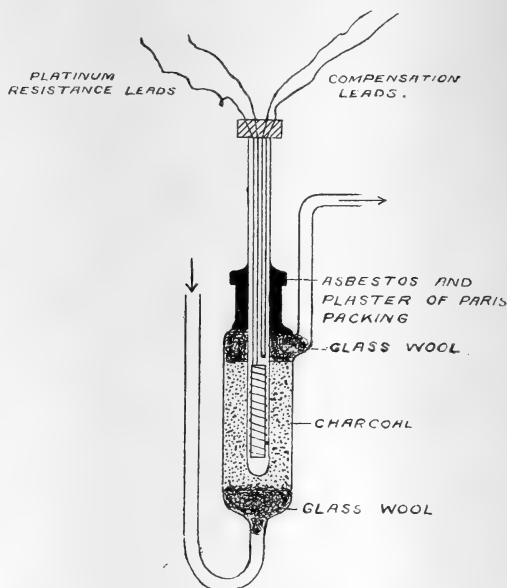
(d) *The temperature of the charcoal.*

The experiments of Dewar showed that when coconut charcoal is used as an absorbent of gases, the absorption is greatly increased by lowering the temperature of the charcoal.

The same law holds for the radioactive emanations when absorbed by any kind of charcoal. To examine this point, experiments were carried out at different temperatures with samples of wood, animal, and coconut charcoal. The weight of the coconut charcoal used was about 1.8 gm.

Temperatures were measured by a Callendar platinum resistance thermometer, the bulb of which was surrounded by the charcoal. The stem of the thermometer was sealed into the tube containing the charcoal by an asbestos and plaster-of-paris packing, so that the joint was perfectly air-tight. The diagram shows the arrangement.

Fig. 8.



The lowest temperatures mentioned were obtained from a mixture of solid carbon dioxide and ether; those next above were the temperatures of the room. The remaining were obtained by heating in a specially constructed oven, so that the temperature variation was not any more than three or four degrees.

The air-current, after passing through the charcoal, was circulated through a short coil of copper tubing immersed in water, in order to ensure that it should not be above ordinary temperature in the testing-vessel.

For coconut, charcoal curves were taken at five temperatures, viz. 185, 124, 69.5, 17.6, and -75.5 degrees centigrade. (See fig. 9, Table IV.)

For wood charcoal three curves were taken, corresponding to 170, 16.1, and -74.2 degrees centigrade. (See fig. 10.) The non-absorption curve is practically the same as that for 170° .

The curves show at once that the above law holds. Wood charcoal, under ordinary conditions, is not a good absorber, yet it is found to be affected by temperature according to the law. The absorption by animal charcoal is affected in a similar manner, but the experimental curves are omitted.

In a special experiment it was determined that the cocoanut charcoal begins to give off carbon dioxide at 211°C. , yet at even higher temperatures than this it showed some power of absorbing the emanation. By using a large quantity of charcoal, and keeping the speed of the air-current constant at 1.65 c.c. per second, it was found that the ionization-current in the testing-vessel kept decreasing, showing a slight and gradually decreasing absorption up to about 300°C. , when the ionization current began to increase. This increase was no doubt due to the loss of absorbing charcoal by oxidization, and to the increased conductivity of the gas in the testing-vessel owing to the presence of carbon dioxide. (For α -ray ionization the conductivity of carbon dioxide is 1.5 times that of air.)

In connexion with the above experiments, it should be noticed that it is hardly probable that the charcoal becomes "saturated" with thorium emanation—that is, that it reaches a state when it cannot occlude any more emanation—as it would in the case of ordinary gases. It must be remembered that the emanation which has been occluded by the charcoal is itself decaying at the rate of half-value in 54 seconds.

In the observations of ionization currents given above, the numbers represent the current when it has reached a *steady* state. At slow speeds the current keeps increasing for a few minutes until it becomes steady, but at high speeds it reaches a steady value almost immediately.

In conclusion, it may be stated that the percentage absorption of thorium emanation by means of charcoal

- (1) depends on the nature of the charcoal used, being greatest for coconut, intermediate for animal, and least for wood charcoal;
- (2) is decreased by increasing the velocity of the gas-current which conducts the emanation through the charcoal;

- (3) is increased by increasing the charcoal absorbing surface ; and
- (4) is dependent on the temperature, following the law for ordinary gases, viz., that a lowering of the temperature of the charcoal causes an increase, and a raising of the temperature a decrease in the amount of absorption.

I desire to express my thanks to Professor Rutherford for suggesting this investigation and for his interest and kindness while it was in progress.

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XXXVI. *An Investigation of the Resistance to the Flow of Air through a Pipe, with the Deduction and Verification of a Rational Formula.* By A. H. GIBSON, M.Sc., Lecturer in Hydraulics in the Manchester University*.

- (1) Introduction.
- (2) Rational Formula for Pipe-flow.
- (3) Experimental Data.
- (4) Conclusions.
- (5) Effect of Temperature Variations.

(1) *Introduction.*

IN an experimental investigation recently completed by Dr. J. H. Grindley and the author†, the resistance to the flow of air through the pipes used, at velocities below the critical, was used to determine the coefficient of viscosity of the air. In a few of these experiments the velocity was allowed to exceed the critical, and in these cases the results obtained were so widely at variance with the commonly accepted laws of pipe resistance that the author was led to a further investigation of the phenomenon, and to a comparison of the results obtained by himself and by other experimenters.

This investigation showed conclusively that any formula of the usual form

$$\delta p = \frac{f l v^2}{2 g m}$$

(where δp is the difference of pressure at the two ends of a pipe of hydraulic mean depth m ($m = \text{diam.} \div 4$) and of length l , due to a flow of air through the pipe with mean velocity v) only applies if the coefficient f is varied not only with the physical condition of the interior surface of the pipe, but

* Communicated by the Author.

† Proceedings Royal Soc., A. vol. lxxx, p. 114 (1908).

also with its diameter, with the mean velocity of flow, with the mean pressure, and with the temperature of the air. The fact that this coefficient depends on five factors, of which three may vary in the same pipe, renders this, and any more elaborate formula involving the same law of frictional resistance, useless as a means of determining the probable fall in pressure along a given pipe line, with any degree of accuracy.

Realizing this, the author attempted to determine some formula which should truly represent the state of affairs in such cases of resisted flow; and the following paper is devoted to a deduction of such a rational formula and to a determination of the constants involved in this, from the results of experiments by himself and by other experimenters.

(2) *Rational Formula for Pipe-flow.*

This is deduced on the assumptions that the resistance to flow along any small element of the pipe depends on the diameter, length, and surface condition of the element; on the viscosity and density of the fluid; and on the mean velocity of flow through the element; and also that it depends on some power of each of these factors.

This being so we may write

$$\delta p = k \cdot d^x \cdot \mu^y \cdot \rho^z \cdot v^n \cdot (\delta l)^a, \quad . \quad . \quad . \quad (1)$$

where δp = pressure difference in lbs. per sq. ft. at two points δl ft. apart along the pipe.

„ d = pipe diameter in feet.

„ μ = coefficient of viscosity of the fluid under the temperature conditions obtaining in the pipe.

„ δl = length of element of pipe in feet.

„ ρ = density of the fluid, at the mean pressure p , and mean temperature (τ° abs.) obtaining in the element.

„ v = mean velocity of flow in this element in ft. per sec.

„ k is a numerical coefficient.

Although this expression contains no term directly marking the effect of the roughness of the pipe surface, this effect is included in the terms ρ^z and v^n . This will be seen if it be granted that the effect of the roughness in increasing the resistance to flow is due to loss of available energy in eddy production, the eddies being formed by the sudden deflexion of particles of fluid in close proximity to the walls.

The mass of fluid thus affected will be greater as the roughness increases and as the velocity of flow increases, and

will also depend directly on its density, while the loss of energy per unit mass will depend on the velocity. If this be granted, it follows that the effect of a variation in the roughness will be felt in the factors involving both ρ and v , and that the values of the indices z and n of these expressions as determined for equation (1) will implicitly involve the effect of the roughness.

If $[M]$, $[L]$, and $[T]$ be the fundamental units of mass, length, and time, (1) may be expressed dimensionally as

$$\left[\frac{M}{LT^2}\right] = k[L]^x \cdot \left[\frac{M}{LT}\right]^y \cdot \left[\frac{M}{L^3}\right]^z \cdot \left[\frac{L}{T}\right]^n \cdot [L]^a.$$

Now experiment shows that the resistance to flow is, other things being equal, directly proportional to the length, so that $a=1$.

Inserting this value the equation becomes

$$[M] \cdot [L]^{-1} \cdot [T]^{-2} = k \cdot [M]^{z+y} \cdot [L]^{x-y-3z+n+1} \cdot [T]^{-(y+n)}.$$

Equating indices of like dimensional quantities we get

$$\begin{cases} x-y-3z = -2-n \\ y+z = 1 \\ y = 2-n, \end{cases}$$

and on solving this we have

$$x = n-3; \quad y = 2-n; \quad z = n-1.$$

Substituting these values in (1), this becomes

$$\delta p = k \cdot d^{n-3} \cdot \mu^{2-n} \cdot \rho^{n-1} \cdot v^n \cdot \delta l. \quad . \quad . \quad . \quad (2)$$

Where ρ and therefore v is constant, the fluid being incompressible, on integrating over the whole length of pipe this gives the Reynolds's formula* for the flow of water through a uniform pipe:—

$$\delta p = k \cdot d^{n-3} \cdot \mu^{2-n} \cdot \rho^{n-1} \cdot v^n \cdot l. \quad . \quad . \quad . \quad (2')$$

In a compressible fluid $\rho = \frac{\text{mass}}{\text{volume}}$, where the volume of unit mass is given by the relation $pV = c\tau$, so that $\rho = \frac{Mp}{c\tau}$.
Substituting $\frac{p}{c\tau}$ (where c has the value corresponding to

* Scientific Papers, Osborne Reynolds, vol. ii. p. 97.

† In the case of air, if the mass of gas is unity (32.2 lb. wt. units), and if p be measured in lbs. wt. per sq. ft., the value of c is $53.18 \times 32.2 = 1710$, while if p be measured in lbs. per sq. ins. c becomes 11.9.

unit mass) for ρ in equation (2), this becomes:

$$\delta p = k \cdot d^{n-3} \cdot \mu^{2-n} \cdot \left[\frac{p}{c\tau} \right]^{n-1} \cdot v^n \cdot \delta l. \quad (3)$$

Writing this in the form

$$p^{1-n} \cdot \delta p = k \cdot d^{n-3} \cdot \mu^{2-n} \cdot (c\tau)^{1-n} \cdot v^n \cdot \delta l,$$

and putting $\frac{v_m p_m}{p}$ for v , where v_m and p_m represent the mean velocity and pressure in the pipe, we get, on integrating,

$$\frac{p_1^2 - p_2^2}{2} = k \cdot d^{n-3} \cdot \mu^{2-n} \cdot (c\tau)^{1-n} \cdot v_m^n \cdot l$$

On putting

$$p_m = \frac{p_1 + p_2}{2}; \quad \delta p = \frac{p_1 - p_2}{2},$$

this gives for the whole pipe

$$\delta p = k \cdot d^{n-3} \cdot \mu^{2-n} \cdot \left[\frac{p_m}{c\tau} \right]^{n-1} \cdot v_m^n \cdot l, \quad (4)$$

as the equation for the resistance to the steady isothermal flow of a compressible fluid.

With stream-line motion, *i. e.* at all velocities below the critical, $n=1$, and equation (4) becomes

$$\delta p = \frac{k\mu v_m}{d^2} \cdot l.$$

This indicates that under these conditions the pressure drop is independent of the mean pressure in the pipe, and, except for the effect of the varying viscosity, also of the temperature, both of which conclusions received verification by the viscosity experiments of Dr. Grindley and the author.

In the majority of the experimental results available to the author, the temperature τ was not given and could be only estimated approximately from a consideration of the climatic conditions presumably obtaining at the time of the experiments. In only one series of experiments (by the author on a lead pipe of small bore) was the variation of temperature known, and of sufficient magnitude to enable the above formula to be fully checked. As, however, if the formula is correct, the variation in resistance is small for a temperature variation not exceeding 20° F. (not above 3 per cent. in any case), it was thought well in the first case to combine the factors involving temperature (either directly or in its effect on viscosity), to take the factors thus obtained as being sensibly constant during the experiments to be first considered, and afterwards to verify the law of variation with temperature by a separate series of experiments on the author's lead pipe of 1/8 inch bore.

On doing this, formula (4), which may be written as

$$\delta p = \frac{k\mu^2 c\tau}{(\mu c\tau)^n} \cdot \frac{p_m^{n-1} \cdot v_m^n}{d^{3-n}} \cdot l, \quad . \quad . \quad . \quad (5)$$

becomes

$$\delta p = \frac{A}{B^n} \cdot \frac{p_m^{n-1} \cdot v_m^n}{d^{3-n}} \cdot l. \quad . \quad . \quad . \quad (6)$$

Writing this as

$$\frac{\delta p \cdot p_m \cdot d^3}{l} = A \left(\frac{p_m \cdot v_m \cdot d}{B} \right)^n,$$

we get, on taking logarithms,

$$\log \left[\frac{\delta p \cdot p_m \cdot d^3}{l} \right] = n [\log p_m \cdot v_m \cdot d - \log B] + \log A. \quad (7)$$

(3) Experimental Data.

The experiments examined are tabulated below :—

(1) Experiments by Messrs. Riedler and Gutermuth on the Paris air-mains. These results are taken from values given by Professor Unwin (Proc. Inst. C. E. vol. cv. p. 190). The pipes were of cast iron, $11\frac{3}{4}$ ins. (.98 ft.) diameter. Owing to allowance having to be made for the effect of numerous draining tanks, traps, and stop-valves on the pipe-line, the values of δp quoted in this table are only approximate. The length of pipes ranges from 10982 ft. to 54270 ft., the mean pressure from 92 to 118 lbs. per sq. in. abs., and the mean velocity from 8.5 to 28.6 ft. per second. The temperature of the air is unknown, but was probably about 65° F.

TABLE I.

Expt.	Pipe.		Pressure—lbs. per sq. in. (abs.).				v_m , ft. per sec.
	Length ft.	Diam. ft.	p_1 .	p_2 .	p_m .	δp .	
1	54270	.98	106.6	80.15	91.9	20.45	28.62
2	"	"	114.4	97.25	96.6	27.15	27.35
3	"	"	119.4	105.4	110.0	14.04	19.35
4	"	"	116.1	109.4	111.6	6.66	13.89
5	"	"	114.4	110.0	111.6	3.98	8.51
6	14460	"	116.1	112.6	114.4	3.45	16.80
7	"	"	119.4	117.3	118.2	2.08	10.94
8	"	"	114.4	107.2	109.9	7.17	24.06
9	"	"	119.4	114.3	116.3	5.06	18.30
10	10982	"	102.9	100.5	101.7	2.40	20.95
11	28776	"	112.5	109.9	110.5	2.63	13.97

(2) Experiments by M. Stockalper on the pipe-line for supplying compressed air to the working face of the St. Gothard tunnel. These results are taken from the Proc. Inst. C. E. vol. lxiii. p. 348. The pipe-line consisted of two parts, the first being partly of cast and partly of wrought iron, while the second was of wrought iron, the whole having flanged joints and rubber joint-rings. The first portion of the pipe was 15092 ft. long and 7·87 inches (·656 ft.) in diameter, the mean velocity ranging from 15·6 to 19·3 ft. per sec. The second portion was 1713 ft. long and 5·9 inches (·492 ft.) in diameter, the mean velocity ranging from 29 to 37 ft. per sec. The mean pressure in the pipes ranged from 53 to 80 lbs. abs. per sq. ins., while the mean temperatures were 70° F. in the large and 79° F. in the small pipe.

TABLE II.

Expt.	Pipe.		lbs. per sq. inch.				v_m . ft. per sec.
	Length ft.	Diam. ft.	p_1 .	p_2 .	p_m .	δp .	
1 a	15092	·656	82·3	77·0	79·6	5·29	19·32
1 b	1713	·492	77·0	73·6	75·5	3·52	37·14
2 a	15092	·656	64·0	60·8	62·4	3·24	16·30
3 a	15092	·656	56·4	53·6	55·0	2·79	15·58
3 b	1713	·492	53·6	52·0	52·8	1·56	29·34

(3) Experiments by Dr. Brix (Proc. Inst. C. E. vol. xliii. p. 184) on wrought-iron pipes 103 ft. long, and having diameters of 3·25, 5·2, and 6·75 ins. (·271, ·433, and ·5625 ft.). The mean pressure was approximately atmospheric throughout, the mean velocity ranging from 23·3 to 42 ft. per sec. In these experiments the temperature is unknown, but is probably about 65° F.

Owing to the comparatively small length of the pipes, the head equivalent of the kinetic energy at exit and of the loss at entrance forms a large proportion of the total pressure drop. The sum of these losses has been taken as approximately $1·5 \frac{v_m^2}{2g}$, and allowance has been made for this in calculating δp .

TABLE III.

Expt.	Pipe.		lbs. per sq. in.				v_m , ft. per sec.
	Length, ft.	Diam. ft.	p_1 .	p_2 .	p_m .	δp .	
1 <i>a</i>	103	·5625	14·74	14·7	14·72	·0439	42·0
2 <i>a</i>	"	·5433	14·75	14·7	14·72	·0475	36·3
3 <i>a</i>	"	·5271	14·75	14·7	14·72	·0526	26·9
4 <i>a</i>	"	·5625	14·73	14·7	14·72	·0340	39·5
5 <i>a</i>	"	·5433	14·74	14·7	14·72	·0395	32·1
6 <i>a</i>	"	·5271	14·74	14·7	14·72	·0440	23·3

(4) Experiments by Dr. Brix (Proc. Inst. C. E. vol. xliii. p. 184) on a lead pipe 0·25 in. (·02083 ft.) diameter and 358·5 ft. long. The mean pressure ranged from atmospheric to 17·4 lbs. absolute per sq. in., and the mean velocity from 12·3 to 28·7 ft. per sec. In these experiments the temperature is unknown, but was probably about 65° F.

TABLE IV.

Expt.	Pipe.		lbs. per sq. in.				v_m , ft. per sec.
	Length, ft.	Diam. ft.	p_1 .	p_2 .	p_m .	δp .	
1 <i>b</i>	358·5	·02083	17·79	14·70	16·25	3·09	20·4
2 <i>b</i>	"	"	18·18	"	16·44	3·48	22·0
3 <i>b</i>	"	"	18·57	"	16·63	3·87	23·5
4 <i>b</i>	"	"	18·96	"	16·83	4·26	25·0
5 <i>b</i>	"	"	19·34	"	17·02	4·64	26·3
6 <i>b</i>	"	"	19·73	"	17·21	5·03	27·5
7 <i>b</i>	"	"	20·12	"	17·41	5·42	28·7
8 <i>b</i>	"	"	15·35	14·05	14·70	1·30	12·3
9 <i>b</i>	"	"	15·67	13·73	"	1·94	16·3
10 <i>b</i>	"	"	15·86	13·54	"	2·32	18·4
11 <i>b</i>	"	"	16·25	13·15	"	3·10	22·3

(5) Experiments by Dr. Grindley and the author on a lead pipe ·12761 in. (·01063 ft.) diameter and 108·8 ft. long, and by the author on a series of lead pipes ·1248 in. (·0104 ft.) diameter and of lengths ranging from 37·2 to 108·8 ft.

In these experiments the mean pressure varied from 14·0 to 17·4 lbs. per sq. inch absolute, the mean velocity from 6·0 to 43·0 ft. per second, and the temperature was varied as required between 32° F. and 212° F.

TABLE V.

Expt.	Pipe.		Mean temp. F.	lbs. per sq. in. (abs.).				v_m , ft. per sec.
	Length, ft.	Diam. ft.		p_1 .	p_2 .	p_m .	δp .	
1 ...	108.8	.01063	66°	15.3	14.7	15.0	.591	6.09
2 ...	"	"	"	15.6	14.8	15.2	.893	8.43
3 ...	"	"	"	18.0	16.7	17.4	1.28	10.92
4 ...	"	"	"	17.2	15.5	16.3	1.67	13.5
5 ...	"	"	"	17.3	14.6	16.0	2.66	19.4
6 ...	"	"	"	16.3	13.9	15.2	2.68	19.9
7 ...	"	.01039	"	15.6	14.8	15.2	.757	7.43
8 ...	74.4	"	"	18.7	14.7	16.7	4.03	34.6
9 ...	"	"	"	18.6	"	16.6	3.87	32.8
10 ...	"	"	"	18.8	"	16.7	4.09	33.4
11 ...	"	"	"	18.6	"	16.6	3.89	32.2
12 ...	"	"	"	17.5	"	16.1	2.76	26.8
13 ...	37.2	"	"	14.7	13.7	14.2	1.02	22.7
14 ...	"	"	"	14.7	13.6	14.1	1.07	26.4
15 ...	"	"	"	17.0	14.7	15.8	2.26	38.2
16 ...	"	"	"	17.3	"	16.1	2.65	43.0
17 ...	"	"	"	18.0	"	16.3	2.85	44.5
18 ...	108.8	.01063	32°	15.5	1.11	10.1
19 ...	"	"	212°	15.5	1.22	10.1
20 ...	108.8	"	32°	15.1	.520	5.5
21 ...	"	"	212°	15.3	.597	5.5
22 ...	"	"	74°	15.6	.889	8.43
23 ...	"	"	212°	15.5	.978	8.43
2 ...	"	"	66°	15.2	.893	8.42
24 ...	"	"	60°	15.2	.732	7.23
25 ...	"	"	212°	15.4	.828	7.23
26 ...	"	"	32°	15.5	1.22	10.95
27 ...	"	"	212°	15.5	1.22	10.10
28 ...	"	"	32°	15.2	.810	7.91
29 ...	"	"	212°	15.4	.810	7.23
30 ...	"	"	66°	15.0	.595	6.10
31 ...	"	"	212°	15.3	.595	5.56
1 ...	"	"	66°	15.0	.591	6.09

In these experiments the pipe was in one continuous length, coiled on a drum 14 inches diameter. The pressures at points in the pipe at measured distances apart were measured by means of water piezometers, great care being taken to ensure these giving true pressure readings. The volume of air passing through the pipe was readily determined,

since this air was displaced by water under pressure from an air reservoir fitted with a calibrated gauge-glass.

After leaving the coil it entered a second measuring reservoir. The pressure in each reservoir was measured by a mercury piezometer, and the mean volume passing the pipe at the mean pressure p_m calculated from the volumes leaving the first and entering the second and from their respective pressures. A sketch and description of the complete apparatus used may be seen in the Proc. Roy. Soc. A. vol. lxxx. 1908, p. 115.

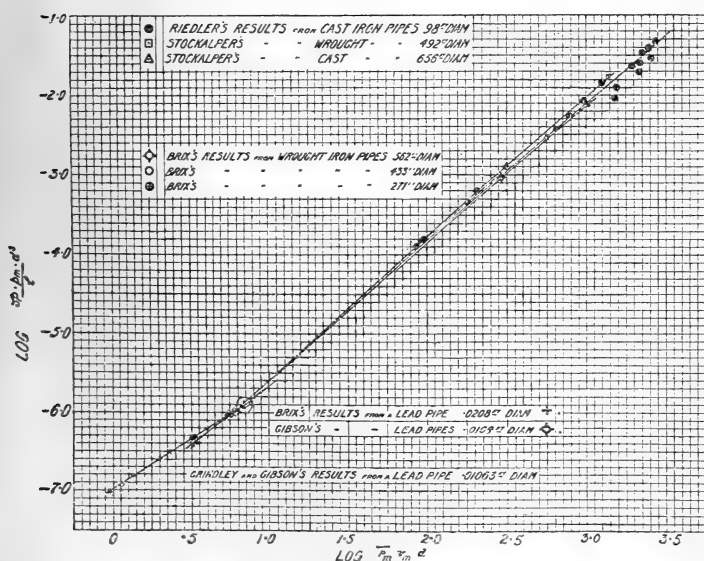
IV. Conclusions.

We have then, available for comparison, the results of the whole of the foregoing experiments which, with the exception of these numbered from 18 to 31 of Table V., may be taken as carried out at substantially the same temperature, viz. 66° F. For each of these the values of

$$\log \left[\frac{\delta p \cdot p_m \cdot d^3}{l} \right] \text{ and of } \log [p_m \cdot v_m \cdot d]$$

have been calculated and plotted, the logarithmic homologues for the whole series of pipes being shown in fig. 1.

Fig. 1.



If the assumptions made in deducing formula (6) are
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justified, the logarithmic homologue for each pipe will be a straight line; and these straight lines will all pass through the same point, the abscissa of which represents $\log B$ while its ordinate represents $\log A$. The plotted results afford ample justification of the assumptions; for certainly within a limit commensurate in magnitude with the limit of accuracy in the experiments, the plotted lines lie on a series of straight lines all of which pass through the point (antilog 6.097, antilog .8200).

This makes $A = 125 \times 10^{-8}$, $B = 6.60$; so that equation (6) becomes

$$\delta p = .00000125 \frac{p_m^{n-1} \cdot v_m^n \cdot l}{6.6^n \cdot d^{3-n}} \text{ lbs. per sq. in.} \quad (7')$$

The following are the values of 'n' as deduced from these experiments.

{	Lead pipe .125 in. diam.	Very smooth inner surface.	n	
		Author	1.28	
	" " .250 in. diam.	Surface condition unknown, probably moderately rough.		
		Brix	1.49	
{	Cast Iron pipe 7.87 in. diam.	Stockalper	1.775	
	" " " 11.75 in. diam.	Brix	1.77	
{	Wrought Iron pipe 3.25 in. diam.	Brix	1.825	
	" " " 5.2 in. diam.	Brix	1.81	
	" " " 5.9 in. diam.	Stockalper	1.81	
	" " " 6.75 in. diam.	Brix	1.79	

As might be expected, with fairly large diameters the condition of the pipe surface only appears to affect the value of 'n' slightly, this value appearing (as might also be expected) to diminish slightly with an increasing diameter.

It would appear from these results, that for either cast-iron or wrought-iron pipes laid under normal conditions as to jointing, &c., n may be taken as having the following mean values.

Diam. (ins.).	3.	5.	7.	9.	12.
n	1.83	1.81	1.79	1.775	1.77

while for all cases of flow through such pipes, where the air is at atmospheric temperature (about 66° F.) the drop in

pressure is given with sufficient accuracy by the formula

$$\delta p = \cdot 00000125 \frac{p_m^{n-1} \cdot v_m^n \cdot l}{16 \cdot 6^n \cdot d^{3-n}} \text{ lbs. per sq. in.}$$

where d and l are in feet and p_m in lbs. per sq. inch.

If δp be measured in inches of water, the other units being unaltered, this becomes

$$\delta p = \cdot 0000346 \frac{p_m^{n-1} \cdot v_m^n \cdot l}{6 \cdot 6^n \cdot d^{3-n}} \text{ ins. of water.}$$

If δp and p_m be measured in kg. per sq. cmm., and if the metre be taken as the unit of length, we get

$$A = \cdot 00260; \quad B = 1010.$$

In the case of the lead pipes examined, the uncertainty as to the condition of the interior surface of the pipe used by Dr. Brix prevents any accurate deductions being drawn. The effect of a variation in the physical condition of the surface in a pipe of such small diameter is very marked in the variation of n from 1·28 in the pipes used by the author (the interior surfaces of which were exactly as left by the die) to 1·49 in the pipe of Dr. Brix. Probably to be on the safe side in practice, and to allow for irregularities in laying and jointing, it would be advisable to take $n = 1\cdot50$ for all lead pipes of small diameter.

V. *Effect of Temperature Variation.*

An increase in temperature tends to affect the resistance to flow along a pipe with a given mean velocity and at a given mean pressure, in two ways.

- (1) By reducing the density of the fluid it tends to reduce the resistance.
- (2) By increasing the viscosity it tends to increase the resistance.

Now if formula (4) is correct for all temperatures, in any two experiments (1 and 2) carried out at the same mean velocity and with the same mean pressures, but with different temperatures, we shall have :—

$$\frac{\delta p_1}{\delta p_2} = \left\{ \left(\frac{\mu_1}{\mu_2} \right)^{2-n} \cdot \left(\frac{\tau_2}{\tau_1} \right)^{n-1} \right\},$$

while if the mean pressures are equal and if $\delta p_1 = \delta p_2$, we shall have

$$\left(\frac{v_{m2}}{v_{m1}} \right)^n = \left\{ \left(\frac{\mu_1}{\mu_2} \right)^{2-n} \cdot \left(\frac{\tau_2}{\tau_1} \right)^{n-1} \right\}.$$

To test this, the series of experiments (16 to 31, Table V.) were carried out by the author, the essential results being as follows :—

No.	t° F.	τ° abs.	v_m f. s.	δp .	$\frac{\delta p_1}{\delta p_2}$ (observed).	$\frac{\delta p_1}{\delta p_2}$ * (calculated).
19 ...	212	673	10.1	1.22	1.10	1.09
18 ...	32	493	10.1	1.11		
21 ...	212	673	5.5	.597	1.15	1.09
20 ...	32	493	5.5	.520		
2 ...	66	527	8.43	.893	.91	.94
23 ...	212	673	8.43	.978		
22 ...	74	535	8.43	.889		
25 ...	212	673	7.23	.828	1.13	1.07
24 ...	60	521	7.23	.732		

In this series of experiments the mean difference between the calculated and experimental results is 3.4 per cent.

No.	t° F.	τ° abs.	v_m f. s.	δp .	$\left(\frac{v_{m_1}}{v_{m_2}}\right)^{1.28}$ (observed).	$\left(\frac{\mu_2}{\mu_1}\right)^{.72} \left(\frac{\tau_1}{\tau_2}\right)^{.28}$ *
26 ...	32	493	10.95	1.22	1.10	1.09
27 ...	212	673	10.14	1.22		
28 ...	32	493	7.91	.810	1.12	1.09
29 ...	212	673	7.23	.810		
30 ...	66	527	6.10	.595	1.12	1.06
31 ...	212	673	5.56	.595		
1 ...	66	527	6.09	.591		

In this series the mean difference between calculated and experimental results is 3.6 per cent.

Since in every case the proportional difference diminishes as δp increases, and hence as the probable proportional error

* As indicated by the experiments of Dr. Grindley and the author, the viscosity of dry air between 32° F. and 212° F. is given very approximately by the formula

$$U = (341 + .518t)10^{-9} \text{ ft. lb. wt. units,}$$

this giving results within 1 per cent.

The following are a few values of μ :—

t° F.....	32°.	66°	212°
$10^9 \times \mu$...	355	375	447

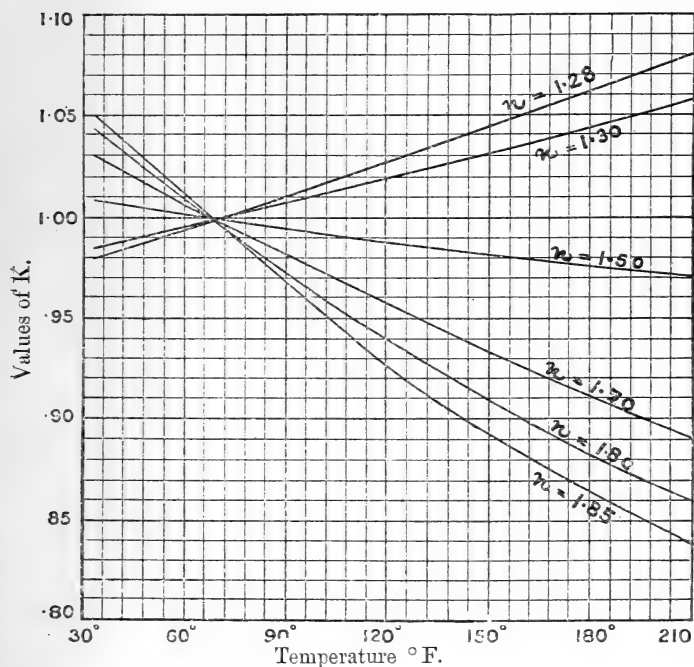
of measurement diminishes, and since if n be taken as 1.26 instead of 1.28 the calculated and experimental results are almost in exact coincidence, the results afford a substantial confirmation of the validity of the formula.

Presumably, too, this formula is equally true for the resistance to the flow of any other gas, if the values of c and μ for the required gas be used. This still requires experimental confirmation.

Modified Form of Equation.

Formula (4) is, however, very cumbersome, and a preferable formula for practical use would consist of (7') together with a numerical coefficient, the latter taking into account the whole effect of temperature variations.

Fig. 2.



This formula thus becomes

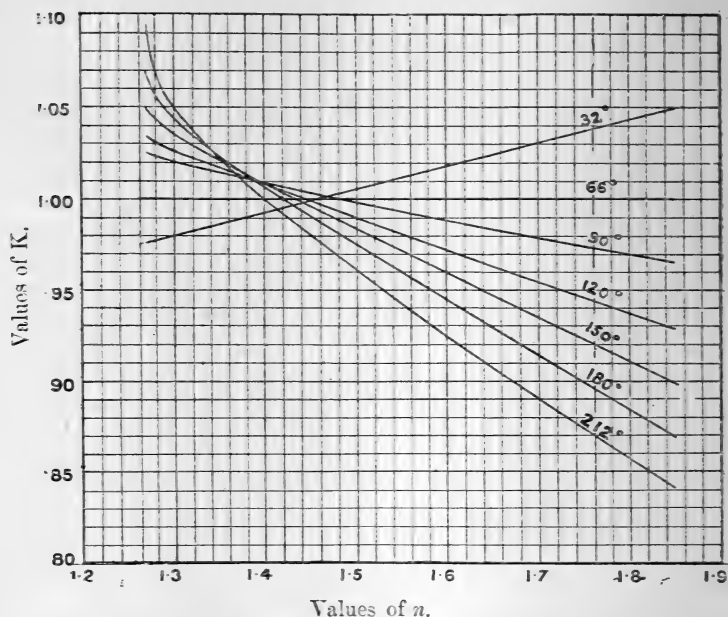
$$\delta p = 0.0000125 K \frac{p_m^{n-1} \cdot v_m^n \cdot l}{6 \cdot 6^n \cdot d^{3-n}} \text{ lbs. per sq. in.,}$$

where, in the case of air, K has the values shown graphically in figs. 2 and 3.

One rather peculiar circumstance becomes evident from

these figures. It will be noted that where the value of n is less than about 1.38, K increases with an increase in temperature, while for greater values of n , K decreases as the temperature increases.

Fig. 3.



The variation of K with temperature becomes greater as n increases or diminishes from 1.38.

Thus when $n = 1.80$ (its mean value for a wrought-iron pipe), the mean variation in K is approximately 1 per cent. per 10°F. temperature variation. This rate of variation of K with temperature decreases as the temperature increases.

The University, Manchester.

XXXVII. *The Use of the Potentiometer on Alternate Current Circuits.* By CHARLES V. DRYSDALE, D.Sc.*

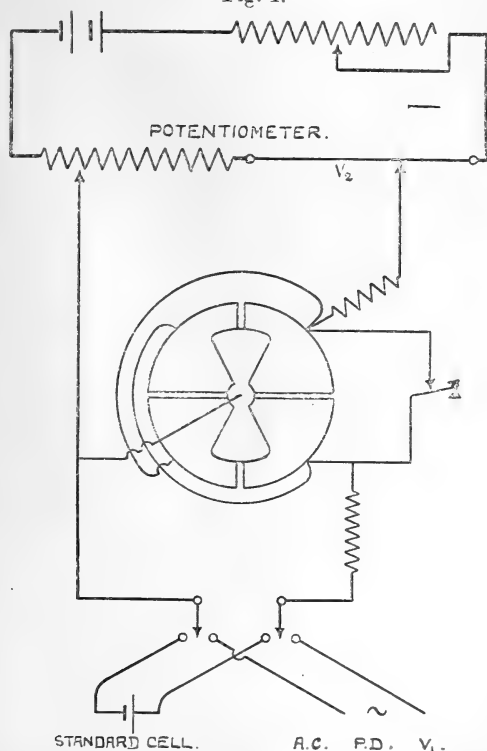
ONE of the greatest inconveniences in connexion with alternate current measurements has always been the question of range. Owing to the fact that all instruments employed for direct indication of alternate currents or P.Ds. have a square law, their range is small and they become impracticably delicate when low voltages are concerned. There is therefore a very great need for some instrument which, like the direct-current potentiometer, should be

* Communicated by the Physical Society: read January 22, 1900.

capable of measuring P.D.s. and currents of any range with accuracy. It would also be a great convenience if this instrument were capable of indicating the phase as well as the magnitude of the P.D. or current tested.

In attempting to apply the potentiometer principle to alternate current measurements, two processes seem to be possible—(a) the balancing of the alternate current P.D. against an equal direct current P.D. by the employment of some differential balancing device preferably of an electrostatic character, or (b) the balancing of two alternate current P.D.s. against one another. Such propositions as have hitherto been made for an alternate-current potentiometer have, the writer believes, always been based on the former idea, owing to the fact that the latter principle would involve equality of phase as well as of P.D. in the two voltages compared. He understands that some time ago Mr. Swinburne suggested the use of a differential electrometer in

Fig. 1.



connexion with a potentiometer, but has not seen any details. An obvious device would be that shown in fig. 1,

in which the P.D. to be measured is applied between the needle and one pair of quadrants, while that derived from the potentiometer is connected between the needle and the other pair of quadrants, in each case through a considerable resistance, say one megohm. By connecting a key to the opposite pairs of quadrants, which normally short-circuits them, but disconnects them when depressed, a deflexion should be obtained to right or left proportional to $V_1^2 - V_2^2$ or $2VdV$ when balance is nearly obtained. Any want of symmetry in the electrometer should be eliminated by connecting the standard cell as shown, so as to obtain a double weighing method.

The procedure in this case is very simple, and exactly resembles ordinary direct-current potentiometry. The sole difficulty lies in the electrometer. It should be possible by multicellular devices, or otherwise, to obtain a fair amount of sensitiveness with 1 or $1\frac{1}{2}$ volts, the P.D. of the standard cell, as has been done by Mr. Addenbrooke, but when we come to a tenth of a volt or less, as is so often necessary in current measurement, electrostatic devices seem hopeless. All other magnetic or thermal devices are incorrect in principle as requiring an appreciable current to operate them, which militates against the accuracy of the potentiometer; but by employing a low-resistance potentiometer, and a differential thermal device, such as could be made from the Duddell thermo-galvanometer employing two junctions and two heaters, or two crossed thermo-junctions in series, a sufficiently sensitive and accurate potentiometer could perhaps be made.

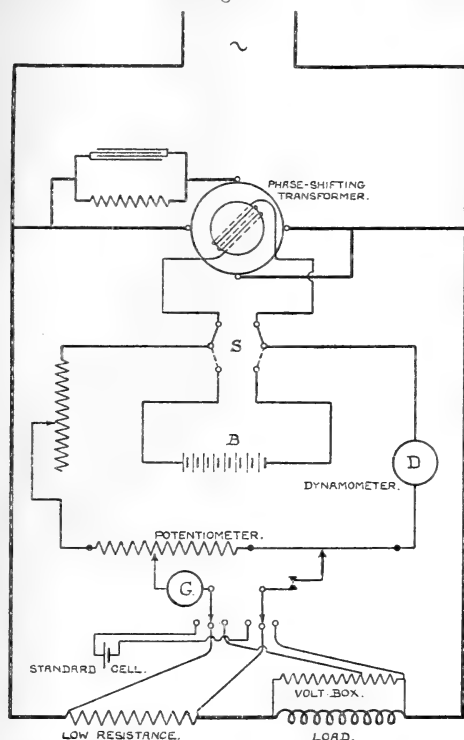
In view of the difficulty of making such devices, however, the writer has turned his attention to the second method, *i. e.* of balancing the alternate current P.D. to be measured against a known alternate current P.D., by the interposition of a sensitive alternate-current detector, such as a telephone or vibration galvanometer. This of course implies that the two voltages under comparison must have the same magnitude, frequency, and phase, and, approximately, the same waveform*.

The equality of frequency is of course secured by deriving the test and comparison voltages from the same source of supply, but some special device is needed for bringing the phases into coincidence. In some experiments recently

* Since this was written the writer has been informed that Dr. Sumpner has somewhere suggested that an alternating-current potentiometer could be made, if any means were available for bringing the phase of the current in the slide-wire into coincidence with that of the P.D. to be measured.

made with the writer's phase-shifting transformer, however *, the accuracy of the phase-variation was found to be so good that it was thought worth while to introduce it into potentiometer work; and fig. 2 shows the diagram of the connexions

Fig. 2.



which have been employed. The phase-shifter is here shown supplied with single phase current through a phase-splitting device, which can be built up with it so that it merely requires connecting straight on to the mains. The current derived from the secondary of this transformer traverses the rheostat coils, and slide-wire of an ordinary potentiometer, such as the Crompton form; and also some form of dynamometer-current measuring device. The writer has found a Weston A. C. *voltmeter* excellent for this purpose, as it gives a very good reading with the current (·05 ampere) employed with the Crompton potentiometer. The remainder of the potentiometer is connected up precisely as for ordinary

* 'Electrician,' Dec. 11, 1908.

direct current working, except that a vibration galvanometer is substituted for the ordinary form. By having an auxiliary battery B and throw-over switch S, the arrangement is always ready for either direct-current or alternate-current comparisons.

The procedure is then as follows :—The throw-over switch is first arranged to supply direct current from the battery B, and the rheostat altered until the standard cell is balanced at its nominal value, as in ordinary direct-current working. The dynamometer reading is then carefully noted, or a fiducial mark is made, which always represents the exact $\cdot 05$ ampere. On now throwing over to alternate current, the rheostat is altered to reproduce the same R.M.S. current as indicated by the dynamometer D ; and the position of the potentiometer contact is shifted and the phase-shifter turned, until the vibration galvanometer shows no vibration. The adjustment is very similar to that of the two resistances in an Anderson's bridge, and presents no difficulties. The voltage and phase are then read off on the potentiometer and phase-shifter respectively. In the illustration a low resistance is connected in series with the load, and a volt-box in parallel with it, and the current and P.D. can be determined directly in the ordinary way. Of course the phase-shifting transformer must always be connected to the same source of supply as the load.

Instead of the vibration galvanometer, a dynamometer or electrometer in which one pair of terminals is connected to the supply so as to be "separately excited," and the other to the ordinary galvanometer terminals, has been employed with good results. The deflexions are then to left or to right as in an ordinary galvanometer. But it must not be forgotten in this case that balance will be obtained not only when the vectors of the two P.Ds. compared are coincident, but also when their vector difference is in quadrature with the P.D. of the supply. For this purpose, if such an instrument is used it must have its "exciting terminals" changed over from one phase to the other, and balance secured in both cases.

Before giving examples of readings obtained in this manner, we must consider what is really measured. It is obvious that this device does not indicate the effective or R.M.S. value of the P.D., except when the supply and tested wave-forms are both sinusoidal or of identical form. When a vibration galvanometer is employed, its sensitiveness to the fundamental wave is so great in comparison with that to the harmonics, that we shall be practically correct in assuming

that it is the fundamental wave only which is measured, and the comparison is really between the mean and not the R.M.S. values. On the other hand, if there is any serious difference of wave-form, the sensitiveness to the upper harmonics should be sufficient to prevent an exact balance being secured, and this warns us when great accuracy is not to be expected.

The effective value of an irregular periodic voltage is

$$\bar{V} = \sqrt{\sum \bar{V}_n^2},$$

where \bar{V}_n is the effective value of the harmonic of order n . If the sum of the squares of the higher harmonics is small in comparison with the square of the fundamental, we may take the approximate square root, and

$$\bar{V} = \bar{V}_1 + \frac{1}{2\bar{V}_1} \sum \bar{V}_n^2,$$

or

$$\bar{V} = \bar{V}_1 \left\{ 1 + \frac{1}{2\bar{V}_1^2} \sum \bar{V}_n^2 \right\},$$

where $\sum \bar{V}_n^2$ in the last two expressions implies the sum of the squares of the effective values of all the harmonics, excluding the fundamental.

In the case of a single harmonic this becomes

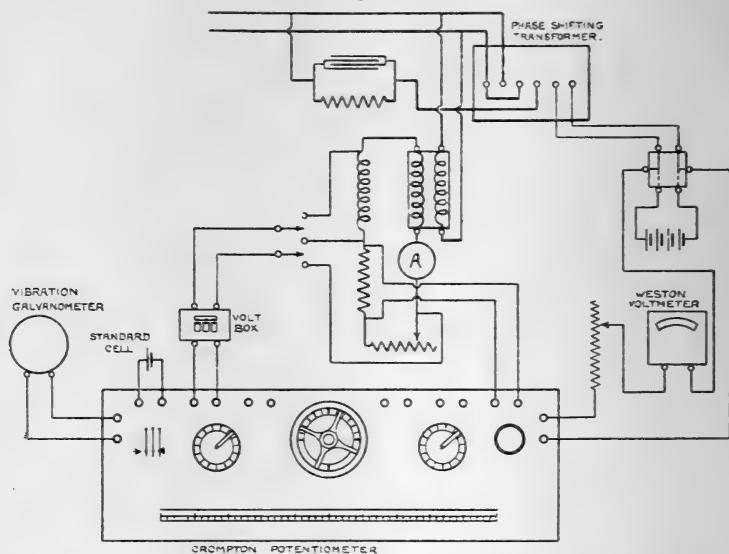
$$\bar{V} = \bar{V}_1 \left\{ 1 + \frac{1}{2} \left(\frac{\bar{V}_n}{\bar{V}_1} \right)^2 \right\},$$

and the expression in brackets only differs 1 per cent. from unity when \bar{V}_n is 14 per cent. of \bar{V}_1 . Such a difference should be fairly easily recognizable, and would probably be sufficient to prevent the obtaining of a perfect balance on the vibration galvanometer; and, moreover, if the distortion of wave-shape is great, there is little object, as a rule, in great accuracy of measurement. If accuracy is required in such cases, it can of course be secured by tracing the wave-shape and making the above correction, but if care is taken to have a supply of fairly sinusoidal wave-form, the measurements may be considered sufficiently accurate for the majority of purposes. It should be observed that this difficulty, or a similar one, is inseparable from any method which can be devised, in which the detector is employed to indicate the *difference* between the test and potentiometer P.Ds. as is done with direct currents.

Measurements.—Two illustrations will be sufficient to show

the possibilities of the method. An ordinary Crompton potentiometer was used with a Weston dynamometer voltmeter in series with it, and the connexions were made as in fig. 3. A Rubens vibration galvanometer was at first used,

Fig. 3.



but was replaced by a Campbell bifilar instrument, which was found much more sensitive and convenient. A check with direct current and a Cadmium cell showed that the normal current was obtained when the dynamometer indicated 82 "volts," and this reading was reproduced in all the alternate-current measurements.

The first test was a comparison of the readings of a Kelvin Dekka-ampere-balance, with the current as given by the P.D. across a Crompton $\cdot 04 \omega$ manganin resistance. The following results were obtained:—

Current by Kelvin.	P.D. by Potentiometer.	Current by Potentiometer.
25	1.002	25.05
20	.8018	20.045
15	.6022	15.055

Frequency of supply 72~.

The readings were obtained with great ease and definiteness, half a division on the slide-wire, or $\cdot 0005$ volt, being easily detected, in spite of the fact that trouble was found from mechanical vibration, as the alternator was in the same room. There seems no reason to doubt that $\cdot 0002$ volt could be easily detected, or that a drop of $\cdot 1$ volt could be measured to an accuracy of $\cdot 2$ per cent. or closer. The readings of the phase angle repeated themselves constantly to within $\cdot 1$ degree.

To give some idea of the accuracy of phase measurements, a resistance and ironless choking-coil were connected in series with the $\cdot 04 \omega$ resistance, and an alternating current of 15 amps. at $60 \sim$ passed through the combination. Readings were then taken of the P.D. across the $\cdot 04 \omega$ resistance, across the choking-coil, total resistance, and the whole circuit, the last three being taken by the aid of a 10×1 volt box. The following are the results :—

Circuit.	First Test.		Second Test.	
	P.D.	Phase-angle, degrees.	P.D.	Phase-angle.
$\cdot 04$ resistance ...	$\cdot 5960$	46.5
Total resistance .	3.117	47.1	3.108	47.3
Choking-coil	4.880	125.9	4.928	125
Mains	6.185	95.3	6.218	94.9

The difference between the first and second readings is due to the fact that a carbon resistance was employed which became heated.

Fig. 4 (p. 410) is a graphical representation of the result in the last case, the horizontal line representing the resistance drop, and the other two lines the drop across the choking-coil and whole circuit respectively. Although the triangle does not quite close, the result may be taken as satisfactory in consideration of the fact that single phase supply was used in this case, and the phase was split by condensers. It should be noted also that errors in the phase angle do not in any way affect the accuracy of the magnitudes of the P.Ds.

The writer has designed a potentiometer in which a small phase-shifting transformer and dynamometer are included in

the case, with a throw-over switch, permitting the instrument to be used as either a direct or alternate current potentiometer.

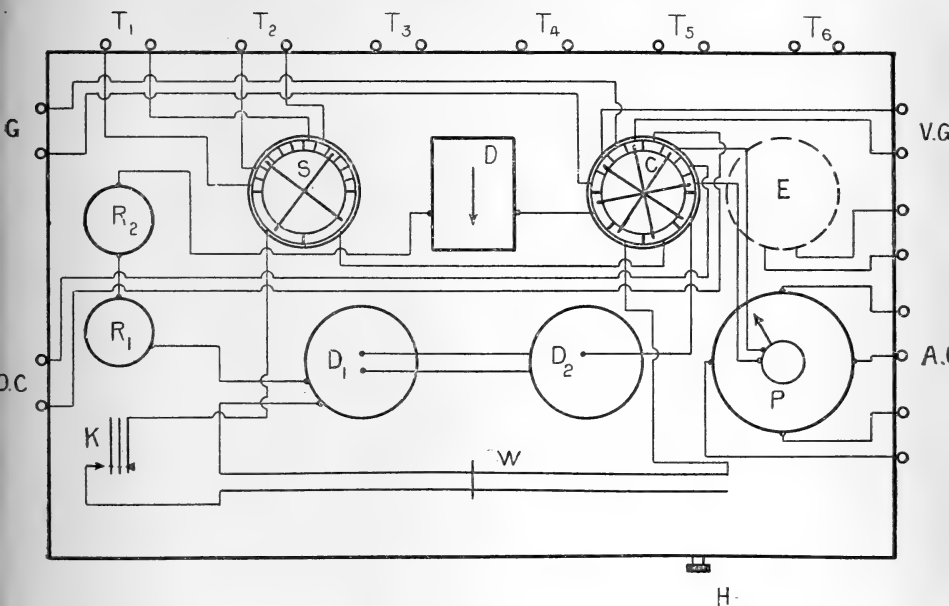
Fig. 4.



Fig. 5 shows the connexions, which need little explanation. The main circuit of the potentiometer, consisting of the rheostats R_1 and R_2 , the two dials D_1 and D_2 , the slide-wire W , and the dynamometer D , is connected to two of the blocks of the change-over switch C , which is similar in form to the well-known selector switch used in the Crompton Potentiometer. The derived P.D. from the dial D_2 and slide-wire contact is led by the key K and the selector switch S to two more of the contacts on C . The remaining eight contacts on C are connected in pairs to the phase-shifting transformer P , the battery terminals D.C., the ordinary galvanometer terminals G , and the vibration galvanometer terminals V.G. The phase-shifter primary is excited from the terminals A.C., which are here shown four in number, for two-phase or split-phase. A single motion of the switch C to right or left connects the main circuit of the potentiometer on to the battery or secondary of the transformer, and simultaneously connects the sliding contacts through the selector switch to the appropriate galvanometer. The key K being provided with a holding-down cam, leaves the hands free to move the potentiometer contacts and rotate the phase-shifter simultaneously. A frequency meter F of the vibrating reed type may be fixed on the base if desired, in which case the instrument is available for inductance and capacity measurements.

The dynamometer D is not provided with a scale, but with a fiducial mark which can be slightly adjusted to suit the standard cell-check. It will also be made astatic to avoid risk of error from stray fields.

Fig. 5.



This apparatus should be capable of giving very good results. In the meantime the results already obtained appear to justify the belief that this method is capable of being used with considerable accuracy and convenience for all P.D., current, and phase measurements.

The numerous applications of such an instrument are obvious, and include P.D., current, phase, and power measurement over almost any range; inductance and capacity tests, and determination of dielectric losses, &c. One of the most valuable applications, however, is to search-coil work with alternate currents, whereby the distribution of main and leakage fluxes in cores, and the propagation of magnetic impulses can be studied. The variation of current and P.D. along a cable fed with alternate currents could also be obtained.

The writer's thanks are heartily tendered to Mr. A. C. Jolley for setting up the apparatus and great assistance in the experiments.

XXXVIII. *The Diffusion of Actinium and Thorium Emanations.* By SIDNEY RUSS, B.Sc., Demonstrator in Physics, Manchester University*.

Introduction.

NO direct determinations of the molecular weights of the radioactive emanations have yet been made. Owing to the very small quantities of these substances available, experimental work on this subject has been almost completely restricted to a study of the way in which the emanations diffuse, and by a comparison of the diffusion coefficients so obtained with those of gases of known molecular weights, values of the molecular weights for the emanations themselves may be inferred by means of Graham's law.

The work of Curie and Danne †, Rutherford ‡, Rutherford and Miss Brooks §, and Makower || on these lines has indicated, for the emanations of radium and thorium, a molecular weight of the order one hundred. Bumstead and Wheeler ¶, however, give a rather higher value in the case of radium, while for the emanation of actinium, Debierne ** obtained a molecular weight of seventy.

On the disintegration theory radium and thorium emanations should have very approximately the same molecular weight, namely, 222. There is thus a very large discrepancy between the theoretical and experimental numbers.

In order to account for this it has been suggested that the ordinary laws of diffusion do not perhaps hold when very minute quantities of the substances in question are being dealt with. There being no obvious remedy in this direction, it was thought that by a study of the diffusion of the emanations in gases differing considerably in molecular weight, any existing discrepancies would be brought to light. With this object in view, the diffusion coefficients of actinium emanation in air, hydrogen, carbon dioxide, and sulphur dioxide were determined. While the work was in progress there appeared in the American Journal of Science, June 1908, a paper by P. B. Perkins, in which the diffusion of radium emanation was compared directly with that of mercury vapour, the experiments giving a molecular weight

* Communicated by the Physical Society: read November 27, 1908.

† *Comptes Rendus*, 1903, p. 114.

‡ 'Radioactivity,' p. 276.

§ Trans. Roy. Soc. of Canada, 1901-2.

|| Phil. Mag. 1905, p. 56.

¶ *Le Radium*, June 1907.

** American Journal of Science, Feb. 1904.

of 235 for the emanation, a number approximating closely to that predicted by theory. The results of these experiments suggested that it might be advisable to determine the coefficient of diffusion of actinium emanation, itself presumably monatomic, into a monatomic gas. For this purpose a quantity of argon was prepared, and the coefficient obtained by two methods.

Whatever the nature of the supposed deviations from the ordinary laws of diffusion referred to above, it might reasonably be expected that they would be exhibited to an equal extent by the different emanations. A direct comparison was therefore carried out over a fairly wide range of pressure between the diffusion of actinium and thorium emanations, a comparison of their coefficients leading directly to the ratio of their molecular weights.

Method of Experiment.

The theory of the method here adopted for obtaining the diffusion coefficient has been given by Rutherford*. If a layer of radioactive material be placed at the bottom of a cylindrical vessel a gradient of the emanation is set up, the partial pressure at any layer distant x from the base being

given by $p = p_0 e^{-\sqrt{\frac{\lambda}{K}} \cdot x}$, where λ is the radioactive constant of the emanation and K its diffusion coefficient. It is not convenient in practice to measure the partial pressure of the emanation, but if a metal rod be placed along the axis of the cylinder in question and kept at a high negative potential, the active deposit produced by the emanation is directed to the rod soon after formation. We have thus a layer of active matter on the rod, the gradient of which is that of the emanation within the cylinder. The distribution of active matter along the rod can be found by measuring the ionization produced by the α rays emitted by successive small segments of the rod. This gives us the variation of p with x in the above equation, and knowing the value of λ , that of K is determined.

Debierne† has shown that the above theory is also applicable when the emanation is allowed to diffuse up into the space between two parallel plates. In an exactly analogous way a deposit of active matter is obtained on the plates, the gradient of which is that of the emanation itself.

Both of these experimental arrangements have been used in the course of this work.

* 'Radioactivity,' p. 275.

† *Le Radium*, June 1907.

Actinium in Different Gases.

For determining the coefficient of diffusion of the emanation in different gases the following experimental arrangement was used. A cylindrical vessel 12·5 cms. long and 7·5 cms. diameter was fitted with a base on which was placed a tray containing a preparation of actinium; about 1 cm. above the surface of the tray which was covered with tissue paper, were suspended vertically two glass plates 10 cms. long, 2·2 cms. broad, and ·2 cm. thick, which were kept separated ·2 cm. from one another. The plates were silvered and metallic connexions supplied, so that an electric field might be applied between the two plates if necessary. It may be stated here that the gradients obtained on the plates with, and without, an electric field were identical, the only difference being that nearly all of the activity was concentrated on the negative plate when the field was applied, while it was shared equally between the two plates with no field.

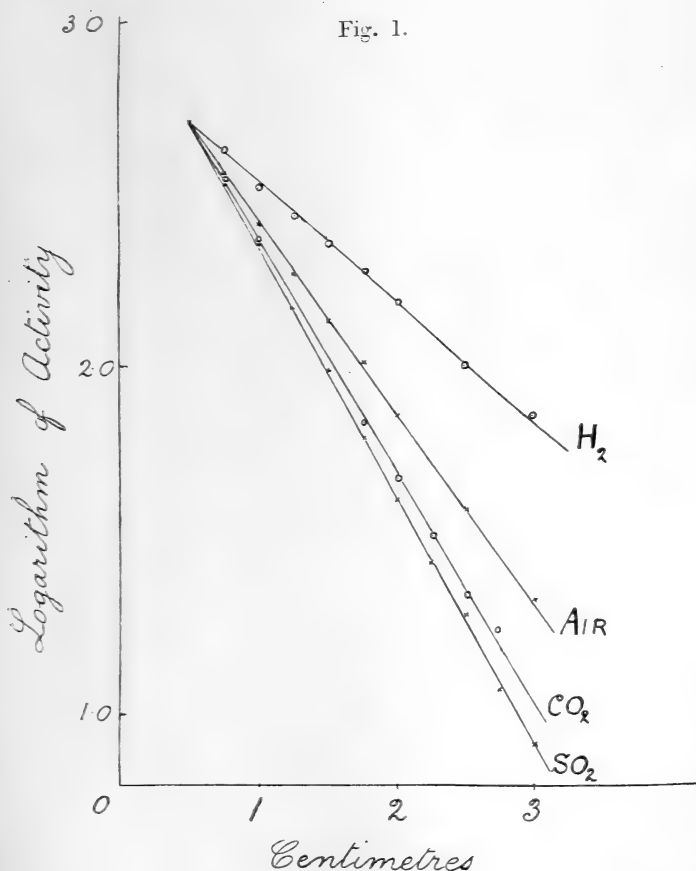
The gases before entering the diffusion vessel were passed through calcium chloride and cotton-wool tubes to render them dry and dust free. The vessel was placed inside a large tank of water, which prevented any appreciable change of temperature occurring during an experiment. During the whole course of the work the temperature varied between 10° and 18° C.

The glass plates were usually exposed for about twenty-four hours, and then they were removed in order to measure

TABLE I.—Diffusion of Actinium Emanation.

Distance in cms.	<i>Air.</i>	<i>Hydrogen.</i>	<i>Carbon Dioxide.</i>	<i>Sulphur Dioxide.</i>
	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.
·5	2·341	2·194	2·581	2·709
·75	2·182	2·115	2·416	2·526
1·0	2·036	1·997	2·233	2·358
1·25	1·901	1·919	2·026	2·173
1·5	1·768	1·837	1·846	1·992
1·75	1·648	1·761	1·710	1·798
2·0	1·494	1·668	1·531	1·616
2·25	1·391	1·439
2·5	1·224	1·496	1·212	1·281
2·75	1·112	1·068
3·0	·964	1·351	...	·912
<i>Diffusion coefficient.</i>	Mean of three	Mean of two	Mean of two	Mean of two
K Experimental...	·096	·330	·073	·062
K Calculated	·363	·078	·064

the gradient of active deposit on their surfaces. One of the plates was placed on a small platform which could be moved along so that small portions ($\cdot 4$ cm. lengths) of it came beneath a slit in a lead sheet, just above which was an ionization vessel. The saturation current produced in this vessel by the α rays emitted from the active deposit on the plate was measured by means of an electrometer, and readings were usually taken for about 4 cms. length of the plate. Corrections were applied for the decay of the active deposit with time, for change in sensitiveness of the electrometer during the measurements, and for the natural leak of the ionization vessel.



A selection of the results for the different gases will be found in Table I. and seen graphically in fig. 1, where the

abscissæ represent distances along the plate and the ordinates the logarithm of the corresponding activity. For convenience in representing the results the initial ordinate has always been taken through the same point. The mean of three values of the coefficient for air was $\cdot 096$, which is somewhat lower than $\cdot 112$, the value found by Debierne. In the light of subsequent work with different diffusion vessels, which gave higher values, it is thought that the rather low value obtained is due to the fact that the preparation of actinium did not completely cover the base of the diffusion vessel, this would cause some of the emanation to diffuse to the sides of the vessel, and consequently make the vertical gradient too steep. This, however, does not interfere with a comparison being made in the different gases.

According to Graham's law the diffusion coefficient of a gas is inversely proportional to the square root of the molecular weight of the gas into which it is diffusing. Hence, taking the value obtained for the diffusion coefficient of the emanation into air, we may calculate its coefficient when diffusing into the other gases which have been worked with. This has been done and the calculated values given at the bottom of Table I. (p. 414).

It will be seen that the differences between the experimental and calculated values get larger as we go from a heavy gas to a light one; but that even for hydrogen there is agreement to within about 10 per cent., so that one may say that if discrepancies do exist in the diffusion processes, they are exhibited fairly equally by these different gases.

Actinium Emanation in Argon.

As already stated, it was thought advisable to observe the behaviour of the emanation when diffusing into a monatomic gas; for this purpose argon was chosen. With the kind assistance of Mr. J. N. Pring about 400 cubic centimetres of the gas were prepared, by passing air, first over red hot copper to remove the oxygen, and then over calcium, heated strongly in an iron tube, to absorb the nitrogen. After passing the gas repeatedly over the calcium a density determination gave the value $19\cdot 7$, which indicated a purity sufficient for the purposes of the experiment. (Density of argon = 20.)

Owing to the small quantity of gas available smaller cylindrical diffusion vessels were made; one with two parallel glass plates suspended above some actinium, spread so as to completely cover the base of the vessel, and the other fitted

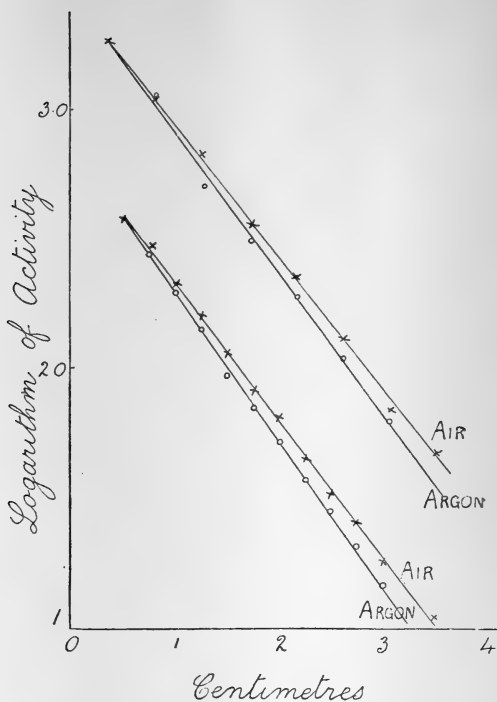
with an ebonite plug 3 cms. long resting on the base of the vessel. A uniform layer of actinium was spread over the plug, a hole through the centre of which served to keep a metal rod along the axis of the cylinder, which was made of brass. The metal rod consisted of small segments of brass sliding on a central steel rod. On applying an electric field between the rod and containing vessel, the active deposit was directed to the rod and the gradient determined by placing each of the small brass segments successively into an α ray electroscope and measuring the ionization thereby produced. The gradient on the plates was determined in the manner already described. The results of the two methods of measurement are given in Table II. and seen in fig. 2 (p. 418), where the gradients obtained with the segmented rod are drawn above those measured from the glass plates.

TABLE II.
Diffusion of Actinium Emanation in :—

<i>Air. Argon.</i>			<i>Air. Argon.</i>		
<i>Parallel Plates.</i>			<i>Segmented Rod.</i>		
Distance in cms.	Log. of Activity.	Log. of Activity.	Distance in cms.	Log. of Activity.	Log. of Activity.
·5	2·550	2·574			
·75	2·454	2·438			
1·0	2·301	2·297	1·37	2·609	2·472
1·25	2·178	2·143	1·82	2·380	2·260
1·5	2·033	1·983	2·27	2·153	1·897
1·75	1·885	1·841	2·73	1·885	1·686
2·0	1·781	1·713	3·18	1·682	1·470
2·25	1·623	1·582	3·62	1·438	1·239
2·5	1·489	1·455	4·07	1·163	·990
2·75	1·382	1·305	4·53	1·014	
3·0	1·231	1·175			
3·5	1·020	·956			
4·0	·896	·612			
Diff. coeff. K	Mean of three			Mean of two	
Experimental	·118	·106		·122	·109
Calculated	·100		...	·104

The coefficients obtained by the two different methods agree well with one another, also with the numbers calculated from Graham's law, making use of the numerical values obtained for air.

Fig. 2.



Variation of the Coefficient with Pressure.

According to the kinetic theory of gases the coefficient of diffusion is inversely proportional to the total pressure of the two diffusing gases; the product of the coefficient and the pressure should therefore remain constant.

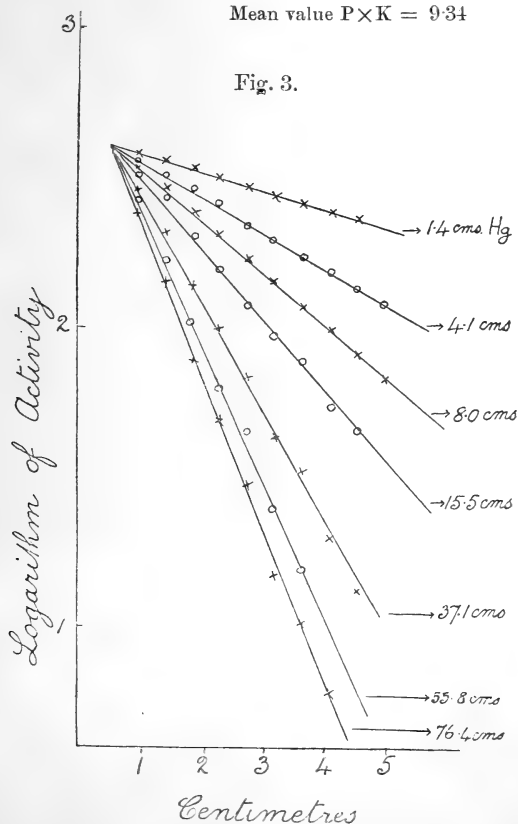
The variation of the coefficient has been determined for pressures between 76.4 cms. and 1.4 cm.; the small vessel containing the segmented rod was used. The results are recorded in Table III. & fig. 3. It will be seen that over this range of pressure, with the exception of 1.4 cm. pressure, the product $P \times K$ is approximately constant, its mean value being 9.34. It is difficult to say whether the high value 10.9 at the lowest pressure indicates a deviation from the above law as the gradient obtained is a very small one, and relatively large errors may be made in estimating it.

TABLE III.—Actinium Emanation in Air.
Variation of Diffusion Coefficient with Pressure.

	1.4 cms.Hg.	4.1 cms.	8.0 cms.	15.5 cms.	37.1 cms.	55.8 cms.	76.4 cms.
Distance in cms.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.
.46	1.618	1.797	1.850	1.254	1.518	2.246	2.609
.92	1.601	1.744	1.759	1.157	1.367	2.057	2.380
1.37	1.564	1.689	1.704	1.074	1.230	1.860	2.153
1.83	1.548	1.646	1.625	.946	1.044	1.650	1.885
2.28	1.509	1.596	1.548	.829	.909	1.434	1.682
2.72	1.485	1.523	1.469	.712	.745	1.286	1.458
3.17	1.445	1.477	1.391	.611	.539	1.024	1.163
3.63	1.425	1.421	1.313	.528	.428	.829	1.014
4.09	1.389	1.380	1.232	.374	.204	.734	.784
4.55	1.375	1.316	1.147	.298	.020		
5.01	1.268	1.071				
Diff. coeff. K	7.81	2.37	1.16	.582	.248	.167	.125
Pressure \times K	10.9	9.7	9.3	9.0	9.2	9.3	9.5

Mean value $P \times K = 9.34$

Fig. 3.



Diffusion of Thorium Emanation in Air and Argon.

Since the time-periods of thorium and actinium emanations are of the same order of magnitude, and previous work (*loc. cit.*) has shown their diffusion coefficients to be similar, the opportunity presented itself of comparing these two emanations under precisely the same conditions of diffusion.

The small diffusion vessels which had been used for actinium were emptied of their preparations and thoria substituted. The measurements were conducted in a similar way, except that the α ray electroscope could not now be used, as the thoria being considerably weaker in emanating power than the actinium, the amount of active deposit collected was smaller. The gradient along the metal rod was consequently measured by means of a rather sensitive electrometer in the manner already described for glass plates.

Observations were made in air between 76 cms. and 8.25 cms. pressure, and in argon at 76 cms.

The results are placed together in Table IV. and may be seen in fig. 4, where the abscissæ represent the distances

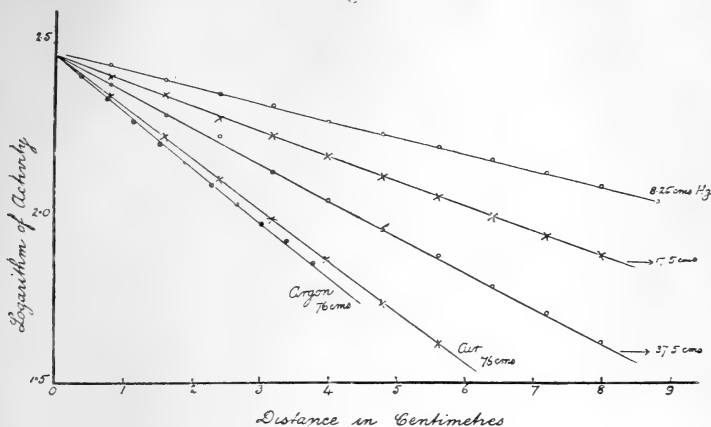
TABLE IV.
Diffusion of Thorium Emanation.

Pressure ...	AIR.				ARGON.	
	8.25 cms.	17.5 cms.	37.52 cms.	76.12 cms.	76 cms.	
Distance in cms.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Distance in cms.	Log. of Activity.
0	2.401	2.472	2.373	2.568	0	1.875
.8	2.375	2.406	2.282	2.453	.38	1.813
1.6	2.327	2.354	2.192	2.329	.76	1.741
2.4	2.289	2.285	2.134	2.203	1.14	1.672
3.2	2.255	2.239	2.028	2.082	1.52	1.608
4.0	2.206	2.176	1.947	1.962	1.90	1.563
4.8	2.169	2.116	1.853	1.833	2.28	1.490
5.6	2.131	2.058	1.775	1.713	2.66	1.434
6.4	2.094	1.989	1.685		3.04	1.369
7.2	2.053	1.937	1.602		3.42	1.327
8.0	2.014	1.874	1.517		3.80	1.254
8.8	1.969					
Diff. coeff. K Pressure \times } diff. coeff. } ($P \times K$) .. }	.966 7.97	.436 7.63	.211 7.94	.103 7.88	K experimental .084 K calculated087	

Mean value $P \times K = 7.85$

along the rod and the ordinates the logarithm of the corresponding activity.

Fig. 4.



It will be seen from the last row of Table IV. that the product pressure \times diffusion coefficient ($P \times K$) remains sensibly constant, the mean value being 7.85.

As in the case of actinium emanation no large deviation from ordinary gas laws is observable when thorium emanation diffuses into a monatomic gas of comparatively low molecular weight like argon; the experimental value obtained for the diffusion coefficient in this case being very nearly that calculated by means of Graham's law.

Comparison of Actinium and Thorium Emanation.

A direct comparison may now be made between the diffusion coefficients of these two emanations, which leads to a ratio of their molecular weights.

Taking the mean value of $P \times K$ for actinium emanation in air as 9.34, and for thorium as 7.85, and dividing each by 76 to reduce to atmospheric pressure, we have:—

$$\frac{\text{Diffusion coefficient of actinium emanation in air}}{\text{Diffusion coefficient of thorium emanation in air}} = \frac{.123}{.103} = 1.19$$

which gives the ratio:—

$$\frac{\text{Molecular weight of thorium emanation}}{\text{Molecular weight of actinium emanation}} = (1.19)^2 = 1.42.$$

It is interesting to compare this ratio with that which is obtained by taking the values of the diffusion coefficients of

thorium and actinium emanations which have been determined by other observers under different experimental conditions.

For thorium emanation we have two sets of experiments; those of Rutherford* giving a diffusion coefficient .09 and those of Makower† giving .109. In the case of actinium emanation there has, up till now (as far as the author is aware), been only one determination of its diffusion coefficient, namely, that of Debierne‡, who obtained the value .112. Taking the mean of the two values quoted for thorium as .09 we obtain for the ratio of the molecular weights $\left(\frac{.112}{.09}\right)^2 = 1.28$.

It will be seen, on reference to Tables III. and IV., that the *mean value* of $P \times K$ does not show as much as 5 per cent. variation from any one of the values themselves, this would indicate that the number given for the ratio of the molecular weights, namely, 1.42 is not subject to as much as a 10 per cent. error.

The conclusion arrived at that actinium emanation is of considerably lower molecular weight than thorium (and therefore also than radium) emanation, is quite in agreement with much of the recent work which has been done, showing that actinium is not one of the products in the *direct* line of descent from uranium to radium §.

Summary.

1. The diffusion of actinium emanation in gases such as air, hydrogen, carbon dioxide, sulphur dioxide, and argon shows no considerable deviations from the ordinary laws of diffusion.

2. The variation with pressure of the coefficients of diffusion of the actinium and thorium emanations appears to be quite regular down to pressures of a few centimetres.

3. A comparison of the diffusion coefficients of the actinium and thorium emanations in air under similar experimental conditions gives the ratio of their molecular weights as 1.42, that of thorium being the heavier.

In conclusion I have much pleasure in thanking Professor Rutherford for several suggestions during the course of the work, and Mr. J. N. Pring for assistance in preparing the argon.

* 'Radioactivity,' p. 276.

† Makower, *Phil. Mag.* 1905, p. 56.

‡ Debierne, *Le Radium*, 1907, p. 213.

§ Rutherford, *Phil. Mag.* 1907, vol. xiv. p. 733; Boltwood, 'Nature,' 1907, p. 544.

XXXIX. *Secondary γ Radiation.* By J. P. V. MADSEN, D.Sc. (Adel.), B.E. (Syd.), Lecturer in Electrical Engineering, University of Adelaide*.

[Plate VIII.]

INTRODUCTION.

AS a result of the passage of γ rays through matter, secondary rays of two types make their appearance. As it will be necessary to distinguish between the secondary rays which proceed from the sides at which the original γ rays enter and emerge from the plate which they penetrate, we shall refer to these as the "incidence" and "emergence" rays respectively.

The secondary radiation consists of β and of γ rays.

The former appear on both sides of the plate; the "incidence" β rays have been recently investigated in some detail by Kleeman (Phil. Mag., Nov. 1907) and by Eve (Phil. Mag., June 1908).

In papers by Professor Bragg and myself (Phil. Mag., May 1908; Trans. Roy. Soc. S. Aus. vol. xxxii. 1908) it is shown that most of the experimental results so far obtained with these rays can be very simply explained on the "material" theory, if we suppose that the β radiation is produced directly from the γ particle and at the outset moves in the direction of the original γ radiation, subsequently undergoing scattering in the ordinary manner of β rays.

The second type of secondary radiation resulting from the primary γ rays, viz., the secondary γ rays, has been investigated on the incidence side of plates of different material by Kleeman (Phil. Mag., May 1908), and later by Eve (Phil. Mag., Aug. 1908).

It was a deduction made by Professor Bragg (Trans. Roy. Soc. S. Aus. Jan. 1908) from the theory of the material nature of X and of γ rays previously propounded by him, that "the existence of modified or softened γ rays might be suspected, since there is an analogous effect in the case of X-rays; and probably they would be found more at the back of the penetrated plate than in front of it." The back and front sides spoken of here refer of course to the sides of emergence and incidence respectively.

* Communicated by the Author. From 'Transactions of the Royal Society of South Australia,' vol. xxxii. 1908. Preliminary account read July 17, 1908.

It will be shown in the present paper that this prediction is fulfilled very exactly ; that the want of symmetry in the amount of radiation from the two sides of the plate is very marked, that a softening of the original rays is effected, and that in addition there is in some cases a lack of symmetry in the quality of the emergence and incidence γ radiation.

Kleeman, from a study of the incidence radiation, has advanced a theory of selective absorption, not only for the secondary but also for the primary γ rays. In the light of the more comprehensive information which can be obtained from a study of both the emergence and the incidence radiation, an attempt will be made in the present paper to show that most of the secondary ray effects can be explained, if we suppose that from the radium there are originally emitted two sets of homogeneous γ rays, which each subsequently suffer modification by the process of scattering, becoming softened and in some cases broken up, giving rise to β rays.

The present paper is intended to give a preliminary outline of the experimental work and general theory.

The measurements are in many cases small and difficult to make with any very great accuracy, but the effects to be described seem well marked. It will be necessary to extend with greater care many of the details of the work before it can be considered as at all complete.

§ I.

The arrangement of apparatus employed in the first experiments is shown in fig. 1 (Pl. VIII.). The radium is placed near the apex of a conical hole made in a block of lead. By means of a powerful magnet, the poles of which are shown, most of the β rays could be prevented from passing out of the conical hole. The magnet and lead block containing the Ra were surrounded by an iron case, to prevent the magnetic field from producing any effect in the ionization-chamber. Resting upon the iron case and suitably insulated from it is placed a lead plate one inch thick, with a circular portion removed from the centre. The ionization-chamber standing upon this lead plate is made of lead 1.25 mm. thick, and contains an insulated electrode in the form of a circular wire ring, suitably protected by sulphur, and connecting to the electrometer or to earth by suitable keys.

In the path of the γ radiation which proceeds from the conical hole, plates of material which are used as radiators

can be placed horizontally as at *e*. When increasing the thickness of radiator the top plate is always kept in the position *e*, and additional thicknesses are placed immediately below and in contact with it. When large thicknesses are required, plugs of the material may be inserted in the conical hole.

The ionization-chamber was filled with ethyl chloride; this increased the effect considerably and worked very satisfactorily.

It was necessary to balance the leak due to natural ionization in the chamber, and to γ radiation which was not completely absorbed by the lead block and plates. For this purpose a balance chamber of about the same size as the ionization-chamber was placed against one of the sides of the iron case insulated from it.

This was connected up in the usual manner, and was found to give quite sufficient balance with the ionization produced by the γ radiation which came through the sides of the lead block containing the Ra. The electrometer was a sensitive instrument of the Dolezalek pattern, giving a scale reading of about 4000 divisions per volt; each division could easily be subdivided to tenths. The charge was allowed to pass into the electrometer for 30 sec., and the mean of the first and second swings was taken as the deflexion.

In fig. 2, curve A shows the effect of increasing the thickness of a Pb radiator. In an experiment, the zero leak was first obtained as a mean of several determinations, and this was subtracted from the measurement obtained with the radiator in position. It will be seen from the curve that the effect increases until a thickness of between 5 and 6 mm. of Pb is reached, after which it decreases gradually.

If, now, a hemispherical dome of Pb 4 mm. thick, with its central portion removed so as not to intercept the original stream of γ rays, be placed in the position shown by *h, h*, fig. 1, and a set of observations be taken as before, subtracting the reading with the radiator in from that with it not in position, the curve B is obtained. Again, a maximum is reached for about the same thickness of radiation as before, but the effect is for the different thickness of radiator about 60 per cent. of its previous value. We may conclude that it is γ radiation which is being intercepted by the dome, as in neither case can any but a very small amount of β radiation pass obliquely through the thickness of 1.25 mm. of lead forming the sides of the ionization-chamber; and that from the radiator are proceeding secondary emergence γ rays

which produce ionization in the testing chamber, as shown in previous papers by means of—

- (a) “tertiary emergence” β rays as at l ;
- (b) tertiary incidence β rays as at k ;
- (c) β radiation produced in the gas of the chamber from the γ rays as they pass through it ;
- (d) possibly some ions produced directly in the gas by the γ rays ; and
- (e) subsequent β and γ ray effects of a higher order.

The results of a similar set of experiments without a Pb dome and with Zn as radiator are shown in fig. 3, curve A. The maximum value of the effect measured is considerably greater than in the case of Pb and occurs at a thickness of about 2 cm. of the radiator ; for increased thicknesses the curve falls as in the case of Pb. It will be noticed that with sufficient thickness of radiator in this case the curve falls below zero. This was found to be due to the fact that a considerable amount of secondary γ radiation, proceeding from the sides of the conical hole in the Pb block containing the Ra, could enter the ionization-chamber as shown by the line c, f (fig. 1). As any radiator is increased in thickness, it not only sends out its own secondary but absorbs that coming from the sides of the conical hole. It will be shown later how proper correction can be made for this effect.

A third set of experiments was carried out with Al as radiator ; the effects were of a similar nature to those from Zn, the maximum value being about the same in amount as for Zn, and was obtained with a thickness of about 4.5 cm. ; but it was found that an appreciable difference was made when a plug of Al was placed at b in place of d as shown in fig. 1, even though the total thickness of Al was kept the same in both cases. It will thus be seen that some secondary radiation can, after emerging obliquely from a depth of several cm. of Al, produce an effect in the ionization-chamber. The present form of apparatus was therefore not suitable for obtaining from Al all the results so far obtained for Pb and Zn ; in these latter it was found that similar errors were very small. It may, however, safely be said for Al as for Zn that the effect produced is considerably greater than for Pb.

We have so far neglected all consideration of the secondary radiation which comes from the sides of the conical hole ; this it will now be shown is by no means small, but the correction which it introduced was found not to be

sufficient to interfere with the deduction just made. Consequently we may say that Zn and Al both give out more secondary emergence γ radiation than does Pb, provided of course that the quality of the radiation in the two cases is not so different as to balance the effect.

In order to find the amount and nature of the radiation which proceeds from the sides of the conical hole, Pb domes were placed as at *h, h*, or Pb cylinders as at *i, i*, fig. 1. When the latter were used, their equivalent thickness was found in terms of corresponding domes. A cylinder was found to be approximately equivalent to a dome 1.5 times its thickness.

The results of experiments upon the radiation from the sides of the cone are shown in fig. 4, curve A, in which thicknesses of dome are shown horizontally and the corresponding leaks per 30 secs. vertically.

The curve A can be represented approximately by the expression $500 + 570e^{-1.25d}$.

That the radiation comes from the sides of the cone can be shown by placing at *e* (fig. 1) Pb plates in which circular holes are cut so as not to intercept the main stream of γ radiation coming from the Ra.

The results now obtained for different dome thicknesses are shown by curve B, fig. 4, which it will be seen can again be represented approximately by the expression $500 + 430e^{-1.25d}$.

It seems reasonable to suppose that besides the constant normal leak in the ionization-chamber, secondary γ radiation from the sides of the conical hole can enter the chamber, and that this radiation may for the purposes of correction to which it is to be applied later on, be taken as equivalent to a homogeneous radiation for which $\lambda = 1.25$.

If now a Pb plug, 4 cm. deep, be placed in the conical hole in the position shown by *a*, fig. 1, so as to intercept the main stream of γ radiation, but still leave the sides of the cone exposed to the ionization-chamber, and a set of experiments similar to the above be performed, the results shown in curve C, fig. 4, are obtained. These again are approximately represented by $500 + 270e^{-1.25d}$.

It is not surprising that the constant normal leak has not been reduced to any appreciable extent, for it will be seen from fig. 1, that the plug would hardly intercept any γ radiation passing directly from the Ra to the ionization-chamber. On the other hand, we obtain a rather important result, for assuming, of course, that the interpretation put upon these experiments is correct, even when the Ra is

completely surrounded by a thickness of 4 cm. of Pb, a secondary radiation escapes, which has a value of λ approx. 1.25. Now it has been shown by Wigger (*Jahrbuch der Rad.* Bd. ii. 1905) that practically all soft γ rays coming from the Ra are stopped by a thickness of Pb considerably less than 4 cm., and that the hard γ rays which can penetrate such a thickness have a value of $\lambda = .24$.

Again, the amount of the secondary radiation has only been reduced from 570 to 270 by the insertion of the 4 cm. plug of Pb, so that we must suppose most of the secondary radiation we are dealing with to be produced from the hard γ rays and very little of it from the soft γ rays. This result, it will be shown later, is supported by other experiments.

It is now possible to apply the necessary correction to the curve A, fig. 2. The result is shown by the full line curve C, the proper correction being obtained by adding a quantity $400(1 - e^{-(1.5 \times 1.25d)})$ to the original values. Part of the secondary radiation from the sides of the cone was in these experiments cut off by suitable Pb screens; the factor 1.5 is introduced to allow for the obliquity with which the rays coming from the sides of the cone cross the radiators.

In order to investigate now the quality of the secondary radiation which comes from the Pb radiators, a Pb plate 5 mm. thick was used as radiator, and the emergence radiation was absorbed by Pb domes. In each experiment the dome having been placed in position, two sets of measurements were taken, one with, the other without, the radiator, and the results of these subtracted. The mean of a number of experiments is shown in Table I., and in fig. 5, curve A, the value of the thickness of the dome multiplied by its density is shown horizontally, and the log. of the corresponding current vertically, the reading with no dome being reduced to 100. It will be seen that the emergence radiation from Pb, within the range of thickness of dome, over which it has been measured with sufficient accuracy, is approximately equivalent to a homogeneous radiation for which $\lambda = 1.33$.

Now these results require correction on account of the absorption by the 5 mm. Pb radiator of the secondary radiation from the sides of the cone. As, however, this latter has been shown to have approximately the same value of λ , viz., 1.25, it is hardly necessary at the present stage to attempt to apply the correction.

The quality of the radiation from Pb was now tested by Zn and by Al domes. The results are shown in the curves B and C, fig. 5. It will be seen that λ/Δ is of much the

same value when the radiation is tested by Zn or Al; but is considerably greater when tested by Pb. This is similar to the results which are obtained when the original γ radiation coming from Ra, which apparently consists of a mixture of hard and soft rays, is tested by small thicknesses of absorbing screen.

TABLE I.

PB DOMES.

Radiation from Pb.

Thickness of Dome, mm.	0	1	4	6	10
	100	80	60	43	26.5

Radiation from Zn.

Thickness of Dome, mm.	0	2	4	6	10
	100	64	46	33	21

Radiation from Al.

Thickness of Dome, mm.	0	2	4	6	10
	100	63	45	32	20

ZN DOMES.

Radiation from Pb.

Thickness of Dome, mm.	0	5.4	11.7	25.3
	100	85	67.5	44

Radiation from Zn.

Thickness of Dome, mm.	0	5.4	11.7	25.3
	100	84	63	34

Radiation from Al.

Thickness of Dome, mm.	0	5.4	11.7	25.3
	100	81	59	33

AL DOMES.

Radiation from Pb.

Thickness of Dome, mm.	0	8	21
	100	88	75

Radiation from Zn.

Thickness of Dome, mm.	0	21
	100	74

Radiation from Al.

Thickness of Dome, mm.	0	21
	100	72

It has already been shown that the radiation from the sides of the cone is approximately of the same quality as that

from a Pb radiator; we may therefore assume without much possibility of error that the value of λ by Zn for this radiation is .34.

It is now possible to apply the necessary correction to the curve A, fig. 3. A quantity $570(1 - e^{-(1.5 \times .34)d})$ has been added to the previous readings; the result is shown in the full-line curve B. The last two readings require a slightly less correction on account of the Zn plug cutting off some of the original γ radiation which produces the effect from the sides of the cone; the necessary correction in this case was determined by a separate experiment.

The quantity of the emergence radiation from Zn and from Al was now tested by Zn and by Al domes. The results are given in Table I., proper correction being applied in each case for the absorption of the radiation from the sides of the cone.

The Zn radiator was 12 mm. thick, the Al 17 mm. Measurements were also made with radiators of other thicknesses, but the results were not sufficiently accurate to enable one to say whether the quality of the emergence radiation varied with the thickness of radiator.

The results are plotted in fig. 5, from which it will be seen that the radiation from Zn and Al is far from homogeneous when tested by Pb domes.

The emergence radiation from Zn and Al appears to consist of two sets of γ rays—one of much the same quality as those which are produced from Pb, the other a very much softer bundle. A rough analysis of either the Zn or Al curve, if such is legitimate, would suggest that it may be derived from a hard and soft bundle of γ rays, for which the values of λ by Pb are 1.3 and 5 respectively, the effect of the hard bundle being initially almost twice that of the soft.

It will be seen from the figure that mass for mass Al and Zn domes are not able to bring out so clearly as Pb domes the distinction between the quality of the secondary radiations from Zn and Al as compared with that from Pb.

This, again, is much the same as is found when one attempts to sort out the original hard and soft γ rays of Ra by such screens.

However, the radiation from Pb always appears somewhat harder than that from Zn and Al, no matter what screen is used.

This result has been checked on different occasions with care, and is of importance, as the results obtained by Klee-man from the secondary incidence γ rays, and the theory of selective absorption founded thereon, and extended to the original γ radiation, should apparently apply also to the secondary emergence rays.

§ II.

To investigate the secondary incidence γ rays the ionization-chamber (fig. 1), was inverted, and the large plates which served as radiators were laid directly upon the ionization-chamber as at *p, p*. This enabled the secondary incidence rays from the radiator to enter the ionization-chamber, not only by the sides *i, i* but also through the top of the chamber, and should more than compensate for any decrease in effect due to the primary γ rays falling upon a larger area than they did in the case where the emergence radiation was measured.

The secondary incidence radiation increased in amount as the thickness of the radiator was increased. In fig. 6 is shown the relation between the mass of the radiator and the amount of the secondary incidence γ radiation for C, Al, Zn, and Pb.

C and Al gave maximum values of nearly the same amount, viz., 300; Zn not quite so much, viz., 270; while Pb gave only 100. Comparing these with the corrected values for the emergence radiation from Pb and Zn respectively, which were 680 and 1400 approximately, it is seen that the lack of symmetry between the effects on the two sides of a plate is very marked—the effect on the emergence side being from about 4.5 to 6.5 times that on the incidence side. The quantities on the incidence side are consequently so small that they become difficult to measure with any great accuracy, especially when an attempt is made to determine the quality of the radiation.

To measure the quality of the incidence radiation the top of the ionization-chamber was covered with a lead sheet about 1 cm. thick, from the centre of which was cut a circular hole 17.5 cm. diameter. One of the larger domes was then inverted and placed so as to intercept the radiation from the radiator, which was placed in the large circular hole just mentioned. A thickness of radiator sufficient to give the maximum effect was used in all cases. There appeared to be very little difference in the quality of the radiation returned from C, Al, and Zn. A Pb dome 4 mm. thick reduced the effect to approximately 20 per cent., indicating a radiation for which the value of λ is 4, if we can suppose it homogeneous.

In the case of Zn and Al it will be seen from Table I. that a 4 mm. Pb dome reduces the effect of the emergence radiation to about 45 per cent. There is thus a considerable difference in the quality of the emergence and the incidence γ radiations from these substances.

That the incidence radiation from these substances is softer than the emergence, is also supported by the experiments of Kleeman and Eve already referred to, both these observers having found the value of λ for the incidence rays from these or similar substances to be of the order 4, measured by Pb. In the case of the incidence radiation from Pb the measurements were not sufficiently accurate to decide whether the incidence radiation was softer than the emergence; it appeared to be of much the same quality, *i. e.*, $\lambda = 1.3$ approximately.

§ III.

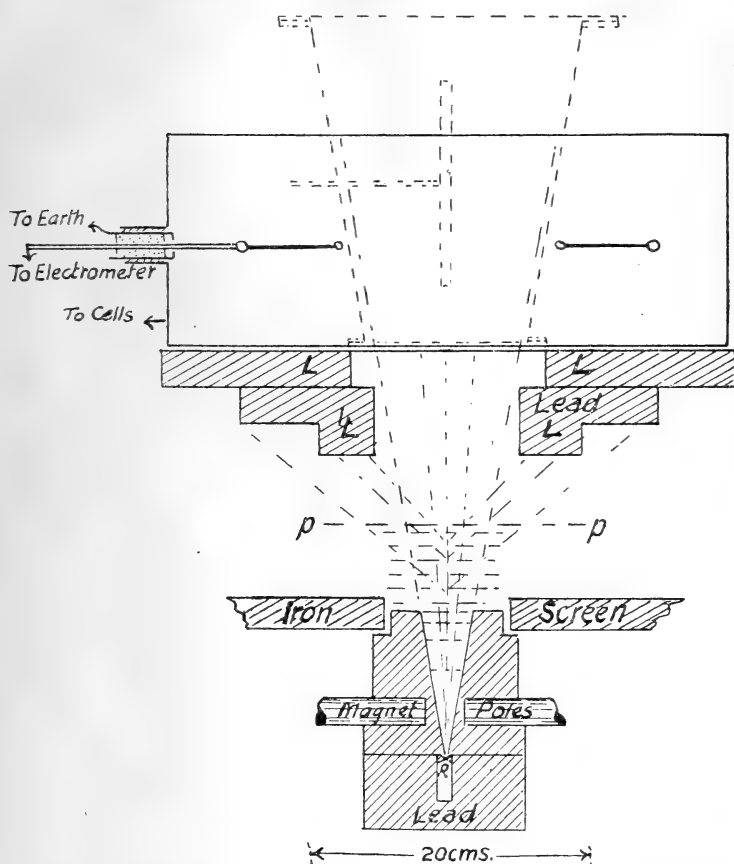
As the apparatus described in fig. 1 was unsuitable for measuring the amount of secondary emergence radiation from substances such as Al and C, and as we do not know that the distribution of the emergence radiation is the same in all directions for all substances, an experiment was set up as shown in fig. 7.

The lower portion of the diagram shows the arrangement of the Ra, magnet, and iron case which is the same as previously described. Some distance above the Ra was supported—upon an insulated stand—a flat cylindrical ionization-chamber. The floor of this chamber was of sheet lead 1.25 mm. thick, the sides were of Zn, and for a lid was used a thin Al plate. The conical chamber, shown by dotted lines, was used in experiments to be described later. The radiators were in the form of flat plates, the first of which in any experiment was placed in the position *p, p*. Additional thicknesses were added immediately below *p, p* and in contact with the top plate, while for very great thicknesses plugs of the material could be inserted in the conical hole in the Pb block, through which the γ radiation streamed.

In the first set of experiments the Pb plates L, L were used in the position shown, the large central hole in them allowing the γ rays to pass without obstruction to the ionization-chamber; while a considerable amount of secondary γ radiation from the radiating plates *p, p* was prevented from reaching the chamber. A strong magnetic field was applied during the experiments. The floor of the ionization-chamber, consisting of Pb 1.25 mm. thick, was sufficient, as has been shown in previous papers, to give the full amount of emergence β radiation. Several readings were taken with each thickness of radiator in position, and the readings were sufficiently large with the chamber filled with air. From any reading it is necessary to subtract an amount corresponding

to the leak produced by the natural ionization in the chamber, to the radiation which comes directly from the Ra and has not all been cut off by the interposed screens, such as L, L, and to secondary γ radiation from surrounding bodies, which

Fig. 7.



The radiators shown by horizontal dotted lines below p, p intercept the radiation from the Ra. By means of Pb plates L, L an amount of the secondary radiation may be prevented from entering the ionization-chamber. The dotted conical chamber was used in a separate set of experiments.

may be able to enter the chamber. This latter quantity was found to be by no means negligible, and so far as I know has not been allowed for in many experiments by previous

observers. To obtain with great accuracy the zero reading necessary for subtraction, is a course exceedingly difficult.

In the present experiment a close approximation to its value was found by inserting a large Pb plug in the conical hole and placing plates of Pb above this, so as to make in all a thickness of about 20 cm. This, as shown by previous observers, should be sufficient to reduce even the hard γ radiation, which streamed through the conical hole, to almost a negligible amount, while it should not affect to any great extent the secondary radiation coming from surrounding bodies.

The results of experiments with different materials used as radiators are given in Table II., and in fig. 8 are shown by full-line curves the thickness of radiator multiplied by its density, plotted horizontally against the logarithm of the corresponding currents vertically. The value of the current with no radiation in position at p , p has in each case been reduced to the common value 100, and the zero for the ordinates has in some of the curves been altered so as to prevent overlapping. It will be seen that Pb, Bi, and Hg give results almost identical; while Sn, Zn, Al, and C all give results very similar to each other, but differing from those of the first group in that a more sudden drop of the curve is shown for the initial thicknesses of radiator. From a value of $d\Delta=7$ onwards the curves for all the substances become practically parallel.

The Pb screens L, L, fig. 7, were now removed, and radiators were used in the form of large, flat plates, the first of which was placed horizontally just below the floor of the ionization-chamber, merely separated from it by a small thickness of ebonite for insulation.

Successive thicknesses of radiator were added immediately below the first. By this means it was thought possible to allow a large proportion of all the emergent secondary γ radiation to add on its effect in the chamber. The results of these experiments, given in Table III. and shown by the dotted curves in fig. 8 (Pl. VIII.) show that this anticipation was realized.

The dotted curves $\textcircled{\text{Al}'}$ and $\textcircled{\text{Zn}'}$ need not be considered at present; they were obtained with a carbon floor introduced into the ionization-chamber, and show the effects which previous work would lead us to expect.

Al, C, Zn, Fe, S, and Sn give results which are all very nearly identical, and the curves show a considerable difference in slope to what they did in the previous experiment.

Pb and Hg, however, are distinctly different from the other substances, of which we may hereafter take Zn as typical.

The curve for Pb has not been altered to such a large extent by the change of arrangement of the experiment as has that for Zn and such substances.

The results for Zn are shown separately in fig. 9 (Pl. VIII.) by curves A and B, in which the values of thickness of absorbing screen multiplied by its density are shown horizontally and the corresponding currents vertically. The difference between the ordinates for any value of $d\Delta$ represents the amount of secondary radiation which it was possible to cut out by the Pb screens L, L in the first experiment. This difference is shown by the dotted curve C. It will be seen that the amount of secondary γ radiation increases with the thickness of absorbing screen or radiator, until a value corresponding to $d\Delta = 18$ has been reached, after which it gradually falls. For certain values of $d\Delta$ the effect produced by the secondary γ rays may be almost as great as that which is being produced by the primary γ rays.

It is thus seen that the effect produced in an ionization-chamber in an ordinary absorption experiment depends largely upon the position of the absorbing plate relatively to the ionization-chamber, and the shape of an absorption curve may consequently be made to vary between considerable limits. The shape of the chamber, however, appears to make very little difference to the shape of the absorption curve, within of course certain limits.

A conical ionization-chamber, as shown by the dotted diagram in fig. 7 (p. 433), gave results for the first few thicknesses of radiator almost the same as those shown by the dotted curves, fig. 8 (Pl. VIII.) ; while, again, with the plates placed at p , p results similar to those shown by full-line curves in the same figure were obtained, without of course the necessity of using Pb screen such as L, L.

This is readily explained if we remember that probably most of the effect produced in the ionization-chamber is due to emergence and incidence β rays coming from the floor and lid of the chamber ; these rays being the product of the original γ rays and of the secondary γ rays. The β rays as a result of the scattering they undergo emerge in all directions from the floor and lid of the chamber, and any alteration in the shape of the chamber will affect the ionization produced by the β rays which come from the secondary γ rays to much the same extent as it will those which come from the original γ rays.

Returning to fig. 8 it will be found that the ordinary absorption curve, corresponding to the dotted curve Pb 1, *i. e.*, in which as much secondary radiation as possible is included, can be represented very accurately by the expression : $30e^{-.32d} + 70e^{-1.35d}$.

Again, the full-line curve Pb 2, from which as much as possible of the secondary radiation has been excluded, is represented by the expression : $24e^{-.32d} + 76e^{-1.35d}$. The difference between these two expressions, *viz.*, $6(e^{-.32d} - e^{-1.35d})$ is a measure, therefore, of the total emergence secondary γ radiation from Pb. Now it has been shown in previous experiments in this paper that the secondary γ radiation from Pb can be approximately represented by a homogeneous radiation, for which the value of λ is 1.25 ; we are therefore able to account for the results which have just been obtained if we suppose that the original hard bundle of γ rays, which proceed from the Ra, makes all, or at least very nearly all, the secondary radiation which comes from Pb ; this is in agreement with the result obtained previously (p. 425).

In the case of substances of the nature of Zn the difference between the two absorption curves, as shown in fig. 9, cannot be explained so simply. The secondary radiation after reaching its maximum value falls off more rapidly than we should expect, had all the secondary been derived from the hard set of original γ rays. We must suppose that in such substances the soft γ rays from the Ra also produce a certain amount of secondary emergence radiation, or that even though such may be produced in all substances it is better able to escape from such substances as Zn than from substances of the nature of Pb. This of course is merely another way of saying that Pb absorbs the softened secondary radiation to a much greater extent than Zn.

Again, it has been shown earlier in this paper that the emergence radiation from such substances as Zn appears, when tested by Pb, to be divisible into two quantities corresponding to a hard and soft bundle of γ rays, the hard having about the same penetrating power as the secondary rays from Pb. As it will be shown presently that there is good reason to believe that two distinct bundles of γ rays—a hard and a soft—are given out from the Ra, we may for the present look upon the two bundles of secondary rays which are produced from Zn and such substances as the products of the corresponding hard and soft bundles of the original radiation.

It will be seen from fig. 8 that when as much secondary

radiation as possible is excluded the curves for all substances very nearly correspond. Let us suppose that had it been possible to exclude all the effects of secondary γ radiation, the curves for all substances would have become identical. This supposition must apparently be very near the truth, for it can be seen from the geometry of the arrangement in fig. 7 that by no means can all the secondary radiation have been excluded in the first set of experiments. We arrive, then, at this result: that, excluding all secondary γ radiation and its effects, we may consider the Ra to give out two distinct sets of γ rays, one more penetrating than the other, each practically homogeneous, with values of $\lambda/\Delta = \cdot 028$ and $\cdot 12$ respectively, these values being practically independent of the nature of the absorbing substance.

When the secondary γ radiation is allowed to produce its effect in addition to that produced directly by the original γ rays, this law may be modified to a very considerable extent. In the usual arrangement of apparatus used for determining the quality of a radiation the absorbing plates are as a rule placed close to the ionization-chamber, and the values of λ deduced from the tangent to the absorption curve—that is to say, it is generally with the dotted-line curves of fig. 8 that we are dealing, and the effects of secondary radiation may, as has already been shown, be in some cases very nearly as important as the effects produced directly from the original γ radiation.

This, for example, affords a ready explanation of the effect observed by Eve (Phil. Mag., Aug. 1908), who states:—“It is noteworthy that the Ra in 2·2 cm. of nickel-steel gives an effect about 1·5 times as great as when the Ra is in 1 cm. of Pb. From the relative densities we should expect 2·2 cm. of steel to be equivalent to 1·5 cm. of Pb, and therefore the Ra in the steel cylinder should give, by the density law, two-thirds of the effect of the Ra in Pb. It actually gives one and a half times as much; thus the primary γ rays traverse steel much more readily than Pb, but the rays passing through iron are subsequently absorbed more readily by Pb than if the Ra were in Pb.”

From the results of the present paper we can say that in going through 2·2 cm. of steel or 1·5 cm. of Pb, the soft primary γ rays will have suffered in each case the same number of collisions, but that the effect of a collision with an atom of Pb, or rather with the constituent of such an atom, is much more definite than in the case of substances of lower atomic weight; that in the case of Pb the collision

has broken the original γ ray up, immediately sending out a β ray, or has so shattered the original γ ray as to make it easy for Pb to subsequently complete the process.

In the case of Zn, &c., the effect of the collision has not been so definite; much scattered γ radiation is produced, somewhat softened by the effects of collision, but still able to suffer further modification as a γ ray before being eventually broken up and sending out the β ray.

It must be noticed that in experiments upon the secondary incidence γ rays, such as have been carried out by Kleeman and by Eve, the nature of the secondary effect may be modified to a very considerable extent if the Ra be covered directly by a considerable mass of a substance such as Zn. Where a similar screen of lead would appear to have hardened the primary beam which is being experimented with, the screen of Zn may very well appear in some cases to have softened the primary beam, more especially if the secondary effects produced by hard and by soft rays do not conform to the same law of distribution.

It may be observed that the value of λ/Δ obtained in the present investigation for the hard γ rays from Ra is $\cdot 028$, whereas the mean value obtained by Wigger was $\cdot 021$. The difference probably arises from the fact that Wigger appears to have taken the natural ionization in the chamber as the amount to be subtracted from all the readings. It seems quite possible that a considerable amount of secondary radiation from surrounding bodies may have been able to enter the chamber, in which case the correction should have been greater, and this applied to all the readings would increase the value of λ/Δ which he obtained.

Again, since the hard γ rays produce a considerable amount of secondary radiation in Pb, it is not surprising that with a long, narrow chamber, such as Wigger used, the shape of the absorption curve for Pb differs from that obtained in the present experiments.

We have now sufficient information to enable us to construct at least a working theory. The main points to be observed are, firstly, the lack of symmetry in the amounts, and secondly, a lack of symmetry in some cases in the quality of the secondary γ radiation.

The asymmetry shown by the secondary emergence and incidence β rays, which are produced from the primary γ rays of Ra, has already been put forward as an argument in favour of the "material" theory propounded by Professor Bragg.

The modification of the æther-pulse theory, recently advanced by Professor Sir J. J. Thomson, may possibly

furnish a partial explanation of the facts ; but there are many difficulties in the way of a pulse theory, even modified to this extent ; some of these have been pointed out in a recent paper by Professor Bragg and myself (Trans. Roy. Soc. S. Aus. vol. xxxii. 1908).

When one attempts to further explain by the pulse theory the lack of symmetry which exists in the case of the secondary γ rays, the difficulties become much greater.

If we are to suppose that the incidence and emergence γ rays are true secondary effects produced by vibrations of electrons in the absorbing material, the effect resembles very closely that of fluorescence. The corresponding optical problem, however, gives rise to no such lack of symmetry ; we find in its case no corresponding asymmetry of distribution or of quality of the secondary effect.

If, however, we attempt an explanation upon the "material" theory, these difficulties at once disappear.

Let us consider in the first place the effects produced by a homogeneous bundle of hard γ rays. These in passing through matter suffer collision ; the effect of such collision is to change the direction of motion of the incident primary ray—in other words, to scatter it ; at the same time the scattered ray loses a certain amount of energy—it has become softened ; this softening may be due either to a change in its speed or to a change in moment of the γ pair, or it may be both. The distribution of the scattered radiation will probably depend upon the nature of collision, and may be influenced largely by the atomic structure of the matter with which the γ particle has collided. We have seen for instance that in the case of Pb, where the secondary radiation is produced mainly from the hard bundle of γ rays proceeding from the Ra, that much more of the scattered γ radiation moves on in the direction of the original rays than returns in the opposite direction.

Consider the case of a fine stream of homogeneous rays moving in the direction x , a plate of absorbing material of thickness L being placed in their path, at right angles to the direction of motion of the particles. Let I_0 be the number of γ particles which are sent out by the Ra per unit time. The number I which are able to proceed a distance x through the absorbing plate without suffering appreciable effect is such that $I = I_0 e^{-\lambda x}$. In a distance dx , the number which have suffered serious collision is $I_0 \lambda e^{-\lambda x} dx$. Of these let a fraction, q , be merely scattered, at the same time softened or reduced to what we may call an intermediate stage, in which their coefficient of absorption (so called) is λ' .

The remainder represented by the fraction $(1-q)$ are turned into β rays at once. Let $i_0 = qI_0\lambda e^{-\lambda x} dx$.

As a result of scattering in a direction inclined to the original at an angle θ , let $i_0 F\theta$ represent the number of particles per sec. which cross the unit area of a spherical surface of unit radius described with centre at the point where scattering occurs; then the number of scattered rays which emerge per sec. from the flat plate of thickness L

$$= \int_0^L \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2\pi \sin \theta \cdot F\theta \cdot qI_0\lambda e^{-\lambda x} e^{-\lambda'(L-x) \sec \theta} d\theta dx.$$

In addition to these rays we have emerging per sec. a number $I_0 e^{-\lambda L}$ which have suffered no scattering.

We are not at present able to evaluate the expression given above, from want of knowledge regarding $F\theta$. However, we may proceed to make an approximate calculation of the effect which might be expected in such a case as we have investigated in fig. 1.

Considering as before a fine pencil of homogeneous γ rays falling normally upon a plate of thickness L , we may express the emergence radiation as—

$$kI_0\lambda \int_0^L e^{-\lambda x} e^{-\alpha\lambda'(L-x)} dx = \frac{KI_0\lambda}{\lambda' - \lambda} (e^{-\lambda L} - e^{-\alpha\lambda' L})$$

where k is a constant, λ the coefficient of absorption of the primary set of rays, λ' a similar coefficient for the secondary rays, and α a constant obliquity factor, which in the present case is taken as 2, a value obtained by considering the geometrical arrangement of the radiators in fig. 1 with regard to the ionization-chamber.

Let us now apply this result to the case of a Pb radiator; the values of λ and λ' are .32 and 1.3 respectively. The curve corresponding to $e^{-.32L} - e^{-2.6L}$ is shown by the dotted line in fig. 2, the maximum having been adjusted to the same value as for the experimental curve.

The agreement between the theoretical and the actual curve is very good.

The maximum value is reached for about the same thickness of radiator, and for thicknesses of radiator greater than that required to give the maximum effect the curves slope away to about the same extent. This, taken in conjunction with previous experiments, seems to show that

practically all the secondary γ radiation obtained in these experiments from Pb may be considered as derived from the original hard γ rays of the Ra.

It must, however, be remembered that the secondary emergence rays have to penetrate the Pb sides of the ionization-chamber, and allowing for obliquity this corresponds to a thickness of about 2 mm., so that it is still quite possible that a fair amount of very soft secondary γ radiation may have escaped detection.

If we now proceed in a similar manner to construct the corresponding curve for Zn, viz., $e^{-21L} - e^{-64L}$, it is found that it by no means corresponds to the experimental curve. If, however, we assume that for a thickness of about 10 cm. of the Zn radiator the primary soft bundle is producing very little of the secondary emergence γ radiation, the curve as calculated above may be placed in the position shown by C, fig. 3. The curve D is now drawn to represent the difference between the curves B and C. We may now see how closely this remainder corresponds to what might be expected if the soft bundle of original γ rays produced a corresponding softer secondary. The maximum in the case of this remainder curve is further to the left than in the case of curve C, and after reaching its maximum value the curve falls away much more rapidly than does C. It corresponds, in fact, very well with what we might reasonably expect to be produced by the soft bundle of γ rays coming from the Ra. Again, although for any given thickness of radiator the ratio of the ordinates to the curves C and D is by no means constant, it is of the order indicated in fig. 5 by the curve A''.

If, now, the curves C and D represent the effects due respectively to the hard and soft γ rays from the Ra, it should be possible to show experimentally that the quality of the radiation from a Zn radiator depends to some extent upon the thickness of the radiator. No serious attempt has so far been made to carry out this investigation. However, we may check the result in another way. The quality of the radiation from Zn should show some change if we absorb some of the soft γ rays coming from the Ra by means of a Pb screen or plug before allowing the radiation to fall upon the Zn radiator. As we have already seen, it is possible in this way to reduce the softer radiation without cutting down the hard to such an extent as would be necessary if a plug of material such as Zn were used.

Using 17 mm. of Al as radiator, with no plug, the reading with no dome was to the reading with a 6 mm. dome as

100:31. With a 9 mm. plug, however, inserted in the conical hole just over the Ra the reading with no dome was to the reading with a 6 mm. dome as 100:39. A similar effect was also obtained with a Zn radiator.

We see, then, that hardening the primary radiation has the effect of hardening the secondary, and this is in agreement with the suggestion put forward, viz., that in the case of Zn, Al, and such substances the secondary emergence γ radiation consists of two bundles corresponding to, in fact derived immediately from, the two corresponding bundles constituting the original γ radiation.

In fig. 9 the dotted curve C has been plotted to a larger scale, and the curve D obtained previously in fig. 3, and shown in that fig. curve B, has been plotted with the same maximum value as C' for comparison with it. It will be noticed that the curve D reaches its maximum for a smaller thickness of radiator than does C'. This we might expect as the secondary rays will be able to emerge from a greater depth of the radiator, when some of them come out more normally, as happens in the case represented by curve C'.

In the experiment from which the curve D is derived the rays emerge very obliquely, and their self-absorption by the radiator is much greater for a given thickness of radiator.

Otherwise the curves are in very good agreement. It is to be noticed that in curve C' certain effects may show themselves which are probably not present to such a large extent in the case represented by D. Soft radiation, for instance, has a much better opportunity of producing its effect in the former case.

Returning now to a brief consideration of the secondary incidence γ radiation. Although no attempt has as yet been made to determine whether the radiation from different substances is strictly homogeneous, it was thought advisable to find out how its amount varied for any substance as the Ra was covered with different screen thicknesses of Pb.

In these experiments C, Zn, and Al showed much the same effect. With no plug, a 5 mm. plug, and a 10 mm. plug over the Ra, the corresponding readings were as 100:60:35. Now, since the value of λ for the soft γ radiation from Ra is 1.35 for Pb, it should have required about 8 mm. of Pb to reduce it to 35 per cent. of its original value, and had the incidence secondary radiation been due entirely to the original soft γ rays it should apparently have been reduced to much the same amount. It is shown above that

10 mm. of Pb, placed in the path of the primary, reduces the secondary effect to 35 per cent., so that it appears that very nearly all the secondary incidence γ radiation from Zn and such substances is derived from the soft bundle of primary γ rays.

In the case of the radiation from Pb the measurements could not be performed with any great accuracy. However, it was found possible to place a 2 cms. Pb plug over the Ra without cutting off all the incidence radiation ; the reading was reduced to about 30 per cent.

The experiment is accurate enough to enable us to say that the secondary incidence radiation from Pb is derived to a large extent from the original hard bundle of γ rays ; its quality, as tested by Pb domes, shows also that it is of much the same nature as the emergence radiation, which appears to be the product of the hard γ rays.

In all these experiments, and especially in those which deal with the incidence radiation, it must be borne in mind that the radiation has to penetrate a thickness of Pb, corresponding to 2 mm. before its effect is measured, and consequently the effect of very soft γ radiation may have been missed.

The lack of symmetry in quality of the emergence and incidence radiation from some materials may be explained if we suppose that the distribution of the scattered radiation which is produced from the hard γ rays is not the same as for that produced from the softer rays. The secondary radiation produced from the hard γ rays appears to move on more in the direction of the original stream than do the secondaries produced from the softer rays.

The distribution of the secondaries may also to some extent depend upon the nature of the medium in which scattering has taken place, and it does not seem at all unlikely that the atomic structure of the radiator should determine to some extent the nature and result of a collision between the γ particle and the constituent part of the atom.

It has already been stated that these experiments give little or no support to the theory of selective absorption advanced by Kleeman ; the effects seem much simpler than we might be led to suppose from that theory. It seems possible, also, to suggest probable causes for the effects which Kleeman has observed. In some cases the absorbing screens used by Kleeman seem not to be of sufficient thickness to preclude the possibility of a certain amount of β rays, given off by the radiator, penetrating the screen. Secondly, it has

been assumed that the incidence γ radiation from all the substances experimented with is homogeneous. If, for example, the incidence radiation from Pb contained a very soft bundle in addition to the hard, many of the results obtained by Kleeman could be immediately explained. With the present form of apparatus I have not of course been able to test this point; it does not, however, seem at all improbable that Pb should give out an exceedingly soft bundle of γ rays, the product of the soft γ rays in the original radiation; indeed, if it is safe to make comparisons from the behaviour of such substances as Zn, it would be surprising if such a soft bundle did not exist, although of course its effect may be small. Thirdly, in dealing with the very soft γ rays, which constitute the secondary incidence radiation, it seems necessary to take into account the effect of tertiary radiation.

In conclusion it may not be out of place to discuss one or two points of some interest which have arisen during the progress of these experiments.

In fig. 8 it will be observed that for a mass of absorbing screen 70 the result of cutting out a considerable amount of the secondary radiation has been to change the value of the effect produced with Zn screens from 56 to 43. In the case of Pb the corresponding change is from 38 to 30. In other words, the ratio of the effect produced by a Zn screen to that produced by a Pb screen of the same mass is apparently much the same whether secondary radiation is included or not. It appears almost as if the effect produced in the ionization chamber were due more to the secondaries produced from the hard γ rays than to direct effects of these hard γ rays. The hard γ rays are not apparently broken up as the result of the first collision; a few may be, but most of the rays seem capable of passing into what we may for convenience speak of as an intermediate stage. A second collision when the ray is in this stage may be more effective in breaking up the γ ray and causing the β particle to be sent out. In some cases the γ ray may still retain the characteristics of the γ ray even after many collisions.

It would seem possible for a γ ray to produce some ionization in a gas as a result of its passage through that gas, slow-speed delta rays being produced from the atoms of the gas. Where high-speed electrons appear they are to be considered as having been produced from the γ ray itself, being originally a part of it, as already suggested by Professor Bragg.

It has been found by Eve (Phil. Mag. 1906) that the γ

rays from thorium have an absorption curve, measured by Pb, very nearly the same as that for Ra. We should expect, then, that thorium gave out two homogeneous sets of γ rays as does Ra, the values of λ by Pb for these groups being $\cdot 32$ and $1\cdot 35$. It has also been found by Eve that the value of λ for the γ rays given out by Ur is $1\cdot 4$, and that these rays appear nearly homogeneous. The γ rays of Ur appear, then, to correspond very closely with soft bundle of rays emitted by both Th and Ra.

Again, it seems rather striking that the secondary rays produced by the hard γ rays of Ra produce in Pb, and apparently in all substances, a secondary, for which the value of λ is of the same order as for the Ur rays and the soft rays of Ra and Th.

In the case of the secondary rays produced from the hard γ rays of Ra it has been possible to treat them as practically homogeneous over the range of thickness of screens which have been used. It will, however, require much more careful research to determine the exact quality of this radiation.

Again, the value of λ/Δ for the soft primary bundle of rays has been shown to be nearly four times that of the hard. It seems possible that this may have some connexion with the result obtained previously by Professor Bragg and myself for the β rays which are produced from the γ rays of Ra. It was found that the value of λ for the hard and soft β rays was approximately as one to four.

SUMMARY.

1. The γ rays of Ra and possibly of Th appear to consist of two distinct homogeneous bundles, the value of λ/Δ for the soft set being approximately four times that for the hard.

2. For each set of rays the value of λ/Δ is constant and practically independent of the nature of the absorbing material with which λ is measured, provided that in the case of the soft rays secondary effects be excluded.

3. Secondary γ radiation appears on both sides of a plate which is penetrated by a stream of γ rays. There exists a marked lack of symmetry between the amount of secondary radiation which proceeds from the two sides.

4. A lack of symmetry exists in the case of some substances between the quality of the radiation on the two sides.

5. The last results seem very difficult to reconcile with a pulse theory. On the "material" theory propounded by Professor Bragg no such difficulty arises.

6. The secondary γ radiation appears to be derived from the primary by a process of scattering. This process generally involving a reduction in the subsequent penetrating power of the ray affected.

7. There appears to be reason to believe that the distribution of the scattered radiation depends to some extent upon the hardness of the radiation which is scattered; also upon the nature of the material in which the scattering is produced. The softer radiation appears to be turned back to a somewhat greater extent than the hard. Materials of high atomic weight seem to be able to produce more complete scattering than those of lower atomic weight.

8. The absorption of γ radiation which has already passed through a thickness of one substance by screens of a different substance may not in all cases give a true measure of the absorption of the original radiation which has been effected by the first screens.

In conclusion, it is a pleasant duty for me to acknowledge my indebtedness and sincere thanks to Professor Bragg for his keen interest and advice during the progress of these experiments.

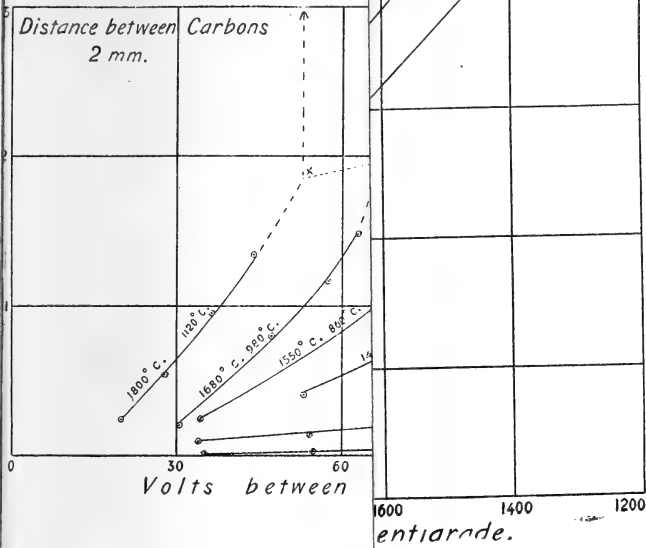
University, Adelaide,
Sept. 1908.

XL. *Notices respecting New Books.*

Annuaire pour l'an 1909 publié par le Bureau des Longitudes.
Paris: Gauthier-Villars. 1 fr. 50.

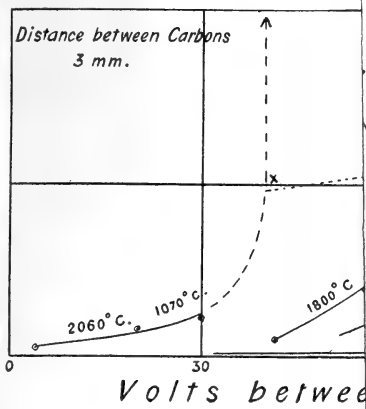
IN accordance with the new plan adopted by the Bureau, of selecting material from different branches of science in alternate years, the present issue contains data respecting Metrology, Money, Geography, Statistics, and Meteorology. The chemical and physical data which appeared in 1908 will be again inserted in 1910. This year we may call attention to the very detailed notices of M. Bigourdan on Variable Stars, and of M. Lallemand on Movements and Deformations of the Earth's Crust.

FIG. 5.



the hot carbon and the critical
in the electrodes necessary to
d 3 millimetres between the

FIG. 6.



2.

a *b* Hot.

ential between the carbons.

FIG. 1.

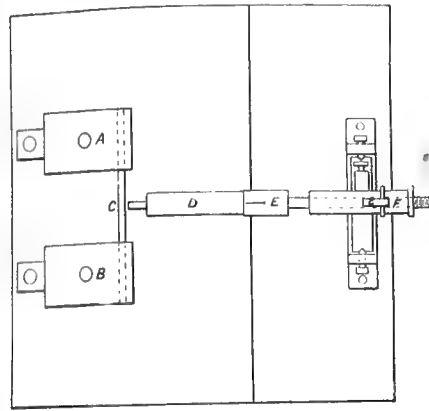
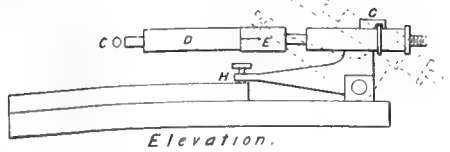


FIG. 2.

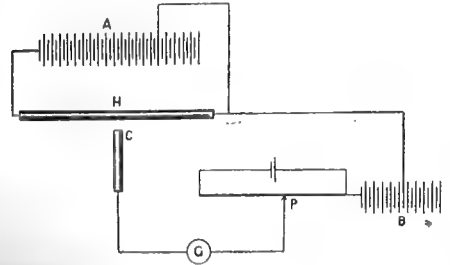


FIG. 3.

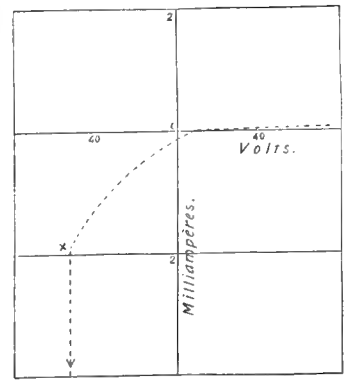


FIG. 4.

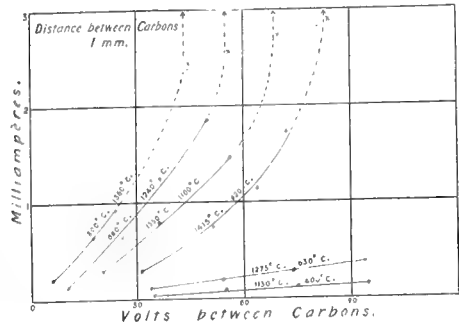


FIG. 5.

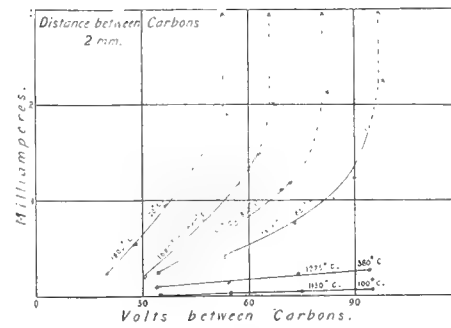


FIG. 6.

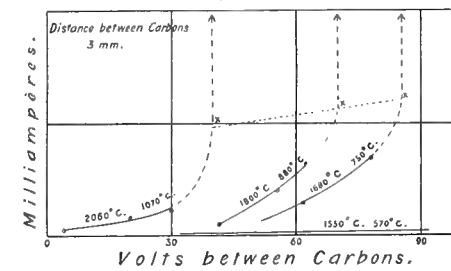
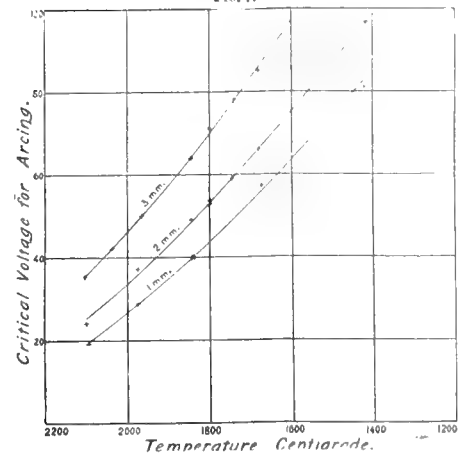
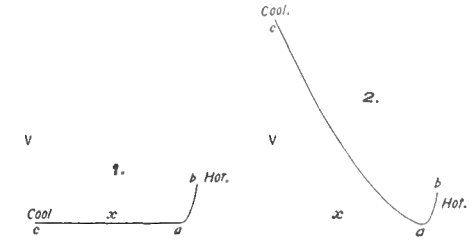


FIG. 7.



The relation between the temperature of the hot carbon and the critical value of the potential-difference between the electrodes necessary to start arcing, for distances of 1, 2, and 3 millimetres between the carbons.

FIG. 8.



Representation of the distribution of potential between the carbons.

FIG. 9.

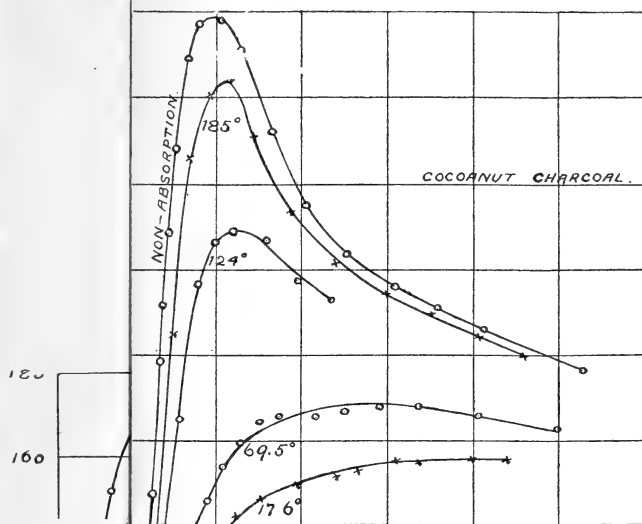


FIG. 5.

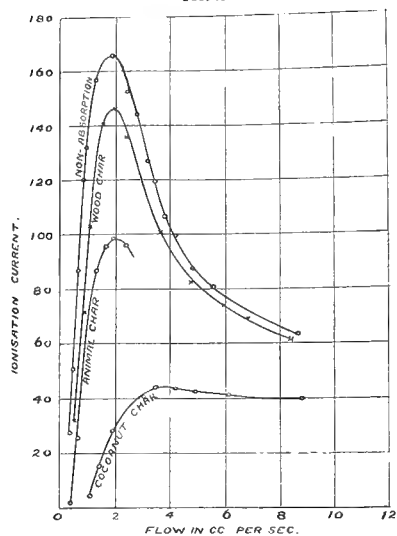


FIG. 6.

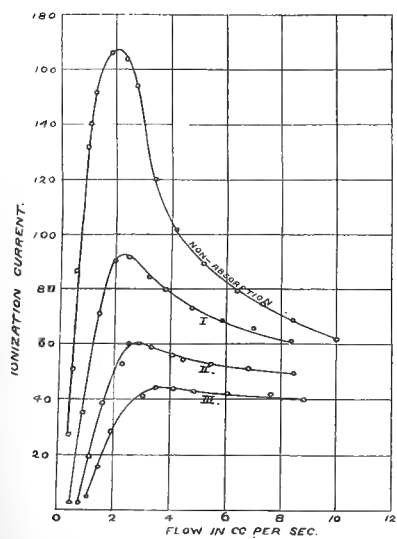


FIG. 7.

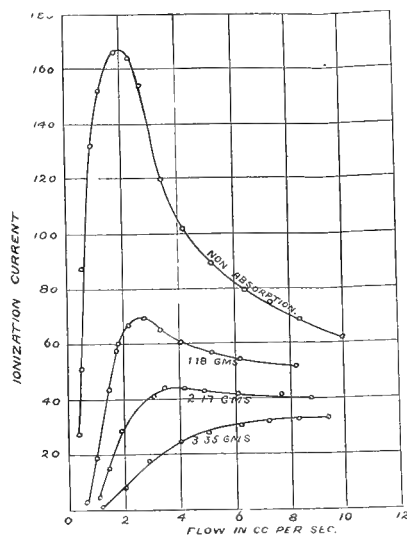


FIG. 9.

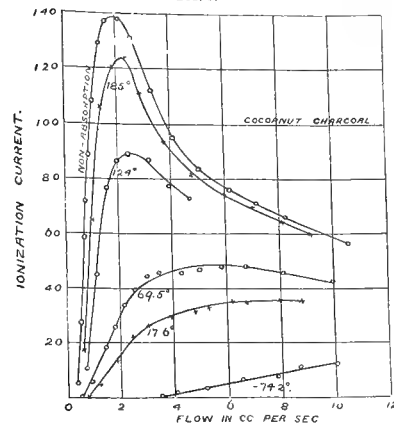


FIG. 10.

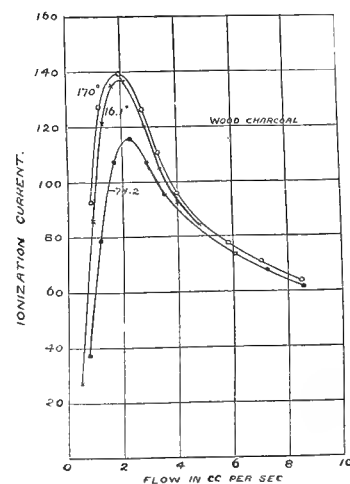


FIG. 6.

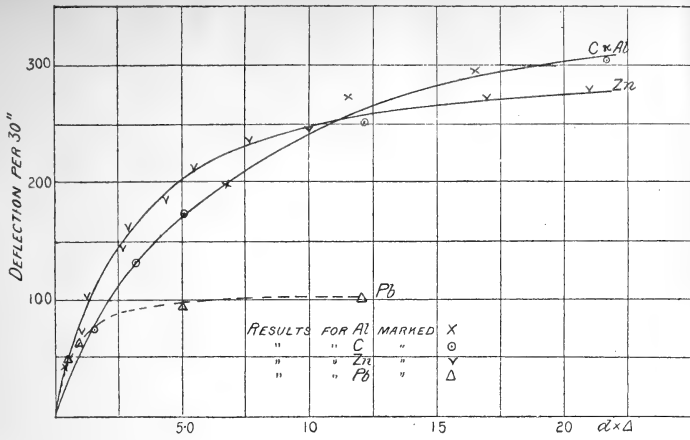
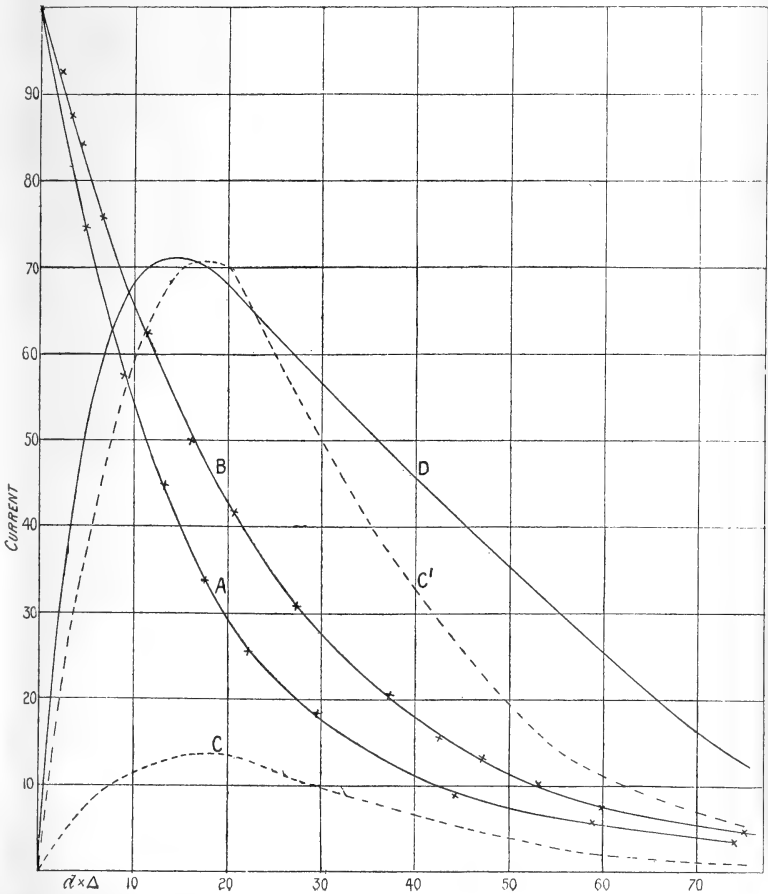
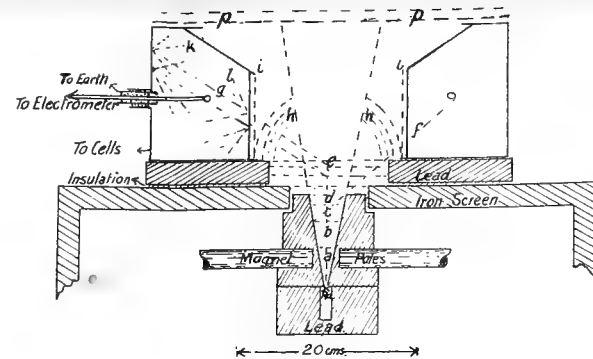


FIG. 9.





The γ radiation proceeds from the Ra through the conical hole. Radiators may be placed in its path as at h, i . The quality of the secondary radiation may be tested by the domes h, i or the cylinders k, l .

When measuring the effects of the secondary incidence radiation the ionization-chamber is inverted and radiators placed as at p, p .

FIG. 2.

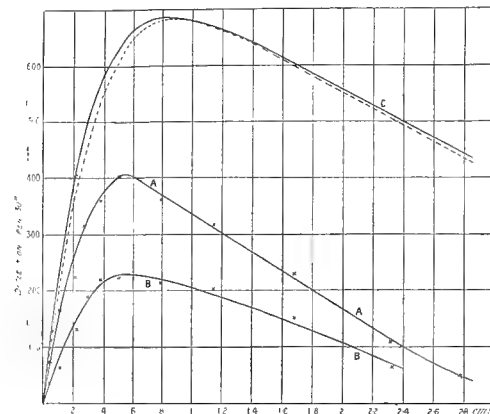


FIG. 3.

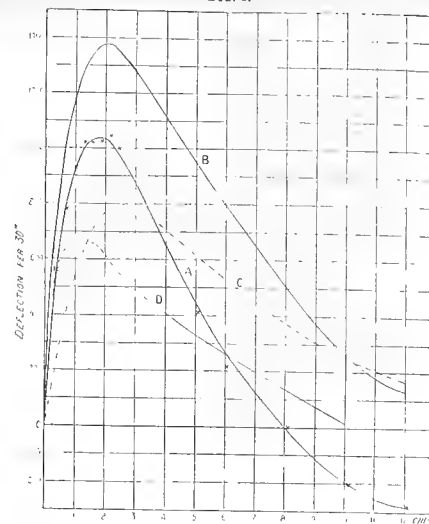


FIG. 4.

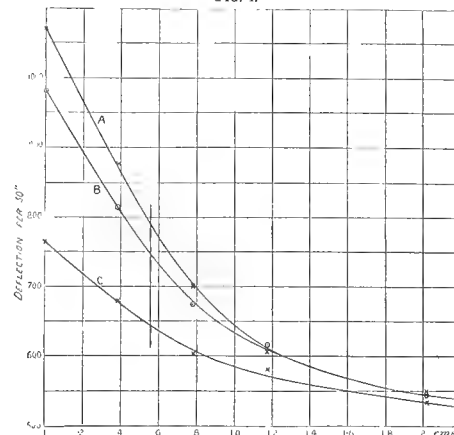


FIG. 5.

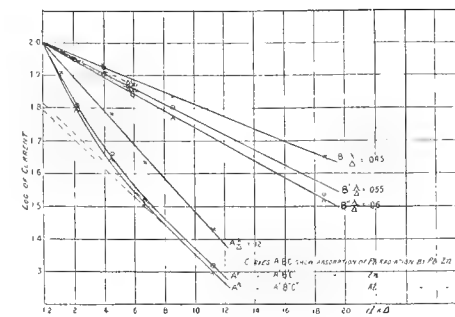


FIG. 8.

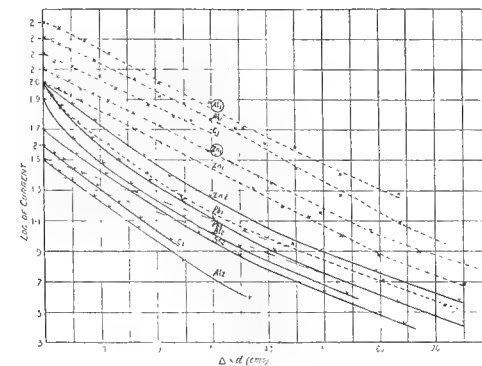


FIG. 6.

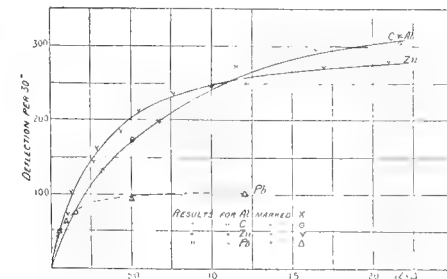
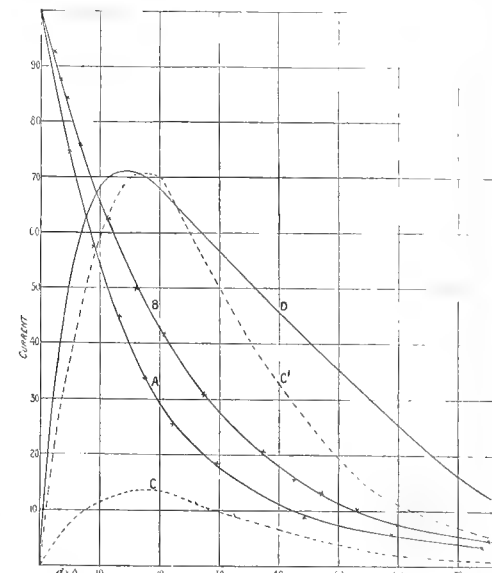
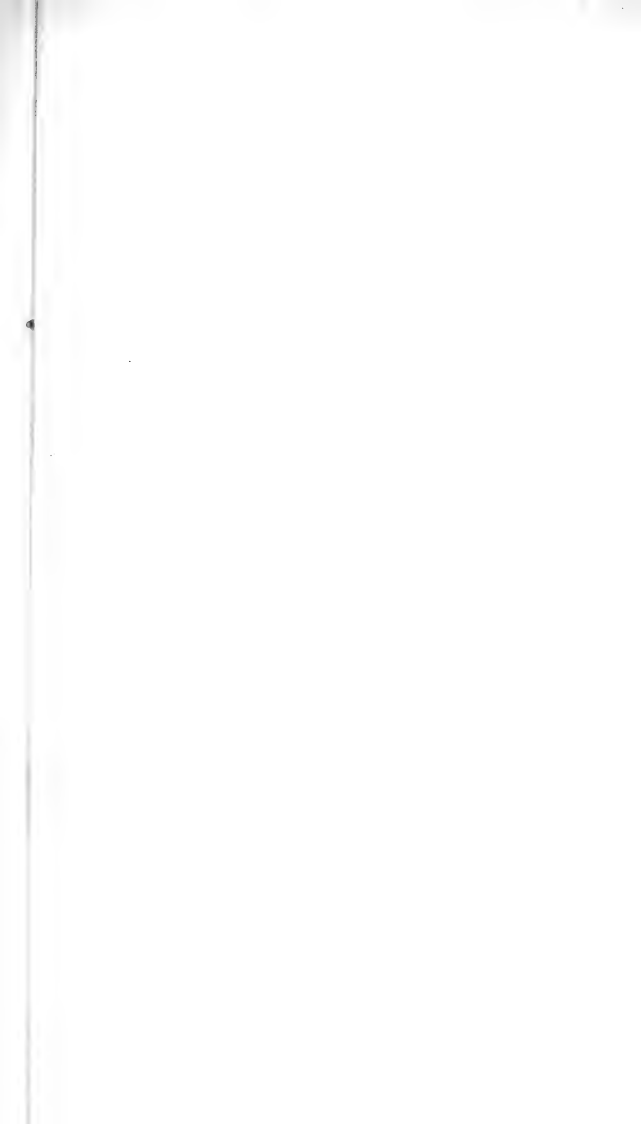


FIG. 9.





THE
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AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

APRIL 1909.

XII. *The Relighting of the Carbon Arc.* By J. A. POLLOCK,
D.Sc., E. M. WELLISCH, M.A., and A. B. B. RANCLAUD,
B.Sc.*

[Plates IX. & X.]

1. *Introductory.*

WHEN the arc between fixed carbons, in a hand-fed lamp, burns itself out, it may be restarted if too great an interval of time is not allowed to elapse, by lessening the distance between the carbon terminals, but without bringing them into contact. Again, if the circuit is broken and reclosed after a short time, the arc may reestablish itself without the carbons being moved.

In connexion with this latter point, Mr. Upson† has given observations of the maximum times of interruption of the circuit within which the arc will restart, for different arc-lengths and for various previous currents, with carbon-carbon, and with copper-carbon arcs in air, and states that in the circumstances of his experiments copper-carbon arcs in coal-gas and in hydrogen did not restart.

The relighting of the arc after a given time of interruption depends, however, not only on the previous current and on the arc-length, but also on the potential-difference established between the electrodes at the moment of reclosing the circuit,

* Communicated by the Authors: read before the Royal Society of New South Wales.

† Upson, *Phil. Mag.* xiv. p. 126 (1907).

and the object of our experiments has been to find the relation between this latter factor and the time interval, for carbon-carbon arcs in air at natural pressure, under various conditions.

The maximum time of interruption of the circuit, under given conditions, within which the arc will reform on re-making the connexions, is astonishingly well-defined, and could in our observations be determined to $\cdot 002$ second. If the interval between the break and the make of the circuit exceeds what may be called the critical time for the given circumstances, after reclosing the circuit a small non-luminous current passes between the carbons; the heating effects associated with this current are not sufficient to maintain the electrodes at their high temperatures, and the current soon dies away as the temperatures diminish. From considerations advanced in a previous paper* with reference to the establishment of the cathode fall of potential which is such a characteristic feature of the developed arc, one is led to think that, on reclosing the circuit, this smaller current always precedes the larger one of the fully formed discharge; the problem of critical relighting is then essentially that of the change from a non-luminous to a luminous current under the circumstances of the experiments.

In the relighting of the arc both carbons are at a high temperature, and the conditions are complicated by the presence, at the moment of reclosing the circuit, of ions at the anode surface as well as near that of the cathode. Simpler conditions are associated with the change of current régime when only the negative carbon is incandescent; this case, involving, previous to the formation of the arc, the flow of negative electricity from a hot to a cool carbon, has been investigated by two of us †, and an explanation reached which seems to account for the phenomena observed.

In the present experiments, it will be seen that the flow of negative electricity at the moment of reclosing the circuit is not always from a hot to a cooler carbon; the conditions of the change from the non-luminous to the luminous discharge are, therefore, in some instances, more complicated than those in the case previously considered, and the explanation of the development of the arc suggested in the paper just mentioned is not sufficient to account for all the features observed in this investigation. Further data are required before a complete description can be given.

* Pollock & Ranclaud, *Phil. Mag.* March 1909.

† Pollock & Ranclaud, *loc. cit.*

The conditions associated with a change from the non-luminous to the luminous discharge, in the case of the ordinary carbon arc, are seen in the wave-forms of current and potential-difference in connexion with alternating-current arc-lamps.

Fig. 1 (Pl. IX.), showing curves of the volts at the brushes of the machine, of current in the circuit, and of the potential-difference between the carbons of the lamp, is copied from fig. 14 of a paper by Mr. Duddell and Professor Marchant on *Experiments on Alternate Current Arcs* by aid of Oscillographs*, in which many other illustrations will be found.

The curves may be described by saying that the potential-difference between the carbons rises from zero, while the current keeps low and non-luminous, until the potential-difference reaches the value, p , necessary to change the state of the current to that of the arc-discharge. The current then rises very rapidly, while the potential-difference falls so that a greater electromotive force may be available along the rest of the circuit, a necessary condition if the increase in the current is to be maintained. On the falling side of the wave, the second maximum of the potential-difference seems to be connected with the gradually diminishing current, rather than with any abrupt change in the nature of the discharge.

The current curve is unsymmetrically placed with reference to the zero points of the potential-difference curve, because on the rising side of the wave the change is from a non-luminous to a luminous discharge, when on account of the smallness of the previous current the temperatures are low, whereas on the falling side the change is in the opposite direction, when, owing to the previous larger current, the temperatures are higher.

2. Experimental Detail.

For all the experiments Conradty carbons, Marke C, were employed; both positive and negative were solid, each 13 millimetres in diameter, the lamp being hand-fed. A heavy pendulum, operating two switches when allowed to swing, opened and again closed the circuit; the distance between the switch levers could be readily altered. The time interval between the opening and the reclosing of the circuit for different lengths between the levers was carefully determined by separate experiments carried out as follows:—The switches

* Duddell & Marchant, *Journ. Inst. Elect. Eng.* xxviii. p. 1 (1899).

were arranged to open and close the circuits of two electromagnetic scribes which marked a smoked plate fixed to the pendulum; the records for various distances between the switch-levers were then compared with that on the same plates of a style attached to the prong of a standardized tuning-fork. A third key, also worked by the pendulum, enabled the battery connexions to be reversed in the interval between the break and the make of the circuit if desired.

A scheme of the connexions is shown in fig. 2 (Pl. IX.), where A is an ammeter, V a voltmeter, R a variable resistance, B the arc, and X and Y the two switches.

An observation consisted in finding, for a given potential-difference between the carbons at the instant of the reclosing of the circuit, the greatest distance between the switch-levers for which the arc would relight. This maximum distance could be determined to within two millimetres, which corresponds to a time interval of about $\cdot 002$ second. The time interval corresponding to the maximum distance may be called the critical time for relighting under the given conditions. This time, under otherwise fixed circumstances, varies considerably with the carbons used, and the results are only directly comparable when they refer to the one pair of carbons.

The observations were made in all cases with "normal" arcs*. The lengths of the arc were measured, on images of the carbons, vertically from the point of the negative to a horizontal line passing through the edge of the crater, the values obtained being reduced according to the magnification of the image.

3. *Relighting with Carbons in Normal Position when Potentials Reversed.*

When the connexions from the battery are reversed during the interval between the opening and closing of the circuit, so that at the remake of the circuit the still existing crater becomes negative to the previous cathode, the crater being on the upper carbon, the phenomena are simpler than in other cases, and will therefore be the first described.

In fig. 3 (Pl. IX.) is shown the relation between the minimum potential-difference for relighting and the interval between the break and the make of the circuit, under the condition of the reversal of the potentials of the carbons, for a previous current of 10 amperes. Three curves are drawn, from observations with the same pair of carbons, for arc-lengths of

* Mrs. Ayrton, 'The Electric Arc,' p. 104.

1.3, 1.9, and 3.1 millimetres respectively, the upper electrode being the positive before the break of the circuit.

In this instance, on reclosing the circuit, the hot is negative to the cooler carbon; the conditions under which the arc is formed are, therefore, nearly allied to those in the simple case previously investigated, see Section 1. The longer the interval between the break and make of the connexions, the lower are the temperatures of the carbons at the moment of reclosing the circuit; taking the fall of temperature of the carbons after the arc is extinguished as nearly proportional to the time, the curves in fig. 3 may be considered as giving, approximately at least, the form of the relation between the critical potential for relighting and the temperature of the hot negative, the temperature of the other electrode being of less importance in this particular case. From this point of view one would expect the curves in fig. 3 to be like those in fig. 7 of the previous paper*, which give the exact form of such a relation under somewhat the same conditions as those under consideration. A comparison shows that the two sets of curves are similar in shape.

Under the conditions of the experiment, the first effect of reclosing the circuit is no doubt a small current of negative ions flowing from the hot to the cooler carbon, and from this non-luminous current the arc may be considered to develop. In the fully formed arc the cathode fall of potential indicates an accumulation of positive ions near the cathode surface. These positive ions, in the growth of the discharge from the non-luminous current, must come in the first instance from the anode. If I is the current of negative ions, E the potential gradient at the anode surface, and λ the length of the last free run of the ions at the end of which they collide with the anode, the energy reaching the anode surface per second is $IE\lambda$; following the suggestion contained in the paper referred to, it is considered that for the arc to form, in the circumstances here contemplated, the potential-difference between the carbons, at the moment of reclosing the circuit, must have the value necessary to make the magnitude of $IE\lambda$ sufficient to raise a portion of the anode surface to such a temperature that positive ions are somewhat freely emitted.

For this particular experiment the suggestion seems sufficient to account for the phenomena observed, but it will be seen that it is not of itself adequate to completely describe the features of the relation in other cases, even where, on reclosing the circuit, the hot is negative to the cooler carbon.

* Pollock & Ranclaud, *loc. cit.*

4. *Effect on the Relighting of Changes in the Relative Positions of the Carbons.*

The form of the relation between the minimum potential-difference for relighting and the time interval between the break and the make of the circuit is influenced by many conditions. With the carbons in a vertical plane they may be situated, before the circuit is broken, either in the normal way, with the crater above, or in the reverse position with the crater below the negative electrode; in addition, the connexions to the battery may be reversed during the interval between the break and the make of the circuit, or left unaltered, so that there are four cases to be considered in connexion with the relative positions of the carbons and the direction of the potential-difference on the remake of the circuit.

In figure 4 (Pl. IX.) are shown the relations between the minimum potential-difference for relighting and the time interval of interruption of the circuit for the four cases mentioned, the current having been 10 amperes when the connexions were broken and the arc-length 1.3 millimetres. The diagrams of the carbons drawn beside each curve indicate by their shape the relative positions of the electrodes before the circuit was opened, while the signs of the potentials, on the reclosing of the circuit, are shown by the usual symbols. All the observations were taken with the same pair of carbons, so the curves are strictly comparable.

Case 4 is the one just discussed in section 3. It is seen that for small time intervals between the break and the make of the circuit, it requires greater potential-differences to restart the arc in the cases 1, 2, and 3 than in that of 4, and that cases 2 and 3 approximate to that of 4 for large intervals of time. Considering that the arc develops from a small non-luminous current of negative ions, an idea which we think must form the basis of any explanation of the critical relighting, in searching for a description of the differences between the curves it has to be noticed that in some cases the negative electrode, on the remake of the circuit, is hotter than the positive, in others the reverse; it is also essential to recognize that in some instances the negative stream of ions is opposed by the convection current of hot gas, while in others it is helped by it, as it has been shown in the previous paper that a change in the relative directions of the stream and current considerably affects the potential-difference necessary for arcing.

The feature of the relations is the evidence, shown by the

curves for cases 2 and 3, of a critical change in the conditions for relighting occurring when the potential-difference attains the value of 90 volts. After reaching this value the minimum potential-difference requisite to start the arc remains for some time practically constant in spite of the fact that for increasing time intervals between the break and the make of the circuit the temperatures of the carbons are diminishing.

The excess of the potential-differences for relighting required in case 1 over those in case 4, for the same time intervals, may be accounted for, perhaps, by the much lower temperature of the negative electrode on the reclosing of the circuit in the former instance, but we have been unable to find, with the data at hand, an explanation of the critical characteristic of the curves for the other cases.

5. Critical characteristic as affected by Arc Length and previous Current.

With a view to finding the influence of arc-length and previous current on the critical characteristic of the curves just mentioned, further observations have been made in connexion with case 3, in which the carbons are in the normal position and the battery connexions remain unaltered during the interval between the break and the make of the circuit. The measurements are given in figures 5, 6, and 7 (Pl. IX.), the currents previous to the break of the circuit being marked on the curves. The observations were thoroughly interlocked with reference to change of carbons, so the curves are comparable.

A comparison of the relations given in figures 5, 6, and 7, shows that the characteristic bend in the curves is more pronounced the higher the previous current. Greater currents mean larger masses of heated carbon with consequent higher temperatures for equal intervals between the break and the make of the circuit.

It is also seen that the greater the arc length, the higher the potential-difference at which the characteristic bend occurs.

6. Photographs of the Relighting.

By arranging an additional lever, in connexion with the pendulum apparatus previously described, to open the shutter of a photographic camera at a short interval after the remaking of the connexions of the arc circuit, photographs showing stages in the development of the arc have been

obtained. The growth of the arc takes place in such a short time that a definite adjustment of the lever was found to be impossible, and the photographs given are only a few of many that have been taken, the remainder showing either no arc, or the arc fully developed.

Figures 1 to 7 in Plate X. show stages in the growth of the arc when the circuit is opened and reclosed without alteration of the battery connexions, while figures 8 to 11 refer to cases where, on the reclosing of the circuit, the still existing crater is negative to the previous cathode. In both instances the glow is seen to develop from the electrode which is positive on the remake of the connexions, a fact which seems to support the view taken in this and the previous paper as to the mode of growth of the arc from the non-luminous discharge. Figure 12 is an example of many of the photographs, showing that, in the case of the reversal of the connexions, the new crater commences on cool rather than on hot carbon.

7. Summary.

In connexion with the relighting of the carbon arc, without movement of the electrodes, when the circuit is opened and reclosed, the relation between the potential-difference, established between the carbons at the moment of the remaking of the connexions, and the maximum time of interruption of the circuit, within which the arc will reform, has been investigated for cases differing as to the relative positions of the carbons before the opening of the circuit, and as to the direction of the potential-difference after the reclose of the connexions. The problem is that of the change from a non-luminous to a luminous discharge in air at normal pressure, of which an explanation, in the case where negative electricity flows from a hot to a cool carbon, has been given in a previous paper. In the present experiments both carbons are at a high temperature, and the conditions of the change are complicated by the presence, at the moment of reclosing the circuit, of ions at the anode surface as well as near that of the cathode. In some of the cases examined the relations show a critical characteristic, but sufficient data are not available to enable an explanation of this result to be given.

We are indebted to Mr. H. L. Watkins, B.A., and Mr. L. A. Cotton, B.A., B.Sc., for help in connexion with the earlier part of the investigation.

The Physical Laboratory,
The University of Sydney,
September 2nd, 1908.

XLII. *On Frictional Electricity.* By MORRIS OWEN, B.Sc.,
Isaac Roberts Student of the University College of North
Wales, Bangor*.

IN a celebrated memoir in the *Annales de Chimie et de Physique* for 1834 †, Péclet describes a long series of experiments in which he measured the intensity of electrification produced by friction under various conditions as to the nature, pressure, and velocity of the rubbing surfaces. The results of the measurements are given as deflexions of a pendulum electrometer connected to a metallic comb placed near one of the rubbed surfaces: they indicate that in each case the surface (if insulating) acquires, after a sufficient amount of friction, a constant surface-density which is independent of the pressure applied during the friction.

Since Péclet's experiments very little attention appears to have been directed to the measurement of the charge produced by the friction of solid bodies, and as it seems desirable to obtain measurements in absolute units the following experiments were arranged with this object and, in particular, to study the electrical charges produced by small amounts of rubbing.

(1) *The Rubbing Apparatus.*

It was thought that the best way of stating the results of these experiments would be to obtain curves showing the charge produced by various amounts of work. The apparatus was therefore arranged so as to enable simultaneous measurements to be made of the work done against friction and the total charge produced on one of the rubbed surfaces.

The rubber consisted of a wheel, of considerable moment of inertia, whose axle turned in fixed bearings and to which a known amount of kinetic energy could be imparted by allowing a weight mg falling through a measured height h to set it into rotation. The weight was attached to a cord which passed over a fixed pulley in the ceiling of the room and was wound round a pulley on the same axle as the wheel. Neglecting the small amount of frictional work at the axles of the pulleys, the kinetic energy communicated to the wheel is then $mgh - \frac{1}{2}mv^2$ or $mh(g-f)$, where f is the acceleration of the falling weight and v the velocity with which it reaches the floor. The acceleration f was small and was measured by observing the time taken by the weight in

* Communicated by Prof. E. Taylor Jones.

† *Ann. de Chimie et de Physique*, lvii. (1834) p. 337.

falling from the ceiling to the floor, a distance of 292 centimetres. The mass of the falling weight was 1035 grammes, and f was 6.45 cms. per sec.²

The rubbed specimens consisted of small disks of ebonite or glass mounted on pieces of sulphur attached firmly to ebonite rods about 5 centimetres long. The ebonite rods were square in section and fitted tightly into a socket in an ebonite plate attached to the lower end of a bent lever. The lever was mounted near the wheel so that the specimen could be made to press horizontally against the rim of the latter, the pressure being produced by placing a known weight in a definite position near the end of the horizontal arm of the lever.

The rub was effected by bringing the specimen evenly into contact with the rim of the wheel at the moment when the falling weight reached the floor. Just before the wheel came to rest the specimen was separated from it so that, neglecting the small quantity of frictional work at the axle of the wheel, the work done against the friction of the specimen was equal to the kinetic energy of the wheel as calculated above.

(2) *Measurement of the Charge on the specimen.*

Immediately after a rub, the ebonite rod supporting the specimen was quickly removed from the socket in the lever and suspended (by a long silk thread always attached to it and passing over a pulley) at a height of about a metre above a tall metal jar into which the specimen could be lowered. The jar was placed inside and insulated by sulphur supports from a larger earth-connected jar which served to shield it from outside influences. The inner jar was connected, by a thin wire passing through a sulphur plug in the wall of the outer jar, to the upper plate of a parallel plate condenser (whose lower plate was earthed) and to a Dolezalek electrometer. The wires leading to the latter passed through cylindrical metal tubes connected to the earth, being supported in the tubes by sulphur plugs at their ends. One terminal of the electrometer was earthed, and by means of a mercury cup key in a block of paraffin wax the other terminal could also be earthed for the purpose of obtaining the zero on the scale. The parallel plate condenser was used for measuring the capacity of the system, but was generally left connected through a mercury key to the jar during the measurements.

The electrometer-needle was suspended by a fine platinum wire of about .01 mm. diameter. It was charged by a small

battery of about 12 storage-cells in some of the experiments, in others by the 200-volt mains. The electrometer was calibrated every day during the course of the experiments by connecting its terminals to various points in a volt box joined up to two storage-cells. The deflexions (observed by a telescope and illuminated scale) due to various fractions of a volt were thus determined. In some cases the electrometer deflexion was proportional to the difference of potential of the quadrants. When this was not so, a calibration curve was drawn and the voltage corresponding to any deflexion read off from this. In measuring the charge on the rubbed specimen the zero of the electrometer was first observed, the earth connexion of the terminal connected to the jar then removed, and the specimen lowered into the jar. The deflexion produced depends on the charge on the specimen, and on the capacity of the system consisting of the outer surface of the inner jar, the electrometer and its connecting wires, and also the parallel plate condenser if this was connected. The capacity of the parallel plate condenser, whose plates were circular, was calculated from the formula

$$C_1 = r^2/4d + r/4\pi \left\{ \log \frac{16\pi r(d+\delta)}{d^2} - 1 + \frac{\delta}{d} \log \frac{d+\delta}{\delta} \right\}^*$$

where r = radius of each plate,

d = distance between the plates,

δ = thickness of each plate.

The capacity C_2 of the rest of the system was measured by communicating to it a charge and observing the deflexions before and after this charge was shared with the condenser.

If the rubbed specimen, on being placed inside the jar, produces a deflexion which indicates a potential V as read off from the calibration curve, then the charge on the specimen is $(C_1 + C_2) V$.

The insulation of all parts of the apparatus was frequently tested, and it was also verified that the specimen lost no appreciable charge during the process of removing it from the lever to the jar. It was in fact possible to take the specimen from the jar, fix it in position on the lever, and then bring it back to the jar, without any appreciable change of deflexion.

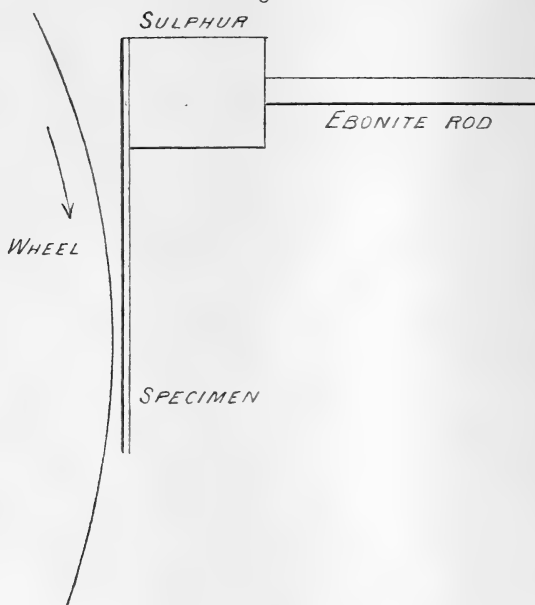
Before each rub the specimen was completely discharged by radiating to its surface for a few seconds with radium.

* Kohlrausch, *Leitfad. d. prakt. Phys.* 8 Aufl. p. 409.

(3) Experimental Difficulties.

One of the chief difficulties in these experiments was that of maintaining a uniform pressure between the specimen and the wheel during the rubbing. If the ebonite rod supporting the specimen did not fit very tightly into the socket in the lever, or if the contact with the wheel was too near the front edge of the specimen, the rubbing became irregular, probably owing to the sliding motion breaking into a series of impacts. This occurred especially with the higher velocities of the wheel. It was invariably accompanied by a jarring sound during the rubbing, and followed by an abnormally small deflexion of the electrometer. Several kinds of mountings for the specimens were tried in order to overcome this difficulty; the most satisfactory, especially for the lower velocities, was one in which the specimen consisted of a rectangular plate, about 4 cms. long, attached at its upper end to the

Fig. 1.



sulphur support from which it projected downwards, the wheel rubbing downwards near the lower end of the specimen (see fig. 1). The only difficulty experienced with this arrangement was that the specimen was apt to become detached from the sulphur support under the greatest pressure when

the wheel was turning rapidly. In the results given below only the results of good rubs, as indicated by a uniform sliding sound, are included.

It was also found very important to ensure that there was even contact over the whole width of the specimen; if the wheel touched it only at one side a small deflexion would result, owing to the small area rubbed.

Another precaution which it was found necessary to take, was to give the specimen a few hours' rest after each rub. If the specimen was rubbed a number of times in succession at short intervals, the work in each case being the same, and less than the amount required to give the maximum deflexion at the first rub, the deflexion invariably increased with the successive rubs up to a maximum value, though the specimen was completely discharged by radium before each rub. This maximum was the same as could be obtained at the first rub with a sufficient amount of work. After an interval of three or four hours, the specimen appeared to have come back to its original state and the same series of increasing deflexions could be obtained with successive rubs. In the results given below the charges produced correspond to the first rub after an interval of three or four hours' rest.

The results were much influenced by the state of the atmosphere; no deflexion was sometimes obtained during wet weather.

(4) *Results.*

In the experiments the results of which are here given the wheel was of slate and its axle was earthed. In some of the experiments the rim of the wheel was covered with a band of copper. The specimens rubbed were of ebonite and glass. Curves were obtained for ten specimens of ebonite of the same size cut from the same sheet, the friction taking place under three different pressures. Seven of the specimens gave very concordant results; the other three gave irregular readings, probably owing to unsatisfactory mounting. Ten specimens of glass were also used with equally concordant results.

In the following tables the first column shows the frictional work in millions of ergs, the second, third, and fourth columns the quantities of electricity Q_1 , Q_2 , Q_3 , in electrostatic units, produced on the specimen of ebonite or glass when the pressure between the wheel and the specimen was 306·8, 1326·8, and 2579·2 grammes weight respectively.

The method was not suitable for determining the surface-density of the charge produced on the specimen, owing to

the difficulty of measuring with any accuracy the area rubbed. The width of the specimen is, however, given in each case.

Readings could not be obtained for glass with the higher velocities and the greatest weight; the specimen always broke when brought into contact with the wheel under these circumstances.

TABLE I.

Ebonite and Slate

(negative electricity produced on ebonite).

Width of specimen = 2.15 cms.

$W \times 10^{-6}$.	Q_1 .	Q_2 .	Q_3 .
4.035	.59	1.76	1.99
8.069	1.04	2.02	2.14
14.121	1.27	2.08	2.38
24.208	1.70	2.24	2.43
40.346	2.11	2.40	2.57
53.459	2.25	2.41	2.58
75.649	2.39	2.54	2.59
99.857	2.47	2.51	2.56
125.074	2.50	2.58	2.55
156.342	2.58	2.57	2.59
202.740	2.53	2.53	2.56
235.018	2.57	2.56	2.52
294.528	2.58	2.55	2.57

TABLE II.

Ebonite and Copper

(positive electricity produced on ebonite).

Width of specimen = 1.75 cms.

$W \times 10^{-6}$.	Q_1 .	Q_2 .	Q_3 .
4.035	.021	.094	.15
8.069	.066	.13	.23
14.121	.19	.26	.37
24.208	.28	.59	.76
40.346	.48	.73	1.07
53.459	.69	1.06	1.16
75.649	.89	.93	1.46
99.857	1.11	1.20	1.44
125.074	1.09	1.54	
156.342	1.40	1.52	
202.740	1.46	1.63	
235.018	1.61	1.65	
294.528	1.58	1.61	

TABLE III.

Glass and Copper
 (positive electricity produced on glass).
 Width of specimen = 1.55 cms.

$W \times 10^{-6}$.	Q_1 .	Q_2 .	Q_3 .
4.035	.002	.026	.047
8.069	.021	.028	.096
14.121	.066	.099	.14
24.208	.076	.14	.26
40.346	.15	.20	.32
53.459	.24	.37	.49
75.649	.35	.42	.51
99.857	.43	.43	.53
125.074	.43	.51	
156.342	.47	.49	
202.740	.46	.52	
235.018	.51	.54	
294.528	.54	.55	

TABLE IV.

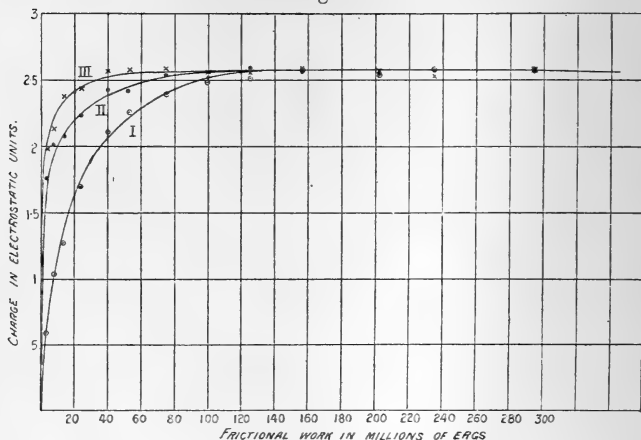
Glass and Slate
 (positive electricity produced on glass).
 Width of specimen = 1.85 cms.

$W \times 10^{-6}$.	Q_1 .	Q_2 .	Q_3 .
4.035	.071	.14	.29
8.069	.12	.33	.43
14.121	.21	.43	.64
24.208	.29	.68	.81
40.346	.52	.77	.93
53.459	.62	.87	.90
75.649	.70	.85	.88
99.857	.83	.95	
125.074	.94	.93	
156.342	.85	.99	
202.740	.85	.93	
235.018	.86	.90	
294.528	.98	.99	

Some of the results are also shown in the curves, figs. 2 and 3 (p. 464).

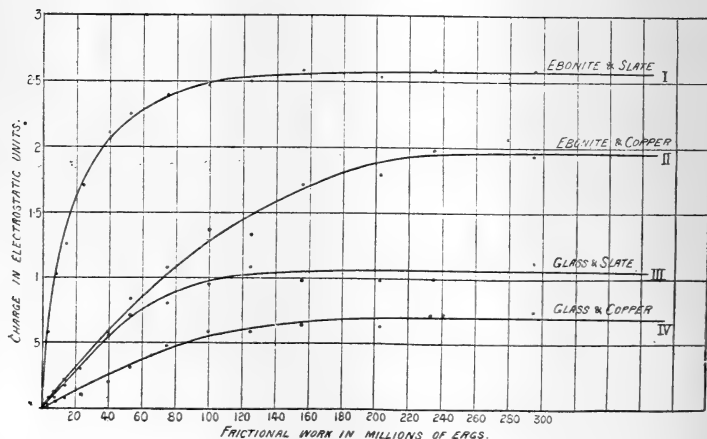
In fig. 2, curves I., II., III. represent the charges produced on the same specimen of ebonite by friction with slate under the smallest, middle, and greatest pressures respectively.

Fig. 2.



Measurements made with specimens of different widths showed that the charge produced by a given amount of frictional work was almost exactly proportional to the width of the specimen, except in the case of very small amounts of work.

Fig. 3.



In fig. 3 the readings obtained with the various glass and

ebonite specimens have all been reduced so as to give readings corresponding to a specimen of 2.15 cms. in width. Curves I., II., III., IV. show the quantities of electricity produced on specimens of equal width in the friction of ebonite and slate, ebonite and copper, glass and slate, glass and copper. The same pressure was applied between the rubbers in each case.

The curves show that with a sufficient amount of frictional work the charge produced reaches a constant maximum value ; that this maximum is independent of the pressure applied during the rubbing, but that the maximum is reached with a smaller quantity of work the greater the pressure.

The view generally held as to frictional electricity is that due to Helmholtz *, viz., that it is of the nature of contact electricity, the frictional work being expended in bringing the surfaces into closer contact. This view appears to be borne out by the observation noted above, that after a number of recent rubs the maximum deflexion may be obtained with quite a small amount of frictional work, an amount much less than that required to give the maximum charge with the first rub. During the three or four hours' rest it might be supposed that the surface of the specimen goes through a process of slow elastic recovery of its original uneven form, or that during this time the surface becomes tarnished by the atmosphere, so that after this period the first contact is not a good one.

It should be noted, however, that in no case, whether after recent rubs or not, did mere contact of wheel and specimen without rubbing produce even the slightest evidence of charge on the specimen.

All the experiments described above were carried out in the Physical Laboratory of the University College of North Wales ; and, in conclusion, I desire to acknowledge my great obligation to Professor E. Taylor Jones for the interest he has taken in the work, and also for much valuable help and advice.

* *Wissenschaftliche Abhandlungen*, Erster Band, p. 860.

XLIII. *An Experimental Investigation of Gibbs's Theory of Surface-Concentration, regarded as the basis of Adsorption.*
(Second Paper.) By WM. C. McC. LEWIS, M.A.*

[From the Muspratt Laboratory of Physical and Electro-chemistry, University of Liverpool.]

IN a previous paper† upon this subject a number of determinations were recorded, which had been carried out with a view to investigate experimentally the expression deduced by Gibbs for the surface condensation of one of the components of a two-phase system at the boundary separating the phases. Two immiscible liquids—namely a hydrocarbon oil and distilled water—were brought into contact, there being a definite interfacial tension existing at the boundary of separation. To the water there were added various substances soluble in water, but insoluble in oil, viz. sodium glycocholate, methyl-orange, and Congo-red. These substances all produced a lowering effect on the interfacial tension—such lowering effect increasing with corresponding increase in concentration of the solute.

Gibbs's expression connecting the surface-concentrating of the solute with the lowering effect on the tension and the concentration of the solute in the bulk of the (aqueous) phase is

$$\Gamma = - \frac{c}{RT} \frac{d\sigma}{dc}$$

where Γ = the mass (in grams) of the solute per square centimetre of the dividing surface in excess of what would be there supposing the concentration to remain the same as in the bulk of the solution ;

c = the bulk concentration of the solute in the aqueous phase ;

R = the gas constant ;

T = the absolute temperature ;

σ = the interfacial tension ; and

$\frac{d\sigma}{dc}$ = the rate of change of the tension with the concentration of the solute.

The negative sign on the right-hand side indicates that $\frac{d\sigma}{dc}$ must be negative in order that Γ may be positive, *i. e.* those substances which lower the interfacial tension will increase in concentration at the interface.

* Communicated by the Physical Society : read November 27, 1908.

† Lewis, Phil. Mag. April 1908.

It was shown how this expression might be applied to the phenomenon of adsorption, which is considered to be more or less a surface condensation. The determinations referred to consisted in measurements of the various quantities occurring in the above expression, and in comparing the values obtained on the two sides of the equation. The general results showed that the actually determined values for Γ far exceeded the calculated values—those found being of the order 10^{-6} grms./cm.², while the calculated ranged from 10^{-7} to 10^{-8} grms./cm.² This discrepancy was observed in the case of all the substances mentioned in the previous paper. The adsorption coefficient (Γ) was measured experimentally under two quite different conditions—namely, at an approximately *plane* surface; and secondly at a very curved surface, with concordant results in both cases.

With the view of throwing some light on the cause of this discrepancy, further determinations of a similar character were carried out and are recorded in the present paper.

It may be pointed out that the substances mentioned are electrolytes (sodium salts of organic acids) and would therefore suffer considerable dissociation in aqueous solution. To account for the discrepancy referred to one possibility suggested itself, namely, the presence of some kind of “double layer” electrical effect. Of course this may or may not be the cause, and it will be shown that for these bodies this is probably *not* the cause, but evidently the first step was to examine other electrolytes—ordinary inorganic salts of simple constitution and comparatively low molecular weight, and compare the effects obtained in these cases with that obtained for sodium glycocholate and the dye-stuffs.

Finally, to carry out adsorption determinations if possible with typical non-electrolytes and compare the results obtained in this case also.

THE INTERFACIAL TENSION BETWEEN OIL AND WATER MEASURED BY THE DROP-PIPETTE.

The pipette was identical with that shown in fig. 1 (previous paper). An expression giving the tension directly in absolute units may be derived as follows:—

Consider the equilibrium of a drop of oil formed in water and just about to break away from the nozzle of the pipette.

Let σ = the interfacial tension;

a = the radius of the orifice;

r = the mean radius of the drop;

ρ_w = the density of water;

ρ_o = the density of oil.

The forces tending to draw the drop off are (1) gravitational, due to the difference in densities of the oil and water

$$= \frac{4}{3} \pi r^3 (\rho_w - \rho_0),$$

and (2) the difference of pressure due to curvature,

$$= 2\pi a^2 \frac{\sigma}{r}.$$

The opposing force is due to the tension and $= 2\pi a\sigma$.

Hence

$$2\pi a\sigma - 2\pi a^2 \frac{\sigma}{r} = \frac{4}{3} \pi r^3 (\rho_w - \rho_0).$$

Applying this to the particular case of the oil in which

$$\begin{aligned} \rho_w &= 1.0, \\ \rho_0 &= 0.899, \\ a &= 0.175 \text{ cm.}, \\ r &= 0.4125 \text{ cm.}, \end{aligned}$$

we obtain $\sigma = 0.0467$ gram per cm.

or 45.8 dynes per cm.

Employing the empiric formula of Lohnstein* and Kohlrausch† the value 40 dynes per cm. is obtained.

The above expression had been already obtained when it was found that an almost identical one had been deduced by G. Guglielmo‡ for liquid-air surfaces. He has applied it to the case of the water-air tension with the following results :—

Radius of orifice of pipette.	Tension in milligrams/mm.
0.8 mm.	7.78
1.1 „	7.60
2.15 „	7.69
} mean 7.69 at 25° C.	

The value obtained by the capillary-tube method is 7.485 at 25°C . The results are therefore in fair agreement.

As in previous determinations, the drop-pipette was employed for the various cases of adsorption dealt with in the present paper, the only difference being that the absolute values for the tension are here calculated by means of the expression deduced above.

* Lohnstein. See previous paper.

† Kohlrausch. See previous paper.

‡ G. Guglielmo, *Accad. Lincei, Atti*, xv. p. 287 (1906).

THE INTERFACIAL TENSION BETWEEN A HYDROCARBON OIL
AND AQUEOUS SOLUTION OF ELECTROLYTE.

The oil employed was similar to that previously experimented with. About twenty electrolytes in all were examined and all showed a *lowering* effect on the oil-water tension—in general monovalent ions of small atomic weight having least effect while copper and barium salts had a marked influence. This lowering of the tension is remarkable in view of the fact that inorganic salts (with the exception of lithium) *raise* the air-water tension*. Further, it is a general property of salts to be adsorbed by charcoal; and if this is regarded as indicative of a lowering of tension at the charcoal surface, one may be perhaps justified in saying that for aqueous solutions of an electrolyte at a liquid or solid† interface, one may expect in all cases a lowering of tension accompanied by adsorption, while on the other hand at the solution-air surface no such generalization can be made.

A certain number of these salts were selected for adsorption determinations—in the case of caustic soda the emulsion method‡ being employed; while with the rest the large-dropping apparatus was used—this being rendered necessary as the adsorption determinations were carried out for each *ion* of the salt separately the, concentration changes being estimated by ordinary volumetric and gravimetric means.

The Adsorption of Caustic Soda on Hydrocarbon Oil.

The values of the interfacial tension for different concentrations of the base (corrected for density) are summarized in the following table.

TABLE I.

Concentration		Interfacial Tension in dynes/cm.
per cent.	in gram-moles/litre.	
0	0	45.81
1.0	0.25	26.93
2.0	0.5	22.49
4.0	1.0	16.76
8.0	2.0	streaming from orifice of pipette — tension too small to be measured.

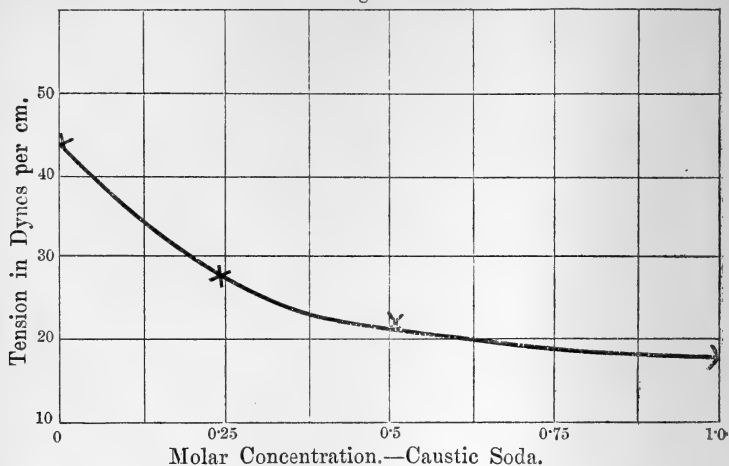
* Whatmough, *Zeitschr. Phys. Chem.* vol. xxxix. p. 129 (1901).

† Using the word in its ordinary sense.

‡ See former paper.

The values of the tension and concentration are plotted in fig. 1.

Fig. 1.



Calculation of Γ by means of Gibbs's expression :—

Take as a particular case the concentration 0.3 per cent.

$$c = 0.003 \text{ gram./cc.}$$

$$R = \frac{2 \times 4.2 \times 10^7 \text{ ergs}}{\text{molecular weight}} = \frac{2 \times 4.2 \times 10^7}{20},$$

assuming complete dissociation
at this dilution.

$$T = 288^\circ \text{ abs.}$$

$\frac{d\sigma}{dc}$, i. e. the tangent to the curve (fig. 1) at the concentration 0.3 per cent.,

$$= - \frac{9 \text{ dynes/cm.}}{0.003 \text{ gram./c.c.}} = 3000.$$

Hence

$$- \frac{c}{RT} \frac{d\sigma}{dc} = \frac{7.5 \times 10^{-9} \text{ grams per cm.}^2}{\text{or } 1.9 \times 10^{-10} \text{ gram-moles per cm.}^2}$$

It has been assumed here in using the term gram-moles that the substance has been adsorbed either as undissociated base or in ionically equivalent proportion. Gibbs's expression for surface concentration does not contain any term allowing for the preferential adsorption of a single ion if any should occur. Also the measurement of concentration changes by means of changes in tension embodies the same assumption.

Experimental Determination of Γ :—

500 c.c. of caustic-soda solution (concentration 0.3 per cent.) were shaken with 1.075 c.c. of the oil until a uniform emulsion resulted. After emulsification a tension determination showed that there had been a decrease in concentration of 0.003 per cent. The diameter of the emulsion particles as measured in the microscope was on the average 3×10^{-5} cm., so that the total adsorbing surface was 107500 cm.²

Hence

$$\begin{aligned}\Gamma &= 1.5 \times 10^{-7} \text{ gram./cm.}^2 \text{ or } 3.7 \times 10^{-9} \text{ gram-moles/cm.}^2 \\ \Gamma \text{ (calculated)} &= \frac{7.5 \times 10^{-9}}{1.9 \times 10^{-10}} \text{ gram-moles/cm.}^2\end{aligned}$$

The probable error in the experimental result may amount to 25 per cent. A comparison of the observed with calculated values shows once more the considerable discrepancy which exists between them. It should be observed, however, that the absolute value of the experimental determination has fallen from the order 10^{-6} (obtained in previous cases) to 10^{-7} gram./cm.² It should also be mentioned that if any slight trace of acid were present in the oil the adsorption of a base would be considerably altered.

Re-determination of the Adsorption of Sodium Glycocholate.

Having obtained a purer specimen of the material than that used in former cases, the values for the tension between the oil and solutions of different concentrations were again determined. There were only small changes noted in the result—the calculated value for the adsorption (at a concentration 0.25 per cent.) being 7×10^{-8} gram./cm.², while in the former case the value was about 5×10^{-8} . The actual value determined by experiment was from 3.5 to 4.7×10^{-6} gram./cm.² while in the previous case it had lain between the limits 3.1 and 5.4 gram./cm.²

The Adsorption of Silver Nitrate.

In the measurements of the adsorption of this substance (and in all subsequent cases) the "large drop" * apparatus was employed. One litre of $\frac{1}{50}$ molar silver nitrate was prepared and about 750 c.c. poured into the tube through which the oil passed in the form of drops (of about 0.3 cm. diameter), and on the surface of these the solute was adsorbed. When the oil-reservoir (of 1 litre capacity) had

* See previous paper.

been exhausted and the total number of drops estimated from the time taken in emptying the reservoir, the solution which had been thus treated was withdrawn and analysed. For this particular substance the silver only was estimated before and after the experiment by precipitation with hydrochloric acid, the Gooch crucible method being employed. An extremely small difference in weight was observed, namely 1.6 milligrams of silver chloride. The total surface area of the oil was 7000 cm.²

Hence the adsorption of the silver is

$$\Gamma_{Ag} = \underline{1.7 \times 10^{-8} \text{ grm./cm.}^2}$$

A repetition of the determination gave

$$\Gamma_{Ag} = 3.4 \times 10^{-8} \text{ grm./cm.}^2$$

The mean value is therefore

$$\Gamma_{Ag} = \underline{2.5 \times 10^{-8} \text{ grm./cm.}^2}$$

It may be mentioned that the measuring vessels used in these and subsequent determinations were recalibrated for 15° C., and all solutions were cooled to this temperature in a bath before measuring out the volume.

It will be noted that the value obtained above is much smaller than for the previously mentioned substances (except caustic soda). One might therefore expect a closer approximation to the value calculated on Gibbs's expression.

Calculation of the Adsorption of Silver Nitrate.

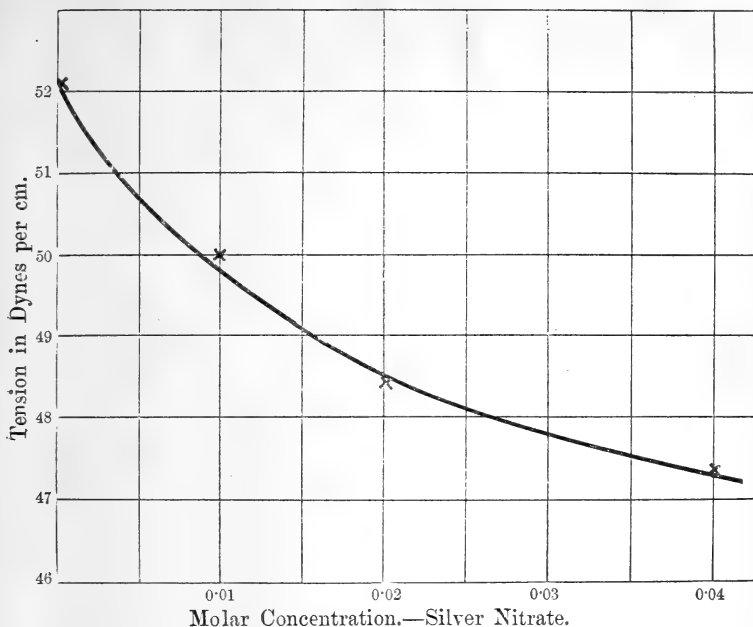
As in previous cases the oil-solution tension curve was obtained by means of the drop-pipette. The oil employed for these measurements was not quite identical with that used with caustic soda. The values obtained (corrected for density) are given in the following table.

TABLE II.
Oil—Silver-Nitrate Solutions.

Concentration. in gram-moles per litre.	Pipette Drop-number.	Tension dynes/cm.
0	135	52
0.01	140	50
0.02	145	48.4
0.04	149	47.3

The values of the concentration are plotted against tension in fig. 2.

Fig. 2.



The calculated adsorption, namely

$$\frac{c}{RT} \frac{d\sigma}{dc},$$

has the following value for a concentration $\frac{1}{30}$ molar :—

$$c = 0.0034 \text{ gram. per c.c.}$$

$$R = \frac{8.4 \times 10^7 \text{ ergs}}{\text{molecular weight}}.$$

The molecular weight at this dilution has been found by Smits* to be 94. $T = 289^\circ \text{ abs.}$

$\frac{d\sigma}{dc}$ is the tangent to the curve at $\frac{1}{30}$ molar concentration and is equal to

$$\frac{1.9 \text{ dyne/cm.}}{0.0034 \text{ gram./c.c.}} = 556.$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = 7.3 \times 10^{-9} \text{ gram per cm.}^2$$

This refers to the salt AgNO_3 , so that on the assumption

* Smits, *Zeitschr. Phys. Chem.* vol. xxxix. p. 418 (1902).

that each ion is adsorbed equivalently we obtain the calculated value for the silver ion to be

$$\Gamma_{\text{Ag}} = 4.5 \times 10^{-9} \text{ gm./cm.}^2$$

$$\Gamma_{\text{Ag}} (\text{found}) = 2.5 \times 10^{-8} \text{ gm./cm.}^2$$

The values for the experimental and calculated adsorption are thus in much closer agreement than in any previous case—the experimental being five times the calculated. The question is : Does this represent a real discrepancy even in this case, or is it to be considered as agreement within the limits of experiment? It is difficult to say, but without laying too great stress upon it, I am of opinion that there is a real (though small) discrepancy between calculated and observed values.

The Adsorption of Potassium Chloride.

Exactly similar determinations were carried out with this substance as in the case of silver nitrate, except that with this salt both the metal and acid were estimated.

The Adsorption of the Potassium of Potassium Chloride.—A very large quantity of oil was allowed to pass through the KCl solution in the large-drop apparatus. The resulting oil-surface area was 16,000 cm.² The quantity of potassium present in the solution before and after adsorption was estimated as potassium sulphate. The difference in weight of the K₂SO₄ precipitate was 3.2 milligrams, whence the adsorption of the potassium is

$$\Gamma_{\text{K}} = 8 \times 10^{-8} \text{ gm./cm.}^2$$

A repetition of the experiment gave

$$\Gamma_{\text{K}} = 2 \times 10^{-8} \text{ gm./cm.}^2$$

Thus the mean value is

$$\Gamma_{\text{K}} = 5 \times 10^{-8} \text{ gm./cm.}^2$$

Calculation of Γ_{K} on Gibbs's formula :—

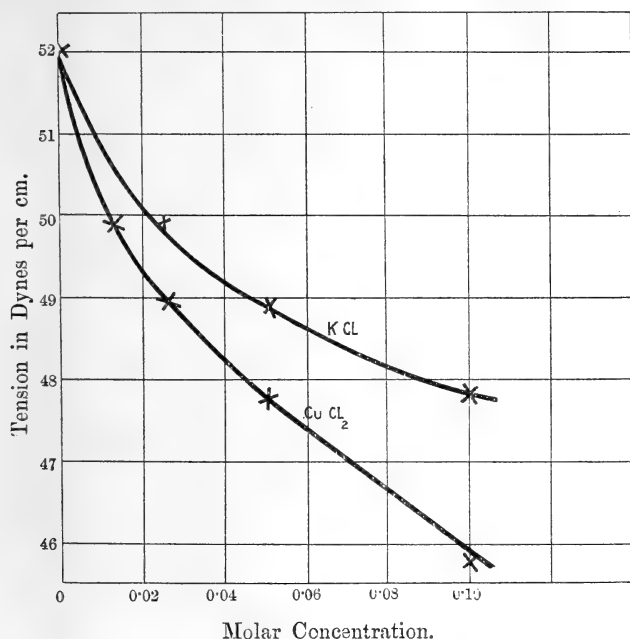
The values of the tension corresponding to different concentration (corrected for density) are given as follows :—

TABLE III.—Hydrocarbon Oil—Potassium Chloride.

Concentration in gram-moles per litre.	Pipette Drop-number.	Tension dynes/cm.
0	135	52
0.0125	139	50.4
0.025	140	49.9
0.05	143	48.9
0.1	147	47.8

The tension-concentration curve is given in fig. 3.

Fig. 3.



As before, the calculated adsorption for $\frac{M}{20}$ KCl is

$$c = 0.0037 \text{ gram./c.c.}$$

$$R = \frac{8.4 \times 10^7 \text{ ergs.}}{37}$$

where 37 is the osmotic molecular weight.

$$T = 289^\circ \text{ abs.}$$

$$\frac{d\sigma}{dc} = 478.$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = \underline{2.3 \times 10^{-9}} \text{ gram KCl per cm.}^2 \text{ surface,}$$

which corresponds to $\underline{1.7 \times 10^{-9}}$ gram. of potassium,

while Γ_K (found) is $\underline{5 \times 10^{-8} \text{ gram./cm.}^2}$

It will be noted that potassium shows a greater discrepancy than silver as regards the calculated value. The results are of about the same order as that obtained for caustic soda.

Estimation of the Adsorbed Chlorine :—

The surface area of the oil was 10,000 cm.² The change in the chlorine concentration was determined as silver chloride, the difference in weight of AgCl precipitate due to adsorbed chlorine being 0.35 milligram. Hence

$$\Gamma_{\text{Cl}} = 8 \times 10^{-9} \text{ gram./cm.}^2$$

Repetition gave $\Gamma_{\text{Cl}} = 3 \times 10^{-9} \text{ gram./cm.}^2$

These quantities are, however, so small that one is only justified in stating that the adsorption of the chlorine is of the order $10^{-9} \text{ gram./cm.}^2$

The calculated value is $1.6 \times 10^{-9} \text{ gram./cm.}^2$, so that the agreement between calculated and observed is fairly good.

These results for potassium chloride appear to point (at least as far as the small changes in concentration observed can be trusted) to the possibility of a certain amount of separation of the ions in the process of adsorption, *i. e.* a preferential cation adsorption. If this is so, it at once suggests some electrical effect taking place which is not taken account of in Gibbs's capillary theory.

The Adsorption of Barium Chloride.

In the case of this substance the chlorine only was estimated, this being carried out volumetrically with AgNO₃ solution. The surface area of the oil was 4000 cm.². In the titration the change in concentration of chlorine due to adsorption was less than 0.1 c.c. AgNO₃ solution. From this one can only say, therefore, that the adsorption Γ_{Cl} is not of greater magnitude than $10^{-8} \text{ gram./cm.}^2$

Calculation of the Adsorption :—

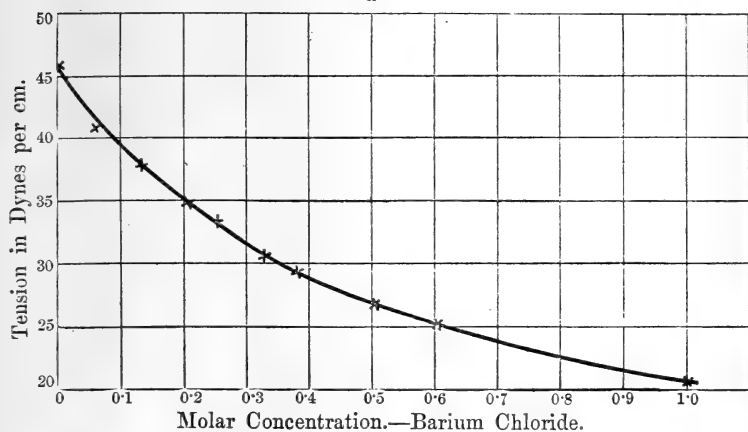
The following table (IV.) contains the values obtained for the tension (corrected for density) at different concentrations.

The values of the tension are plotted against concentration in fig. 4. By taking the tangent to this curve the value of $\frac{d\sigma}{dc}$ is obtained for a particular concentration.

TABLE IV.

Concentration of the Barium Chloride		Density of the Solutions.	Drop-number.	Relative Tension.	Tension in dynes/cm.
Gram-moles per litre.	Per cent. anhydrous.				
0	0	1.000	153	1.000	45.81
0.062	1.29	1.009	171	0.895	40.99
0.125	2.59	1.019	188	0.829	37.98
0.2	4.14	1.038	210	0.757	34.68
0.25	5.18	1.048	222	0.722	33.07
0.33	6.90	1.066	246	0.663	30.37
0.375	7.76	1.073	255	0.644	29.50
0.5	10.35	1.099	290	0.580	26.57
1.0	20.70	1.216	419	0.444	20.34
(1.7) satd. soln.	34.00	1.289	585	0.338	15.48

Fig. 4.



For a solution of strength $\frac{M}{50}$ we have

$$c = 0.0041 \text{ gram./cc.}$$

$$R = \frac{8.4 \times 10^7}{75},$$

where 75 = osmotic molar weight,

$$T = 289^\circ \text{ abs.}$$

$$\frac{d\sigma}{dc} = 427.$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = \frac{5.4 \times 10^{-9} \text{ gram BaCl}_2 \text{ per cm.}^2,}{1.6 \times 10^{-9} \text{ grm. chlorine,}}$$

$$\Gamma_{\text{Cl}} (\text{found}) \nless 10^{-8} \text{ grm./cm.}^2$$

The result obtained here is of too qualitative a nature to show whether there is agreement or not.

Adsorption of Copper Chloride.

With this substance both copper and chlorine were separately estimated.

The Adsorption of the Copper :—

The oil-surface area was 8000 cm.² The copper in the solution was estimated electrolytically, both before and after the adsorption. The change in weight of the electrode was only 0.2 milligram.

Hence

$$\Gamma_{\text{Cu}} = 2.5 \times 10^{-8} \text{ grm./cm.}^2$$

A repetition of the determination gave

$$\Gamma_{\text{Cu}} = 4.5 \times 10^{-8},$$

or mean value

$$\Gamma_{\text{Cu}} = \frac{3.5 \times 10^{-8} \text{ gram per cm.}^2}$$

Estimation of the Chlorine :—

As in previous cases, this was estimated as silver chloride. The difference in weight of the AgCl precipitate was 6 milligrams. The surface-area = 8000 cm.², and hence

$$\Gamma_{\text{Cl}} = 1.8 \times 10^{-8} \text{ grm./cm.}^2$$

A repetition of the determination gave

$$\Gamma_{\text{Cl}} = 2.3 \times 10^{-8};$$

so that the mean value is

$$\Gamma_{\text{Cl}} = \frac{2 \times 10^{-8} \text{ grm./cm.}^2}$$

Hence

$$\Gamma_{\text{Cu}} + \Gamma_{\text{Cl}} = \frac{5.5 \times 10^{-8} \text{ grm./cm.}^2}$$

Calculation of the Adsorption of CuCl_2 :—

The following table gives the value of the tension (corrected for density of the solution) at different concentrations :—

Hydrocarbon-Oil and CuCl_2 .

Concentration in gram-moles per litre.	Pipette Drop-number.	Tension in dynes/cm.
0	135	52
0.0125	140	49.9
0.025	143	48.9
0.05	147	47.8
0.1	154	45.8

These values are plotted in fig. 3.

For a solution of concentration $M/50$

$$c = 0.0027 \text{ gram./cm.}^2,$$

$$\frac{d\sigma}{dc} = 518,$$

$$R = \frac{8.4 \times 10^7}{51}.$$

Where 51 is the molecular weight at this concentration as determined by Biltz *.

$$T = 289 \text{ abs.}$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = 4.0 \times 10^{-9} \text{ gram CuCl}_2 \text{ per cm.}^2,$$

$$\Gamma_{\text{Cu}} + \Gamma_{\text{Cl}} (\text{found}) = 5.5 \times 10^{-8} \text{ gram./cm.}^2$$

We have here once more inequality between observed and calculated values. It should be noted that it is the experimental value of Γ which is always the greater, and there are indications that the cation is adsorbed in somewhat greater quantity than the chemical equivalent of the anion.

Further determinations with other electrolytes must be undertaken before one could definitely state whether this important fact—the separation of the ions—takes place or not.

Before going on to discuss in more detail, however, the data already given, the experiments carried out with non-electrolytes will first be briefly recorded.

* Biltz, *Zeit. Phys. Chem.* vol. xl. p. 199 (1901).

THE ADSORPTION OF NON-ELECTROLYTES.

The chief difficulty at the outset with regard to almost all non-electrolytes is their small solubility in water. It is not easy, therefore, to draw a comparison between the tension effects produced by these bodies and the effects produced by electrolytes. Altogether seventeen non-electrolytes were examined as regards their influence on the interfacial tension. Two of these—aniline and p-nitrosodimethylaniline—were found to be soluble in the oil, and to this is probably due their comparatively marked effects on the tension*.

The following table (p. 481) contains the values obtained from the various substances examined.

The value obtained for cane-sugar is worthy of note. The lowering effect is small, although a more concentrated solution was employed than in the case of any other substance. The result is in agreement with that observed at the air-surface, in which case the presence of cane-sugar has no appreciable effect.

In the case of all these substances (except iodine which was practically without effect) there is a lowering of tension at the interface which on Gibbs's theory must be accompanied by adsorption. The same general rule, therefore, as regards the effect at the oil-surface seems to hold for non-electrolytes as for electrolytes.

* *The Tension between two partially miscible phases.*

The Laplace theory of capillarity shows that the work required to form unit area of interface between two liquids A and B is given by the expression

$$\frac{1}{2}(\rho_A - \rho_B) \int_0^{\infty} z\psi(z) dz = \sigma_{AB},$$

where ρ_A and ρ_B = the densities of the liquids A and B respectively and the integral is supposed constant for all substances. This expression leads to the following relationship:—

$$\sqrt{\sigma_{AB}} = \sqrt{\sigma_{BC}} + \sqrt{\sigma_{CA}},$$

which has not been verified by experiment, but shows in a general way what takes place at the surface. Lord Rayleigh has on this hypothesis calculated the effect of making the transition gradual between A and B by the interposition of n liquids whose densities are in arithmetical series, in which case it is shown that the energy due to surface-tension is reduced to $(n+1)$ of its original value. Hence any diminution in the abruptness will diminish the energy due to surface-tension. The solubility of the aniline in the oil causes the diminution in abruptness, and consequently we find very marked lowering of tension.

TABLE IV.
Interfacial Tension: oil-solution.

Substance.	Per cent. Concentration.	Molar Concentration.	Relative Tension.
Water	1.00
Iodine	0.035	$\frac{1}{360}$	slightly >1.00
Hydroquinone	1.0	$\frac{1}{11}$	„ <1.00
Boric acid	0.1	$\frac{1}{62}$	0.97
Urea	0.8	$\frac{1}{7}$	0.97
Diphenylthiourea	<0.1	...	0.96
Milk sugar	1.0	$\frac{1}{36}$	0.96
Succinimide	0.2	$\frac{1}{48}$	0.94
Phenol	0.1	$\frac{1}{94}$	0.94
Mercuric cyanide	1.0	$\frac{1}{25}$	0.91
p-nitrosodimethylaniline...	<0.1	$<\frac{1}{150}$	0.89
Cane-sugar	6.0	$\frac{1}{6}$	0.83
Amygdalin	1.0	$\frac{1}{51}$	0.82
Ethyl acetate	3.6	$\frac{2}{5}$	0.70
Caffeine	1.0	$\frac{1}{21}$	0.71
Aniline	3.0	$\frac{1}{31}$	0.63

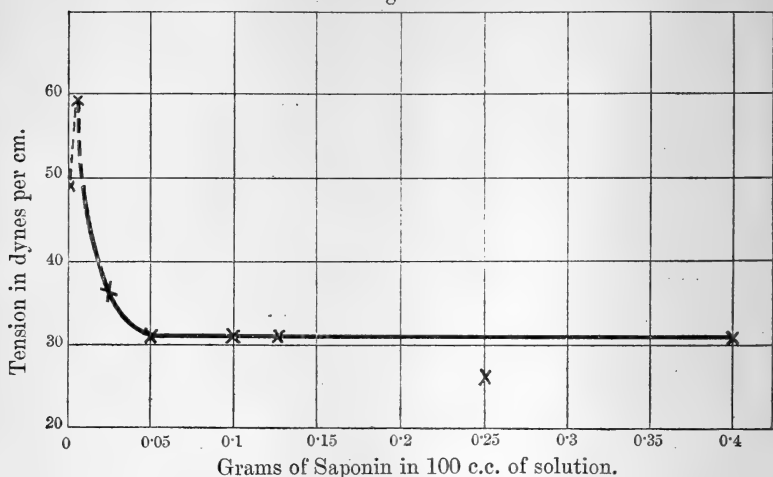
The Anomalous Behaviour of Saponin.

Saponin is a non-electrolyte whose aqueous solutions are characterized by foaming, thus pointing to considerable lowering of tension *. It has also been noticed that bubbles blown with saponin solution appear to gelatinize, so that when once formed they show a crinkled appearance on contraction. Solutions of this substance showed the following effects at the oil-water interface. On increasing the concentration, there was first of all a rise of tension reaching a maximum at about 0.005 per cent. This was followed by a rapid fall and at 0.025 per cent. a slower fall to 0.05 per cent., beyond which the tension remained constant. Measurements were taken up to 0.4 per cent., at which point streaming

* See a paper bearing upon this subject by S. A. Shorter, *Phil. Mag.* vol. xi. p. 317 (1906).

effects began to be observable, so that no further readings could be taken. These peculiarities are shown in fig. 5.

Fig. 5.



It seems probable that the gelatinizing effects which have been observed at the air-surface are also effective at the oil-interface. Experiments were carried out to test whether the substance was soluble in the oil. It was found to be insoluble.

Of the substances enumerated in Table III., it will be seen that caffeine shows the most marked effects in proportion to its concentration. It was therefore selected as a suitable substance wherewith to carry out adsorption determinations.

Determination of $\frac{d\sigma}{dc}$ for Caffeine.

The material was obtained from Kahlbaum. Its molecular weight was determined by lowering of the freezing-point of water, and was found to be in agreement with the formula (weight 194). The substance employed is the monohydrate and crystallizes from water in long needles. That caffeine is a good example of a non-electrolyte follows from the value for its dissociation constant determined by Wood *, namely, 4×10^{-14} , which is of the same order as the constant for water.

The sample of caffeine was first tested, as in previous cases, for any solubility in the oil. The same drop-number was obtained with the oil filtered from caffeine as for the oil fresh from the stock. Caffeine is therefore insoluble in hydrocarbon oil.

* Wood, Journ. Chem. Soc. vol. lxxxix. p. 1844 (1906).

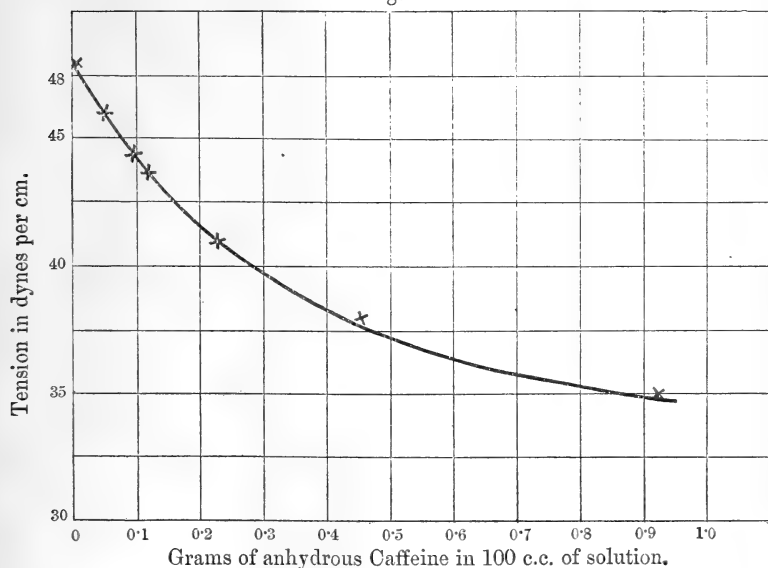
The following table (Table V.) contains the determinations of the interfacial tension for caffeine solutions of different concentrations, the tension being measured by the drop-pipette. It will be noted that this oil is not quite identical with that previously employed.

TABLE V.

Per cent. Concentration of Anhydrous Caffeine.	Drop- number.	Relative Tension.	Tension in dynes/cm.
0	146	1.000	48
0.046	153	0.954	45.8
0.092	158	0.924	44.4
0.114	161	0.907	43.5
0.229	171	0.853	40.9
0.457	184	0.793	38.1
0.915	201	0.726	34.8

The interfacial tension is plotted against concentration,

Fig. 6.



giving the curve on fig. 6. The tangent to the curve gives the value of $\frac{d\sigma}{dc}$ at the desired concentration.

Calculation of the Adsorption Coefficient :—

As before, the expression for the adsorption is

$$\Gamma = - \frac{c}{RT} \frac{d\sigma}{dc}.$$

For a solution of concentration 0.114 per cent. (anhydrous) caffeine the right-hand side of this expression has the following numerical value :—

$$c = 0.114 \text{ per cent.} = 0.00114 \text{ grm./c.c.,}$$

$$R = \frac{2 \times 4.2 \times 10^7}{194} \text{ ergs,}$$

$$T = 288^\circ \text{ abs.,}$$

$$\frac{d\sigma}{dc} = \frac{3 \text{ dynes/cm.}}{0.00114 \text{ grm./c.c.}}$$

$$\therefore \frac{c}{RT} \frac{d\sigma}{dc} = \underline{2.4 \times 10^{-8} \text{ grm./cm.}^2}$$

It will be noted that this value is approximately of the same order as the similarly calculated values for previous substances. The substitution of a non-electrolyte for a simple inorganic salt does not seem therefore to have any marked effect on the theoretical value for the adsorption coefficient. It would seem, therefore, that in both cases we are dealing with a factor of the same magnitude as regards its effect on the tension and the consequent value of $\frac{d\sigma}{dc}$. It will be remembered, however, that the results obtained in previous cases pointed to the existence of certain discrepancies between observed and calculated values of Γ . It is of particular interest, therefore, to attempt to measure Γ directly for caffeine, since in this case there is no possibility of separation of ions.

Determination of Γ for Caffeine in Aqueous Solution.

The emulsion method* was employed in these determination.

Solutions of caffeine approximately 0.125 per cent. were prepared, and 500 c.c. shaken in steamed-out glass vessels for three days with about 1–2 c.c. of oil. The resulting emulsions were very uniform under the microscope, the average diameter of the particles being 6×10^{-5} cm. It is worthy of note that this is of the same order as the average diameter of the particles in the previous cases.

* See previous paper.

The results of two emulsion experiments are given below :—

Caffeine solution (approx. 0.125 per cent.).

Experiment no.	Drop-number before emulsification.	Drop-number after emulsification.	Area of oil-surface.
1	160	160, 160 $\frac{1}{4}$	87,130 cm. ²
2	160, 160 $\frac{1}{4}$	160 $\frac{1}{2}$, 160 $\frac{1}{4}$	127,310 cm. ²

The results show that there was *no readable difference in the concentration of the solution before and after emulsification—the difference being a small fraction of a drop*. Special experiments were undertaken to determine the accuracy of the readings made with the pipette, and it was found that for a small drop-number such as 160 the readings were reproducible to within about $\frac{1}{8}$ th of a drop.

No very reliable quantitative relations can therefore be obtained from these determinations, as we are evidently dealing with a very small change in concentration, *i. e.*, a very small adsorption. The important feature is this—that *the calculated value of Γ (viz. 10^{-8} gram./cm.²) would also correspond to a small fraction of a drop as indicating concentration change*; while a value for Γ of the order obtained in the case of sod. glycocholate and the dyes (viz. 10^{-6} gram./cm.²) would have been indicated by a change of 4–5 drops, which would of course have been perfectly readable. These experiments therefore show that *caffeine is very much less adsorbed than sodium glycocholate, Congo red, or methyl orange*.

It is perhaps going beyond the limits of accuracy warranted by the drop-pipette method to make an approximate calculation of the Γ from the data given above. Taking the second experiment, the change in concentration corresponds to about $\frac{1}{8}$ th of a drop. This is only approximate, and in all probability would give a *maximum* value for Γ . From a curve* obtained by plotting the drop-numbers against concentrations given in Table IV. it is found that $\frac{1}{8}$ th of a drop corresponds to a change in concentration of 0.00094 per cent., and hence the total amount of caffeine removed from the 500 c.c. emulsified is 0.0047 gram. The total adsorbing surface is 127310 cm.², hence

$$\Gamma = 3.7 \times 10^{-8} \text{ gram./cm.}^2,$$

$$\Gamma \text{ calculated} = 2.4 \times 10^{-8} \text{ gram./cm.}^2$$

* The curve is a straight line between drop-numbers 155–200. A fraction of a drop can therefore be read fairly accurately.

An attempt was at once made to increase the adsorbing area in order to get more trustworthy data. It was found, however, that the limit had almost been reached in the case of the second emulsion—only a little more than 2 c.c. of oil can be emulsified by 500 c.c. of caffeine solution of 0.1 per cent. concentration. The method is therefore limited in this direction. A further attempt at more accurate measurement was made by employing a pipette of much larger oil-capacity (about 500 c.c.). The drop-number against water was 2042, in place of 146 in the case of the smaller pipette. The drop-number against a solution of 0.1 per cent. concentration was 2383. One result was obtained in which the change of concentration before and after emulsification amounted to 5 drops, which corresponded to 3.5×10^{-8} gm./cm., but on attempting to reproduce these values quite variable results were obtained, owing to the change in wetting of the pipette during the long course (over two hours) of a single determination.

It appears probable from the foregoing attempts to measure Γ in the case of caffeine, that we are dealing with a quantity too small to be determined with accuracy by the method hitherto pursued. It may be possible by employing, for example, colloidal mercury instead of the oil to determine the caffeine adsorption at a mercury surface with accuracy.

Behaviour of Sodium Glycocholate and Caffeine in contact with Charcoal.

In view of the considerable difference between the values for the adsorption of these two substances at the oil-surface, it was of interest to compare the results obtained with charcoal. The material was the same as that employed by Freundlich—namely, Merck's blood-charcoal purified by acid. Freundlich has observed that the further washing of this material does not affect the result of the adsorption determinations, and this was confirmed in the present instance by the fact that there was not the slightest change in the oil-water tension in the case of distilled water which had been shaken up with a quantity of charcoal.

Adsorption of Sodium Glycocholate:—

500 c.c. of a solution 0.3 per cent. concentration were treated with one gram of charcoal, and the decrease in bulk-concentration of the solute amounted to 0.125 per cent. The change in concentration was measured by means of the change in value of the oil-solution tension as given by the drop-pipette. The drop-difference (before and after) was 47.

Hence 1 gram charcoal adsorbs from a 0.3 per cent. solution 0.625 gram sodium glycocholate.

Adsorption of Caffeine:—

500 c.c. of solution 0.218 per cent. concentration (*i. e.*, approximately equimolar with the sodium glycocholate solution) brought into contact with one gram of charcoal gives a drop-difference of 10, which corresponds on the caffeine curve to a change in concentration of 0.108 per cent.

Hence 1 gram charcoal adsorbs from a 0.218 per cent. solution 0.324 gram of caffeine.

It will be noted that the behaviour of caffeine and sodium glycocholate towards charcoal and oil is very different. In contact with charcoal there appears to be no very marked difference as regards the magnitude of the adsorption for both substances. Possibly this may be due to oxidation of the caffeine by means of the oxygen adsorbed on the charcoal surface.

Calculation of the Range of the Concentration Effects at the Oil-surface.

Consider the case of an infinite mass of solution (of a non-electrolyte in water) whose bulk-concentration is c_1 and surface-concentration c_2 . Suppose one gram-mole of solute transferred from the bulk into the surface-layer. The osmotic work done is

$$RT \log \frac{c_2}{c_1}.$$

There is a corresponding decrease in the surface energy due to adsorption of the solute given by

$$-\frac{d\sigma}{d\Gamma};$$

and equating these two expressions,

$$RT \log \frac{c_2}{c_1} = -\frac{d\sigma}{d\Gamma}.$$

Experimental case :—

If we assume that the solution of caffeine obeys in all probability Gibbs's expression for surface-condensation, from the data previously given we may construct the following table :—

Aqueous solution of Caffeine at 15° C.

Per cent. Bulk-concentration (anhydrous).	Tension in dynes/cm.	Γ calculated from Gibbs's equation in grm./cm. ²
0.0	48	0.0
0.046	45.8	1.2×10^{-8}
0.092	44.4	1.8 „
0.114	43.5	2.0 „
0.229	40.9	2.3 „
0.457	38.1	2.8 „
0.915	34.8	3.0 „

The tension was plotted against Γ , and the tangent to the curve taken at the point corresponding to a bulk-concentration 0.114 per cent. The tangent gives

$$\frac{d\sigma}{d\Gamma} = \frac{7.25 \text{ ergs/cm.}^2}{1 \times 10^{-10} \text{ gram-moles/cm.}^2} = 7.25 \times 10^{10}.$$

At this point

$$\Gamma = 2 \times 10^{-8} \text{ gram./cm.}^2 = 1 \times 10^{-10} \text{ gram-moles/cm.}^2$$

The equation therefore becomes

$$\log_e c_2 = \log_e c_1 + \frac{7.25 \times 10^{10}}{RT}.$$

$$\begin{aligned} \therefore \log_{10} c_2 &= \log_{10} 0.0064 + 0.044 \\ &= -2.1498. \end{aligned}$$

$$\begin{aligned} \therefore c_2 &= 0.0071 \text{ gram-moles per litre} \\ &= 0.00138 \text{ gram./c.c.} \end{aligned}$$

Now if D is the range through which the surface-concentration effects are practically complete, it follows that the average concentration in the surface-layer is given by the expression

$$c_2 = \frac{\Gamma}{D},$$

$$\text{or } D = \frac{\Gamma}{c_2};$$

and on substituting the values obtained above it is found that

$$D = 14 \times 10^{-6} \text{ cm.}$$

It is evident that we have here obtained a value for the "range of molecular action," as may be seen by comparing it with other estimates of the same quantity.

		<i>Method.</i>
Quineke	5×10^{-6} cm.	"Wedge method."
Reinold & Rücker.	1.2×10^{-6}	Limiting thickness of soap film.
Wiener	1.2×10^{-6}	Effect on phase of reflected light.
Plateau	11.4×10^{-6}	Limiting thickness of glycerine film.
Drude	1.7×10^{-6}	" " soap film.
Parks	13.4×10^{-6}	Wetting of powders.
Johonnot	from 0^{-6} to 10^{-8}	Soap film.

DISCUSSION OF RESULTS.

The conclusions to which the foregoing experiments point are :—

- (1) Caffeine in aqueous solution, in all probability obeys Gibbs's law quantitatively.
- (2) Ordinary inorganic salts—Potassium chloride, Silver nitrate, Barium chloride, and Copper chloride—are adsorbed in quantities which are of the same order as the Gibbs's calculated effect, though the experimentally found values are in all cases *greater* than the calculated. Caustic soda shows a more marked discrepancy between calculated and observed values.
- (3) Complex organic salts—Sodium glycocholate, Sodium oleate *, Congo red, and Methyl orange—show a very large discrepancy between observed and calculated values. The first thing is therefore to consider the various possibilities of accounting for this large discrepancy.

The Adsorption of Sodium glycocholate, &c.

First, can the excess exhibited by these substances be explained by an extension of the capillary theory ?

The expression

$$\Gamma = - \frac{c}{RT} \frac{d\sigma}{dc}$$

is deduced on the assumption that we are dealing with *one* component only which suffers surface-concentration, *i. e.*, the solute. Since the above-mentioned substances are salts, they will be dissociated, and almost entirely so, at the dilution worked with. Milner †, in deducing the Gibbs's expression, has allowed for dissociation by the introduction of van 't Hoff's factor (*i*) thus :

$$\Gamma = \frac{c}{iRT} \frac{d\sigma}{dc}$$

This is, however, really the expression employed in the foregoing experiments, since in all cases for the evaluation of *R* (the gas constant) the molecular weight of the solute in

* Reference will be made later to this substance.

† Milner, Phil. Mag., Jan. 1907.

solution has been employed. By simply making this allowance for dissociation, therefore, the discrepancy between observed and calculated values for adsorption is still very marked—as a matter of fact, such a correction acts in the opposite direction.

The next possible explanation may lie in incorrectness in the assumption (which has been made throughout) that the water suffers no surface density-change.

It is very probable that water has not quite the same density in the surface-film as in the bulk of the liquid. Evidence for this is to be found in the phenomenon of heat evolution observed by Parks and others, when water is poured upon finely-divided powders of silica, quartz, glass, &c., where chemical action is of course excluded. Also one must not overlook the fact that in the adsorption of sodium glycocholate and the dyes we are dealing with surface-concentrations far exceeding the ordinarily accepted values for their solubility. Of course in using the term "solubility" we assume that the surface-layer water possesses the same properties as water in bulk. For a solution of sodium glycocholate of 0.25 per cent. bulk concentration, Γ was found to be 5×10^{-6} grm./cm.² The value of the range through which the surface-concentration is sensibly different from the bulk has been shown to be approximately 14×10^{-6} cm. Hence the *average* surface-concentration is 0.37 gram per c.c. or 37 per cent., while the ordinarily accepted value for the solubility of sodium glycocholate is 3.9 per cent. There was, however, no visible colour change on the oil surface due to any precipitation.

Again, methyl orange* showed an adsorption for which $\Gamma = 5.5 \times 10^{-6}$, whence the average concentration is 39 per cent. The actual solubility is only 0.078 per cent.

The substances therefore which have shown very great discrepancies as regards Gibbs's theory are *those whose surface concentration greatly exceeds their solubility in the solvent*. For the other substances examined, viz. caffeine and the inorganic salts, *in no case is the ordinary solubility exceeded*, and the adsorption of these substances is in much closer agreement with theory. Thus the surface-concentration of caffeine was found to be 0.26 per cent., while its solubility at ordinary temperature is 1.35 per cent. For silver nitrate the surface-concentration was approximately 0.28 per cent., while its solubility is 122 per cent. at 0° C. Similarly for potassium chloride, barium chloride, and copper chloride.

* See previous paper, *l. c.*

Another possibility however presents itself—namely, the existence of some electrical effect, such as an electrostatic attraction, causing the discrepancy, since sodium glycocholate and the dyes are electrolytes. If this were so, one would also expect somewhat of the same order of discrepancy manifesting itself in the case of the ordinary inorganic salts. Such does not take place. Caustic soda alone has marked effects, and shows an adsorption 20 times greater than the calculated. The other substances generally show a discrepancy of about 5 to 8 times. To whatever cause this smaller discrepancy is due, it is unlikely that the same reason is to be assigned to the very much larger discrepancies observed for sodium glycocholate.

The only other possible means, apparently, of explaining the want of agreement is on the assumption that we are dealing with some irreversible phenomenon of the nature of gelatinization upon the oil surface. The fact of the surface-concentration being much greater than the ordinarily accepted value for the solubility, is in agreement with this view. It should be pointed out, however, that the overstepping of the solubility limit does not seem to be essential to gelatinization, as will be seen by referring to the determinations carried out with saponin. The solubility of this substance is great; according to Beilstein it is completely miscible with water, and although this cannot actually be the case, the constancy of the tension at comparatively dilute solutions cannot be explained by the solubility having been reached. This is a very striking case showing the sensitiveness of the tension to gelatinizing effects. The hypothesis of gelatinization has been put forward by Milner* to explain the behaviour of sodium oleate, although in this case the tension curve was continuous and quite similar to the sodium-glycocholate curve. From some unpublished experiments by Mr. H. E. Potts in this Laboratory, it was found that sodium oleate showed an adsorption at the oil surface, the magnitude of which was of the order 10^{-6} grm./cm.², while the calculated was of the order 10^{-8} grm./cm.², that is to say a discrepancy very similar to sodium glycocholate.

Mention may also be made that Freundlich and Losev† have found that certain dyestuffs produced amorphous precipitates on the charcoal surface during adsorption determinations. This was an irreversible process (the precipitates being insoluble in water), and as such might be considered as analogous to gelatinization.

* *Loc. cit.*

† Freundlich and Losev, *Zeit. Phys. Chem.* vol. lix. p. 284 (1907).

It is possible, therefore, that some irreversible effect of the nature of gelatinization is the real cause of the discrepancy between observed and calculated values in the adsorption of sodium glycocholate, Congo red, and methyl-orange.

The Adsorption of Inorganic Salts from Aqueous Solution.

It has already been pointed out that for certain purely inorganic salts, discrepancies are found between observed and calculated values of the adsorption, but that the existence of these discrepancies might be considered as not quite proved owing to possibility of great experimental error. On the whole, however, it is believed that there really does exist a small discrepancy which requires explanation. Attention might be drawn to the fact that in the cases measured the metal, *i. e.* the cation, shows an excess, while in the case of potassium chloride, and possibly also in barium chloride, the chlorine (the anion) is much nearer to the calculated value. This at once suggests some separation of ions, that is partial selective adsorption due to some electrical effect. The value, however, for the adsorption of the chlorine in copper chloride is scarcely in accord with the preceding statement. Granting that there is a tendency for selective ionic adsorption, it is evident that the phenomenon cannot be completely covered by Gibbs's capillary expression, in the deduction of which no allowance is made for any such ionic separation. Of course for non-electrolytes, where no ionic separation is possible, Gibbs's expression ought to hold quite readily, and experiment confirms this, although the determinations are unfortunately of a more qualitative character than one could wish. In electrolytes, the separation of cation from anion may possibly be regarded as local electrolysis at the surface of the oil—that in fact we are dealing with a cell of small dimensions the poles of which are the two homogeneous masses, and the heterogeneous layer separating the masses functions as the "electrolytic medium." If this is so, we must treat the question as one of electrolytic conduction, but first of all it must be shown that there exists an electrical potential-difference between the electrodes, *i. e.* between the oil and the water. Owing to the non-conducting nature of the oil one cannot employ the usual methods, but by emulsifying the oil (*i. e.* converting it into small particles suspended in water) and placing the emulsion in an electrostatic field, it can be

observed whether any motion of the particles takes place or not. This was done experimentally as follows:—

A simple Nernst form of U-tube, identical with that employed by Burton*, having a vertical capillary tube affixed at the bend through which water slowly runs so as to form a definite surface of separation (in both limbs) between the water and the oil-water emulsion, is fitted with platinum electrodes dipping into the emulsion. A potential-difference of 230 volts was maintained between the electrodes, and the movement of the emulsion-water boundary down in one limb and up in the other was read off on a scale attached to the vessel. It was found that the particles moved towards the anode, being therefore negatively charged. The velocity of the particles was 4.3×10^{-4} cm./sec. under a gradient of 1 volt/cm.

Burton† has deduced an expression for the potential existing between the particles of colloidal metals and the water in which they are suspended, viz.:

$$V = \frac{4\pi \eta v}{K \bar{X}},$$

where V is the p.d. required (in electrostatic units);
 K is the specific inductive capacity of the medium;
 η is the viscosity of the medium;
 $\frac{v}{\bar{X}}$ is the velocity of the particles under (electrostatic) unit potential gradient,
 \bar{X} being the p.d. maintained between the poles.

Applying this to the present case of oil suspended in water, we obtain

$$V = 0.00048 \text{ electrostatic unit,}$$

or 0.15 volt.

There is thus quite a measurable potential existing between the oil and water; and in considering the adsorption of electrolytes we must not overlook this fact. It may be mentioned, in passing, that the above value for the potential is quite of the same order as that obtained by Burton for various colloidal metals.

Since the oil has been shown to be negatively charged, one would not be surprised, speaking generally, that the cation should be selectively adsorbed.

* Burton, Phil. Mag. ii. p. 434 (1906).

† Burton, *loc. cit.*

SUMMARY.

- (1) The adsorption of various substances on a hydrocarbon-oil surface has been measured. The results are collected in the following table:—

Substance.	Adsorption per cm. ² of the substance assumed to be in the form of <i>undissociated salt</i> or in chemically equivalent ionic proportions.			
	Found value.		Calculated value.	
Sodium glycocholate. .	5×10^{-6} grm./cm. ²		7×10^{-8} grm./cm. ²	
Congo-red	3.7×10^{-6}		1.1×10^{-7}	
Methyl-orange	5.5×10^{-6}		1.2×10^{-7}	
Sodium oleate	10^{-6}		10^{-8}	
Caustic soda	1.5×10^{-7}		7.5×10^{-9}	
Caffeine	3.7×10^{-8}		2.4×10^{-8}	

Substance.	Adsorption of cation per cm. ²		Adsorption of anion per cm. ²	
	Found value.	Calculated value.	Found value.	Calculated value.
Silver nitrate	grm./cm. ² 2.5×10^{-8}	grm./cm. ² 4.5×10^{-9}	grm./cm. ²	2.3×10^{-9}
Potassium chloride. .	5×10^{-8}	1.7×10^{-9}	10^{-9}	1.6×10^{-9}
Barium chloride	3×10^{-9}	not $> 10^{-8}$	1.6×10^{-9}
Copper chloride ...	3.5×10^{-8}	2×10^{-9}	2×10^{-8}	2×10^{-9}

- (2) The large discrepancy between observed and calculated values for the first four substances is possibly due to gelatinization upon the oil-surface.
- (3) Caffeine obeys Gibbs's law within the limits of experimental error.
- (4) There is evidence of slight *selective* adsorption, the cation more than the anion, probably due to electrical effects, since the oil is shown to be negatively charged, the potential-difference between the oil and water being approximately 0.05 volt.

In conclusion, I gladly take this opportunity of expressing my gratitude to Professor F. G. Donnan for his advice and assistance in this work.

XLIV. *Historical Note on the Discovery of the
Ultra-microscopic Method.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IT is essential that, when the work of earlier investigators is found to have anticipated the “discoveries” of later workers, due justice be done to the former. That is the reason why I venture to draw the attention of the readers of the *Philosophical Magazine* to the forgotten pages of the ‘*Indian Engineering*,’ a weekly journal, in the numbers for April 7th, 14th, and 21st, 1888, of which were published three papers by Mr. G. Dubern. In these papers was described a method devised by him for observing and studying the properties of particles too small to be visible under microscopes even of the highest powers, when used with the ordinary arrangements for illumination. The method briefly described was the following:—The particles were held on a glass plate under a microscope; a powerful beam of convergent light entered the glass plate obliquely through a polished slant-end-surface, and being incident at an angle greater than the critical angle on the surface of the plate, suffered a series of total internal reflexions inside the glass plate. Under these circumstances, the ultra-microscopic particles appeared as tiny bright specks on a dark field. Mr. G. Dubern’s arrangement is identical in all particulars with Mr. Cotton’s modification of the ultra-microscopic method of Siedentopf and Szigmondy, and a comparison between the figures illustrating Mr. Dubern’s arrangement in the ‘*Indian Engineering*,’ and fig. 306 on page 491 of Prof. Wood’s ‘*Optics*,’ representing Mr. Cotton’s apparatus, will convince even the sceptical that Mr. Dubern’s apparatus is fully entitled to the name “Ultra-microscope,” and that he must be regarded as having more or less completely anticipated Siedentopf and Szigmondy.

I have the honour to be, Gentlemen,

Your most obedient Servant,

C. V. RAMAN.

Science Association Laboratory,
Calcutta, 17th December, 1908.

XLV. *A Laboratory Machine for Applying Bending and Twisting Moments simultaneously.* By Professor E. G. COKER, M.A., D.Sc., of the City and Guilds of London Technical College, Finsbury*.

[Plates XI. & XII.]

APPARATUS for applying simple tension or compression stresses, and also for applying bending or twisting moments to materials, are in common use in laboratories, but such machines are, as a rule, not very well adapted for experiments on combined stresses, although the frequency of such cases in practice make it very desirable that experiments on the effects of combinations of stresses should be carried out by engineering students. This is particularly the case with shafts which are generally subjected to stresses due to combined bending and twisting moments. The present paper describes a machine built by students of the City and Guilds Technical College, Finsbury, in which uniform bending and twisting moments can be applied simultaneously over the whole length of the specimen, and in any desired proportion to each other.

The machine is similar to one designed by the author for the testing laboratory at McGill University, Montreal, but with some modifications suggested by experience with the earlier machine. The principle on which the design is based is illustrated by fig. 1 (Pl. XI.), in which a rod R is suspended at intermediate points A, B, by wires C, D, depending from a fixed support E. The equal overhanging ends of the rod are loaded by weights W, so that the applied couple between the points of support is uniform and of amount Wa , where a is the length of the lever-arm. The rod is also twisted by weights W_1 attached to equal arms of length b , so that there is a uniform twisting moment of amount W_1b between the points of suspension. The two systems of loading are independent and their ratio can be adjusted to any value desired.

In carrying out this arrangement in practice it is convenient to arrange that one of the levers for applying the twisting moment shall always remain in a horizontal position, and that the other shall be capable of turning through an arc to bring the first lever back to zero after each application of the load. The most convenient way of carrying this out is to replace the adjustable lever by a worm and worm-wheel gear secured in a casing and turned by a hand-wheel. To allow freedom for bending the worm-wheel casing must be

* Communicated by the Physical Society: read February 26, 1909.

pivoted to rotate around a line intersecting the axis of the specimen and perpendicular thereto, and this method of pivoting must also be adopted at the horizontal lever. This arrangement only differs from that of the perfectly freely suspended arrangement shown by fig. 1 (Pl. XI.) in fixing one point of the rod, and this has the indirect advantage of stilling vibration which is troublesome in the freely suspended bar.

The arrangement described above is carried into effect in the manner indicated by figs. 2, 3, 4, and 5, showing the apparatus in side elevation, end elevations, and plan respectively.

The various parts are supported in a built up frame consisting of two planished steel shafts, A secured in cast-iron cross frames, B mounted on four standards, one of which latter is adjustable in height to secure steadiness on an uneven floor. Upon the steel shafts are two castings C, D, each of which has a cylindrical bearing E encircling one of the shafts and resting with a flat face F in line contact with the other shaft, and secured in position by a cross-bar G threaded on studs. This connexion is perfectly rigid, since it removes all degrees of freedom and it is readily released by simply turning back one of the cross-bar nuts, leaving the casting free to slide into a new position. It also has the advantage that no accurate fitting is required for the supporting frame. The casting C carrying the worm-wheel gear W has trunnion bearings H at right angles to and intersecting the axis of the specimen. The bearings are fitted with friction rollers, and when the machine is used simply for torsion the worm-wheel is kept in a vertical position by an arm I keyed to the bearing H and locked in position by a thumb-screw. A weight J attached by an arm to the second bearing balances the pivoted casing in all positions.

The weigh levers are supported from a vertical standard K of the frame D by a wire L, terminating in a thin plate M with a keyhole slot encircling the spindle N. Formerly a roller bearing was used for this spindle, but this is an unnecessary refinement as the friction is extremely small, and can be easily taken into account. The casting supported in this way has three levers, P, Q, and R, the first two of which are for the application of twisting moments S, and the third R, in the line of the specimen, is for applying a bending moment.

All the loading levers are provided with knife-edges S, fig. 6, of circular form, made by turning an ordinary Whitworth nut down to form a disk with a V-shaped edge. These disks carry rings T with wide angled V-shaped recesses on

the inner sides, and light rods V screwed into these rings carry the weights. This arrangement of knife-edge is very easy to adjust accurately, and when bending and twisting stresses are applied simultaneously the rolling line contact adjusts itself to the bending and twisting of the specimen. The bending of the specimen causes a change in the effective arm of the bending levers, which is generally negligible, but a correction may be necessary with a very long specimen. For if a is the length of the lever-arm and b is the radius of the circular knife-edge, an angular deviation of amount θ will cause a change of $a - (a \cos \theta + b \sin \theta)$ in the lever-arm, and this is zero when $\theta = 0$ and also when $a = a \cos \theta + b \sin \theta$.

In the machine described a is 10 inches and b is 0.5 inches, and the angles $\theta = 0$ and $\theta = 5^\circ.75$ both correspond to an effective length of 10 inches. The maximum correction between these values is easily shown to be at an angle θ given by the equation $b \cos \theta = a \sin \theta$, in the present case $2^\circ.9$ approximately, for which value the correction is 0.12 per cent. For values of θ greater than 6° the correction increases more rapidly, and its amount may be obtained from the diagram, fig. 7, which shows the percentage error for all angles up to 10° . In the majority of tests the angular change at the ends rarely exceeds 5° , and the correction is therefore so very small as to be practically negligible.

The worm-wheel W and the casting V for the weigh-levers are bored out to receive the ends of the specimen, and are provided with fixed keys which slide in corresponding key-ways cut in the specimen. When tubes are subjected to stress they are provided with solid ends secured by transverse pins, thereby avoiding brazed joints since these latter are troublesome owing to the state of the metal being altered by the brazing. The end of the specimen projecting through the worm-wheel is fitted with a lever X for applying bending moment, and both levers for bending may be loaded independently or by a cross-bar suspended from stirrups as shown in fig. 2.

A photograph of the machine is shown by fig. 8 (Pl. XII.) with a specimen inserted which has failed under the combined effect of bending moment and twisting moments.

Measurement of the Strains.

The worm-wheel is graduated in degrees and a vernier circle enables 0.1 of a degree to be read with ease, while in order to start with a zero reading this vernier is carried on a ring sliding in a groove in the casing so that it can be

adjusted to any angular position. This arrangement measures the twist on the whole specimen, and includes any motion due to back-lash in the keys and keyways during a test, it is therefore only suitable for measurements with long wires having substantial ends, and for plastic strains in which the end effects are negligible in comparison.

For observations within the elastic limit the author prefers to use an instrument* which is secured to the specimen and is self-contained.

This instrument was originally designed to measure the angle of twist within the elastic limit, and with some recent alterations it can be adjusted in a few seconds for measuring the angular change due to bending. The calibration of the readings is effected on the specimen and serves for both bending and twisting. Fig. 9 (Pl. XI.) shows the apparatus in part longitudinal section.

It consists of a graduated circle A mounted on the specimen B by three screws C in the chuck-plate D. A sleeve E provided with three screws grips the specimen at a fixed distance away from the first set.

The spacing of these two main pieces on the specimen is effected by a clamp, not shown in the figure, which grips the double cones F, G, and maintains them at the correct distance apart, until the set screws are adjusted.

The clamp is afterwards removed, leaving the plane of the graduated circle perpendicular to the axis of the specimen and the sleeve correctly set and ready to receive the reading-microscope H.

The vernier plate carries a sliding tube I, on which a wire J is mounted, and the movement of this latter due to bending or twist is measured by a scale in the eye-piece K, the divisions of which are calibrated by reference to the graduated circle. It is found convenient to have the microscope-tube pivoted about an axis perpendicular to its central line at L, so that any slight difference due to imperfect centering can be adjusted by the screw M to make the calibration value agree for a series of specimens.

The observation wire may be set at any convenient position for calibration, but for observations of the angle of twist when the specimen is also subjected to a uniform bending moment the wire should be in the central plane perpendicular to the specimen. For if the bending is in the plane containing the axis of the specimen and the observation wire, it has the effect of causing new parts of the wire to come

* "On Instruments for Measuring small Torsional Strains," *Phil. Mag.* December 1899.

into view on the scale, but no error is caused thereby. If the specimen is bent in a plane at right angles to the former, then the change in the reading is $(\theta - \phi)l$, where θ and ϕ are the alterations of angle at the ends and $2l$ is the length of the specimen under observation. Since the bending is uniform $\theta = \phi$ and no correction is necessary. Bending in any other plane can be resolved into components in the vertical and horizontal planes, and therefore falls under the preceding cases. In order to effect the adjustment required, both the wire and the microscope slide in adjustable tubes provided with graduated scales, and the movement to bring the wire into focus is divided between them. To check the setting of the wire in the central position it is convenient to apply a uniform bending moment, and then to observe if any change takes place in the reading. The position for no change in the reading can be found in a few seconds.

In experiments where the bending moment is constant and the twisting moment is varied, no adjustment is practically required during the elastic life of the specimen; and even when the bending moment is variable the adjustment is practically negligible, as the length of the specimen under test is only a few inches.

The instrument is used for observations of the angular change due to bending by adjusting the wire in the horizontal plane passing through the axis of the specimen, and at a fixed distance away from the central plane, as shown in fig. 7. Thus if the wire is at a distance x from the central plane, and the specimen is subjected to a uniform bending moment, the reading will be $(l+x)\theta - (l-x)\theta = 2x\theta$, and this is a measure of the angular change θ between the ends, since any correction involves higher powers of θ which are negligible for elastic strains.

The instrument may therefore be used for measuring strains due to bending or twisting, and the single calibration required for both sets of readings is effected when the instrument is in position on the specimen.

With the usual notation the value of Young's modulus can be calculated from the equation $EI_1 dy^2/dx^2 = M$, from which we obtain $EI_1\theta_1 = Ml$ for determining E . Similarly the rigidity modulus for specimens of circular sections may be calculated from the equation $C = T/I_2\theta_2$. The ratio of C to E can also be readily obtained from the value $a\theta_1/\theta_2$, where a is an instrument constant: values of Poisson's ratio may be determined in this way without calculating C and E separately. As an example, we may take the case of a test

on a steel tube 6 inches long, having an external diameter of 0.748 inch and a wall-thickness of 0.024 inch.

The calibration value of the eyepiece scale where referred to the graduated circle was found to be 10.35 divisions for one minute of arc. The following readings were obtained:—

Twisting Moment. Inch-Pounds.	Reading	Δ .
0	0	
100	237	-237
200	474	-237
300	712	-238
400	920	-238
500	1188	-238

and the value of $C=12,400,000$ in inches and pounds was determined from these observations.

The observation wire was then rotated into a horizontal position and set at a distance of one inch from the central plane. Bending moments were applied and readings were observed as follows:—

Bending Moment. Inch-Pounds.	Reading	Δ .
0	0	
100	59	-59
200	118	-59
300	178	-60
400	239	-61
500	297	-58

from which the value of E was calculated. The angular change θ_2 of one end of the specimen was determined from the readings R by the formula $R=2x\theta_2$, where $x=1$ inch, and account was taken of the fact that the effective radius of arc was changed from r_1 to r_2 in the new adjustment of the instrumtment so that the angle θ_2 must be

multiplied by r_1/r_2 . In this instance r_1/r_2 was 1.61 and $E/C = r_2/r_1 \cdot \theta_1/\theta_2 = 2.485$ or $E = 30,800,000$ and Poisson's ratio = .243.

A tension-test on a similar piece of tube gave $E = 30,700,000$.

In addition to its applications for bending and twisting the apparatus may be used for testing a variety of cases of combined stress if a pump is added to give a fluid pressure in the interior of tubes.

The accompanying diagram (Pl. XI. fig. 10) shows the results of tests to failure of bicycle-tubing when subjected to (I.) bending, (II.) twisting, and (III.) twisting combined with a uniform bending moment.

XLVI. *The Viscosity of Water.*

By RICHARD HOSKING, B.A. (Camb.).*

[Plate XIII.]

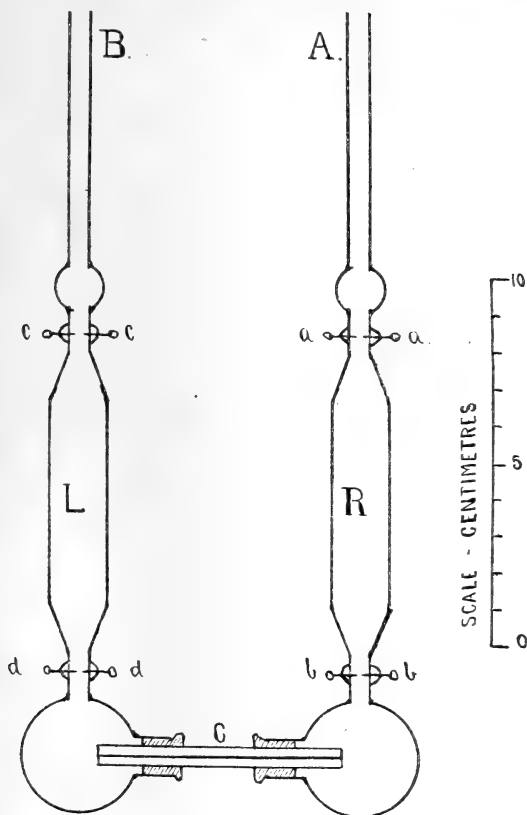
IN my previous experiments on the determination of viscosity by the efflux method †, I have always arranged to have the rate of flow of liquid in the capillary tube very small. The kinetic energy correction in the well-known reduction formula was thus always small in comparison with the first term. In the present experiments, however, I have purposely increased this rate of flow in order to test the formula in cases where the kinetic energy correction is much greater. The glischrometer used in these experiments was of the same form as those previously used by me, but the bulbs were larger. It is shown in fig. 1. At *a, b, c*, and *d*, platinum wires are inserted, which are almost touching inside the tubes. The capillary C is fitted to the limbs by rubber bands. The volume of the bulb R at 0° C. is 10.2801 ccs., and that of L at the same temperature 10.3201 ccs. Four capillaries were carefully selected for separate use in the glischrometer, and their ends were ground with fine emery, in a lathe. Their lengths, measured directly with callipers, were 5.570, 6.494, 5.408, and 6.456 cms. respectively at 0° C., and their radii approximately were .019, .019, .020, .020 cm. respectively, at 0° C. (The exact determination of the equivalent radius of each capillary was made at the end of all the experiments. It involved the cutting up of the tube and the careful measurement of the sections.) Sufficient freshly distilled water was put in to fill up the glischrometer from *b* to *c*.

* Communicated by the Author. Read before the Royal Society of N. S. Wales, June 3, 1908.

† Phil. Mag. March 1900; May 1902; May 1904.

The ends A and B were connected to the reservoir of compressed air or the outside air by means of three-way taps. Measurements of viscosity were taken first with the water flowing out of R into L, and secondly with the water flowing in the opposite way, under the pressure of air in the reservoir. The average of the two determinations was taken as an absolute value of the viscosity. The pressure was measured by means of (1) a water manometer 200 cms. long, (2) a mercury manometer in cases where the pressure was greater than 200 cms. of water.

Fig. 1.



flowing in the opposite way, under the pressure of air in the reservoir. The average of the two determinations was taken as an absolute value of the viscosity. The pressure was measured by means of (1) a water manometer 200 cms. long, (2) a mercury manometer in cases where the pressure was greater than 200 cms. of water.

To facilitate the reading of the water manometer, a pair of small parallel mirrors was attached to each of the arms. These were inclined so as to make an angle of 45° with the vertical manometer scale. One was fixed near the centre of

the arm, and opposite a telescope; the other could be moved along the arm and clamped in front of the water surface. When the mirrors had been set, it was thus possible to read off the positions of the two water surfaces by means of the telescope, at the same instant; for both images were arranged to be side by side in the field. The pressures employed varied between the limits 100 cms. (water) and 42 cms. (mercury).

The time of flow was in most cases very short, the average being about one minute, but in extreme cases it was as low as 22 seconds. Special means had to be employed to register the time intervals correctly. The chronograph used was kindly supplied by the Sydney Observatory. It consisted of two electromagnets side by side. The armatures were provided with needles. Paper tape was fed through rollers immediately over the needle points at the rate of about 5 cms. per second. A special spring enabled the needles to travel forward a little on piercing the tape, and prevented the tearing of the tape. One needle was used for recording seconds by direct reference, through electrical contacts, to a standard clock. The other needle punctured the tape when a key was pressed at the transits of the meniscus in the glischrometer at the points *a* and *b* in the one case; or at *c* and *d* in the other. These transits were observed always through telescopes.

The procedure in determining the viscosity was as follows. The bath temperature was arranged to be as close to the desired temperature as possible, and the heating flame was adjusted. The pressure of air in the reservoir was raised or lowered to the proper level. Double readings of pressure, time of flow, and temperature were taken. The pressure was next altered, and more readings were taken. In most cases, the determinations were repeated. Another capillary was then placed in position in the glischrometer and the series was repeated.

The Reduction Formula.—In the Journal and Proceedings of the Royal Society of New South Wales are published three most important papers by G. H. Knibbs*, dealing with the history, theory, and determination of the viscosity of water by the efflux method. Knibbs has shown that the reduction formula is

$$\eta_t = \frac{\pi R^4}{8VL\left(1 + n\frac{R}{L}\right)} g \cdot \rho h \cdot T - \frac{m \cdot \delta \cdot V(1 + 2kt)}{8\pi L\left(1 + n\frac{R}{L}\right)T} \quad (1)$$

* Vols. xxix., xxx., and xxxi.

In this formula L is the length and R the radius of the capillary, and T the time taken for volume V of the liquid of density δ to flow through the capillary under a pressure gph ; nR is a small length of tube producing a loss of pressure equivalent to that arising from the friction at the ends, its value must be calculated for each series of experiments; m is the numerical factor in the kinetic energy correction, which has a theoretical value of 1.12 , but which has a practical value which must be determined. The factor m has been neglected in so many recent determinations of viscosity, that it is worth the while to repeat the information given by Knibbs respecting it. In 1860 Neumann deduced the value $m=1$, and Jacobsen used it in his 'Introduction to Hæmodynamics' (1860). Hagenbach deduced the value $m=1$ in the same year. Reynolds (following Bernoulli) in 1883 used the value $m=\frac{1}{2}$. Couette in 1890 independently obtained the value $m=1$. Boussinesq (1891) obtained a more accurate value $m=1.12$. Gartenmeister stated (1890) that Finkener had in an unpublished treatise shown that Couette's value was the correct (?) one. Wilberforce (1891) pointed out the defect in Hagenbach's reasoning, and he used the value $m=1$. Knibbs has shown that theoretically Neumann's correction as deduced by Boussinesq is correct, and that experimentally its value varies considerably. Knibbs has deduced values of m from Jacobsen's results, and stated that individual results show how, even under circumstances in which uniformity might be expected, it is not realized; and that if the correction be of sensible magnitude, the deduced viscosity is, to the extent of this uncertainty, unreliable.

Determination of m and n.—Preliminary experiments were made in order to obtain correct (experimental) values for the constants m and n . The temperatures were kept as close to 50° C. as possible, and, under different pressures, the times of flow were recorded. The pressures were reduced to equivalent pressures at 50° C. Knibbs has shown that the reduction formula, for experiments carried out at a certain temperature, may be expressed in the form

$$C + c/T = \rho hT, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{where} \quad C = \eta_i(1 + nR/L) \frac{8VL}{\pi g R^4}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{and} \quad c = \frac{m\delta}{g\pi^2} \frac{V^2}{R^4}(1 + 2kt). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (2) is that of a straight line such that if $1/T$ be taken as abscissæ, and corresponding values of ρhT as ordinates, the line passing through the points so determined

will intersect the axis of ordinates at a distance C from the origin, and make with the axis of abscissæ an angle whose tangent is c . When c has been obtained m is deduced by means of equation (4).

Equation (3) may be written in the form

$$\frac{C\pi g R^4}{8VL} = \eta_t \left(1 + n \frac{R}{L}\right).$$

The left-hand side of the equation will have different values for different capillaries in the glischrometer.

Calling the left side K we have

$$K = \eta_t + n\eta_t R/L \text{ or } K = \eta_t + lR/L, \quad . \quad . \quad (5)$$

where $l = n\eta_t$.

This is the equation of a straight line such that if values of R/L be taken as abscissæ, and corresponding values of K as ordinates, the line passing through such points will intersect the axis of ordinates at a distance η_t from the origin, and make an angle with the axis of abscissæ whose tangent is k . From the value of k obtained in this way, n is at once deduced.

Typical Observations with Tube I. in Glischrometer.

Temp. ($^{\circ}$ C.).	Manometer Reading (h) cms.	Temp. of Manometer $^{\circ}$ C.	Time of Flow (T) secs.	Bulb being emptied.	ρhT (reduced to $50^{\circ}) \times 10^{-3}$.	$\frac{1}{T}$.
50.05	*108.29	22.9	71.38	R	7.713	.01401
50.03	108.27	"	71.46	L	7.718	.01400
50.00	108.31	"	71.45	R	7.707	.01400
50.00	108.31	"	71.33	L	7.697	.01402
50.10	150.28	26.0	54.34	R	8.145	.01840
50.10	150.30	"	54.37	L	8.150	.01839
50.00	196.20	22.9	43.98	R	8.598	.02274
50.00	196.14	"	43.74	L	8.546	.02286
50.00	196.10	24.2	44.03	R	8.600	.02272
50.00	196.17	"	43.70	L	8.540	.02288
50.04	+20.30	23.9	33.58	R	9.239	.02978
50.03	20.30	"	33.52	L	9.220	.02984
50.02	20.30	"	33.58	R	9.236	.02978
50.01	20.30	"	33.53	L	9.232	.02985
50.10	25.40	26.0	28.35	R	9.764	.03527
50.10	24.40	"	29.24	L	9.674	.03420
50.05	30.38	23.9	24.78	R	10.204	.04035
50.06	30.38	"	24.93	L	10.267	.04011
50.07	30.38	"	24.72	R	10.184	.04045
50.08	30.38	"	24.95	L	10.280	.04008
50.00	35.40	22.9	22.32	R	10.701	.04480
50.00	35.40	"	22.38	L	10.726	.04468
50.00	35.40	"	22.28	R	10.683	.04488
50.00	35.40	"	22.38	L	10.726	.04468
50.10	40.40	22.9	20.36	R	11.171	.04912
50.10	40.40	"	21.40	L	11.737	.04673
50.10	40.40	"	20.39	R	11.188	.04904
50.10	40.40	"	21.40	L	11.737	.04673

* Water Manometer.

† Mercury Manometer.

Typical Observations with Tube II. in Glischrometer.

Temp. (° C.).	Manometer Reading (<i>h</i>) cms.	Temp. of Manometer ° C.	Time of Flow (T) secs.	Bulb being emptied.	ρhT (reduced to 50°) $\times 10^{-3}$.	$\frac{1}{T}$.
50.08	*108.41	26.7	80.41	R	8.716	.01244
50.08	108.25	"	80.27	L	8.676	.01246
50.08	108.13	"	80.81	R	8.729	.01237
50.08	108.15	"	80.48	L	8.690	.01242
50.05	151.12	26.7	60.40	R	9.113	.01656
50.06	151.02	"	60.30	L	9.084	.01658
50.07	150.96	"	60.40	R	9.118	.01656
50.07	150.96	"	60.18	L	9.065	.01661
50.02	197.10	26.7	48.58	R	9.549	.02058
50.04	197.20	"	48.15	L	9.470	.02077
50.05	197.04	"	48.45	R	9.531	.02064
50.05	197.08	"	48.15	L	9.468	.02077
50.00	†20.40	26.7	36.79	R	10.153	.02718
50.00	20.38	"	36.68	L	10.113	.02726
50.00	20.38	"	36.93	R	10.180	.02708
50.00	20.38	"	36.85	L	10.157	.02714
50.10	30.42	26.7	27.00	R	11.145	.03703
50.10	30.42	"	27.16	L	11.200	.03682
50.02	30.42	"	27.29	R	11.237	.03664
50.02	30.42	"	26.97	L	11.116	.03707
50.10	40.39	21.9	22.07	R	12.107	.04552
50.10	40.39	"	21.97	L	12.053	.04511
50.10	40.39	"	22.17	R	12.152	.04564
50.10	40.39	"	21.91	L	12.020	.04585

* Water Manometer.

† Mercury Manometer.

Typical Observations with Tube III. in Glischrometer.

Temp. (° C.).	Manometer Reading (<i>h</i>) cms.	Temp. of Manometer ° C.	Time of Flow (T) secs.	Bulb being emptied.	ρhT (reduced to 50°) $\times 10^{-3}$.	$\frac{1}{T}$.
50.07	*108.28	23.5	54.56	R	5.897	.01833
50.07	108.18	"	54.60	L	5.897	.01831
50.07	108.15	"	54.57	R	5.891	.01833
50.07	108.17	"	54.56	L	5.893	.01833
50.20	151.14	23.4	41.69	R	6.316	.02399
50.24	151.14	"	41.64	L	6.325	.02401
50.22	151.08	"	41.72	R	6.321	.02397
50.22	151.08	"	41.58	L	6.300	.02404
50.10	197.40	23.5	34.00	R	6.705	.02941
50.10	197.30	"	33.93	L	6.690	.02948
50.10	197.14	"	33.94	R	6.684	.02947
50.10	197.26	"	33.95	L	6.691	.02946
50.07	†20.30	23.2	26.55	R	7.311	.03767
50.07	20.30	"	26.47	L	7.290	.03779
50.07	20.30	"	26.52	R	7.303	.03772
50.07	20.30	"	26.44	L	7.282	.03783
50.08	30.39	23.9	19.98	R	8.237	.05006
50.08	30.39	"	19.86	L	8.187	.05036
50.08	30.39	"	19.98	R	8.237	.05006
50.08	30.39	"	19.82	L	8.171	.05046
50.10	40.42	24.2	17.51	R	9.606	.05711
50.10	40.42	"	16.43	L	9.016	.06088
50.10	40.42	"	17.54	R	9.624	.05701
50.10	40.42	"	16.22	L	9.900	.06165

* Water Manometer.

† Mercury Manometer.

Typical Observations with Tube IV. in Glischrometer.

Temp. ($^{\circ}$ C.).	Manometer Reading (h) cms.	Temp. of Manometer $^{\circ}$ C.	Time of Flow (T) secs.	Bulb being emptied.	$\frac{\rho h T}{50^{\circ}} \times 10^{-3}$ (reduced to 50°)	$\frac{1}{T}$
50.10	*108.08	26.7	61.26	R	6.614	.01632
50.10	108.10	"	60.80	L	6.561	.01645
50.10	108.09	"	61.83	R	6.675	.01617
50.10	108.10	"	61.21	L	6.607	.01634
50.04	150.97	26.7	46.49	R	6.995	.02151
50.01	151.07	"	46.30	L	6.967	.02160
50.00	150.96	"	46.50	R	6.990	.02150
50.00	150.96	"	46.30	L	7.955	.02160
50.00	197.17	26.0	37.44	R	7.348	.02671
50.00	197.07	"	37.35	L	7.326	.02671
50.00	197.04	"	37.54	R	7.359	.02664
50.00	196.98	"	37.37	L	7.326	.02677
50.10	+20.92	25.7	28.25	R	8.023	.03539
50.10	20.96	"	28.00	L	7.998	.03571
50.10	20.92	"	28.32	R	8.042	.03531
50.10	20.92	"	28.10	L	7.980	.03559
50.10	30.98	25.7	21.30	R	8.954	.04698
50.10	30.99	"	22.80	L	9.594	.04386
50.10	31.00	"	21.40	R	9.010	.04671
50.10	31.00	"	22.58	L	9.498	.04429
50.00	40.47	25.7	20.78	R	11.378	.04812
50.00	40.47	"	21.33	L	11.682	.04688
50.00	40.47	"	20.43	R	11.185	.04895
50.00	40.47	"	21.05	L	11.530	.04750

* Water Manometer.

† Mercury Manometer.

The numbers in columns 6 and 7 were used to obtain the curves given in figs. 2, 3, 4, & 5 (Pl. XIII.). It will be noticed that for a considerable distance the lines are straight, indicating constant values for c in equation (2), and therefore for m for the particular capillary. This constancy is remarkable when we consider the enormous speed with which the water is forced through the tubes in many cases. The individual observations show also that there is no variation in any particular case, and that the value of m in the general formula can be relied upon, when determined in this way. There is, therefore, no necessity to keep the kinetic energy correction small in comparison with the first term in determining viscosities by the efflux method, provided, of course, that the time of flow and pressure can be measured with sufficient accuracy. This will be shown later, where values have been worked out. In certain curves it will also be noticed that at a particular point, there is an abrupt change in the direction of the line, indicating either a largely increased value for m , or a change in the nature of the flow. This is most marked in both the curves for Tube IV. and in one of the curves for Tube III.

The following values for C and c in equation (2) were

obtained from the curves, and the corresponding values of m were deduced by equation (4).

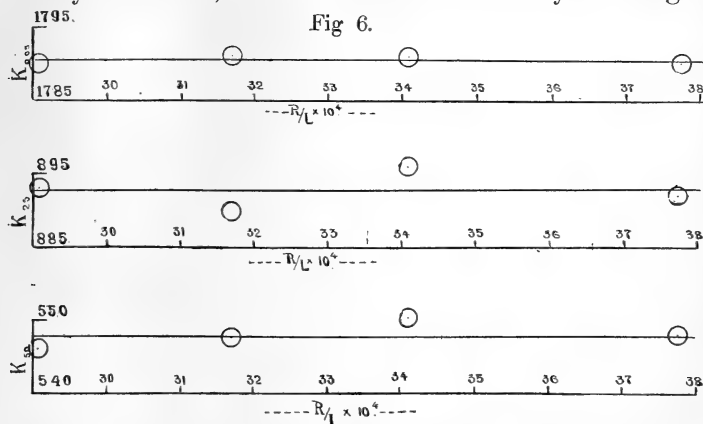
	Tube I.		Tube II.		Tube III.		Tube IV.	
	R to L.	L to R.	R to L.	L to R.	R to L.	L to R.	R to L.	L to R.
C	6350	6330	7400	7440	4570	4570	5400	5390
$10^{-4} \times c$	9.60	9.81	10.00	9.90	7.78	7.78	7.24	7.60
m	1.130	1.164	1.164	1.162	1.128	1.136	1.166	1.216

The values for m are all greater than the theoretical value. There are also two values for each capillary according as the liquid flows in at one end or the other. This fact is most marked in the case of Capillary IV.

The above values for C were used in calculating K in equation (5) for the four tubes. R/L was also calculated. These values are collected in the following table:—

	Tube I.	Tube II.	Tube III.	Tube IV.
K (mean)	·005504	·005463	·005480	·005478
R/L	·003407	·002908	·003774	·003172

It is evident that there is no linear relation between K and R/L . When the above values are plotted in the way already mentioned, it will be noticed that they lie along a



straight line parallel to the axis of abscissæ; that, therefore, K in equation (5) has zero value, *i. e.*, the value for n is zero. See fig. 6,

A set of readings was taken at 25°C. and also at 0°C. , and values for K_{25} and K_0 were found. The results are tabulated below, and the corresponding graphs appear in fig. 6. They bear out the conclusions arrived at in connexion with the results at 50°C.

	Tube I.	Tube II.	Tube III.	Tube IV.
$K_{(25^{\circ})}$	·00896	·00893	·00892	·00890
$K_{(0^{\circ})}$	·01791	·01790	·01790	·01791
R/L	·003407	·002908	·003774	·003172

In the general reduction formula, the most difficult constant to measure accurately is R , the mean radius of efflux. Capillaries are not generally right circular cylinders, nor even elliptical cylinders; and as the degree of precision with which R must be calculated is always four times as great as that required in the deduced viscosity, the examination and measurements of the capillaries must be carried on with extreme care.

Tubes I., II., III., and IV. were in the first place selected from a large number on account of their uniformity of bore—tested with a small mercury column—and their circular end sections.

The first method of measuring R was by contained volumes of mercury. The values obtained (at 0°C.) for the mean radii were ·018968, ·018926, ·020416, and ·020482 cms. respectively.

At the conclusion of all the experiments sections about $\frac{1}{2}$ cm. in thickness were cut from the tubes at regular intervals; they were ground, polished, and mounted in a brass plate. Three independent sets of readings of their dimensions were obtained by me—firstly, by direct comparison with a micrometer eyepiece in a microscope; secondly, by means of a microscope fitted to the dividing engine belonging to the Physics Laboratory, Melbourne University; and, thirdly, by means of a micrometer microscope at the Sydney University. The following average values were obtained for the radii of the sections reduced to 0°C. :—

Capillary I. (Circular).	First Method.	Second Method.	Third Method.	Mean Values.
Section 1	·01905 cm.	·01879 cm.	·01880	·018838
„ 2	·01929 „	·01919 „	·01899	·019106
„ 3	·01932 „	·01933 „	·01877	·019048
„ 4	·01926 „	·01880 „	·01881	·018882

[Mean radius (by mercury) = ·018968 cm.] Mean = ·018968 cm.

Capillary II. (Circular).	First Method.	Second Method.	Third Method.	Mean Values.
Section 1	·01899 cm.	·01889	·01880	·018862
„ 2	·01896 „	·01861	·01880	·018763
„ 3	·01912 „	·01910	·01891	·019009
„ 4	·01906 „	·01909	·01890	·018990

[Mean radius (by mercury) = ·018926 cm.] Mean = ·018906 cm.

Capillary III. (Elliptical).	First Method.	Second Method.	Third Method.	Mean Values.
Section 1	·02023	·02066	·02069	·020603
„ 2	·02016	·01969	·02020	·020023
„ 3	·02083	·02064	·02066	·020682
„ 4	·02016	·02009	·01993	·020022
„ 5	·02124	·02119	·02069	·020948
„ 6	·02043	·01983	·01996	·020000
„ 7	·02097	·02032	·02075	·020643
„ 8	·02043	·02010	·01998	·020095

Mean radius by mercury = ·020416.

Mean $\left\{ \begin{array}{l} (a) \cdot 020719 \\ (b) \cdot 020035 \end{array} \right.$

Capillary IV. (Elliptical).	First Method.	Second Method.	Third Method.	Mean Values.
Section 1	·02138	·02125	·02088	·021087
„ 2	·02023	·02070	·01976	·020152
„ 3	·02130	·02118	·02086	·021040
„ 4	·02043	·02007	·01984	·020015
„ 5	·02144	·02143	·02068	·021057
„ 6	·02023	·01954	·01976	·019765
„ 7	·02183	·02125	·02079	·021042
„ 8	·02023	·01997	·01994	·019914

Mean radius by mercury = ·020482.

Mean $\left\{ \begin{array}{l} (a) \cdot 021056 \\ (b) \cdot 019914 \end{array} \right.$

In determining the mean value for each section, the values obtained by the three methods were weighted in the following manner: Method 1, weight 1; Method 2, weight 2; Method 3, weight 3; thus for Capillary 1, Section 1, by adding

together .01905 cm., twice .01879 cm., and three times .01880 cm., and dividing the sum by 6, we obtain the value .018838 cm. The mean obtained in this way is, I consider, the best value the individual results will produce, taking in account the experimental difficulties in measuring such small bores in the three cases.

The mean values by measurement were then combined with the values obtained by mercury column, and in this way the final values were obtained.

Capillary I., circular cylinder, radius .018968 cm. at 0° C.

„ II., „ „ „ „ .018916 „

„ III., elliptical cylinder, semi-axes (a) .020762 „

(b) .020076 „

„ IV., „ „ „ „ (a) .021061 „

(b) .019919 „

„ III., mean radius of efflux $R \left(\text{where } R^4 = \frac{2a^3b^3}{a^2 + b^2} \right)$

= .020413 cm.

„ IV., mean radius of efflux R = .020474 cm.

An accuracy of 1 in 1000 was aimed at throughout the experiments. The values for the constants in the reduction formula having been obtained, the viscosity was determined from the various observations which had been made from time to time, including those already mentioned.

A special set of experiments at 0° C. was made at a fixed pressure in order to obtain accurate values of the viscosity at 0° C. The thermometer registered 0°·05 C. throughout the series. With Tube I. in the glischrometer, under a pressure of 197 cms. of water, the average time of flow was 109·7 secs. The first term of the reduction formula was found to be .018690, and the second term .000769. The value for the viscosity at 0°·05 was .01792; which reduced to 0° C., becomes .01795. A second determination gave the same values practically. The observations and reductions are given on p. 513, also a similar set at 25° C. p. 514.

Most of the experiments, however, were carried out at 50° C. with the water flowing under different pressures. In the following tables (pp. 515–517) are collected the various results. The pressures are given in centimetres of mercury, the times of flow in seconds, the values of viscosity (double observations) in absolute measure, and the kinetic energy correction—second term—also in absolute measure. The pressures and times are approximate, and the viscosity values are reduced to the even temperature 50° C.

Temperature 0° C.

Tube in Gliscro- meter.	Temp. (<i>t</i>).	Pressure (<i>h</i>).	Time, T.	Direction of Flow.	First Term.	Second Term.	ηt .	η reduced to 0° C.
I.	0.05	197.00	109.99	R to L	·018690	·000769	·01792	·01794
	0.05	197.00	109.55	L to R	·018690	·000769	·01792	
	0.05	197.00	109.84	R	·018690	·000770	·01780	
	0.05	197.00	109.55	L	·018690	·000769	·01792	
II.	0.05	196.90	127.74	R	·018410	·00576	·01783	·01789
	0.05	197.00	127.30	L	·018425	·00576	·01735	
	0.05	196.90	127.87	R	·018430	·00577	·01783	
	0.05	196.90	127.63	L	·018465	·00574	·01789	
III.	0.05	196.95	80.60	R	·018960	·001066	·01789	·01795
	0.05	196.95	80.42	L	·018990	·001065	·01793	
	0.05	196.95	80.72	R	·018990	·001066	·01792	
	0.05	196.95	80.36	L	·018980	·001065	·01792	
IV.	0.05	197.20	93.69	R	·018770	·00809	·01786	·01793
	0.05	197.10	93.88	L	·018770	·00805	·01796	
	0.05	197.10	93.81	R	·018683	·00808	·01787	
	0.05	197.10	93.65	L	·018723	·00806	·01792	

These results give mean value for viscosity of water at 0° C. = ·01793.

Results at 50° C. with Tube I. in Glischrometer.

Pressure (<i>h</i>).	Time of Flow (T).	Viscosity values $\eta \times 10^5$.	Viscosity (mean) $\eta \times 10^5$.	Second Term $\times 10^5$.
7 cm.	71 sec.	$\begin{Bmatrix} 552 \\ 551 \\ 554 \\ 554 \end{Bmatrix}$	553	118
11 "	54 "	$\begin{Bmatrix} 552 \\ 554 \\ 553 \end{Bmatrix}$	553	156
15 "	44 "	$\begin{Bmatrix} 552 \\ 552 \\ 551 \end{Bmatrix}$	551	193
20 "	33 "	$\begin{Bmatrix} 550 \\ 550 \end{Bmatrix}$	550	250
25 "	28.8 "	551	551	292
30 "	24.9 "	550	550	338
31 "	24.4 "	547	547	348
33 "	23.5 "	549	549	357
35 "	22.4 "	$\begin{Bmatrix} 552 \\ 554 \\ 556 \end{Bmatrix}$	554	376
40 "	20.9 "	$\begin{Bmatrix} 590 \\ 592 \end{Bmatrix}$	591*	400

Results at 50° C. with Tube II. in Glischrometer.

Pressure (<i>h</i>) (Approx.).	Time of Flow (T) (Approx.).	Second Term $\times 10^5$.	Viscosity $\times 10^5$.	Mean Viscosity $\times 10^5$.
7.3 cm.	80.5 sec.	91	$\begin{Bmatrix} 549 \\ 550 \end{Bmatrix}$	550
11 "	60.3 "	121	$\begin{Bmatrix} 549 \\ 549 \\ 549 \\ 549 \end{Bmatrix}$	549
15 "	48.3 "	151	$\begin{Bmatrix} 549 \\ 548 \end{Bmatrix}$	548
20 "	36.8 "	199	$\begin{Bmatrix} 549 \\ 548 \end{Bmatrix}$	549
30 "	27 "	269	$\begin{Bmatrix} 553 \\ 553 \end{Bmatrix}$	553
40 "	23 "	316	$\begin{Bmatrix} 557 \\ 561 \end{Bmatrix}$	559*
42 "	21.5 "	340	565	565*

Mean $\cdot 00550$

With Tube I. in the glischrometer, the formula gives constant values up to a pressure of 35.4 cms. of mercury, when the time of flow is about 22 seconds. It will be noticed that at this point the kinetic energy correction is more than 60 per cent. of the viscosity, and the average velocity in the tube is 400 cms. per sec.

In the case of Tube II. the formula breaks down suddenly at a pressure between 35.7 cms. and 36.9 cms., the time of flow being about 24 secs., and the correction at pressure 35.7 cms. more than 55 per cent. of the viscosity. The velocity in this case is 370 cms. per sec.

Results at 50° C. with Tube III. in Glischrometer.

Pressure (<i>h</i>) (Approx.).	Time of Flow (T) (Approx.)	Second Term $\times 10^5$.	Viscosity (double readings) $\eta \times 10^5$.	Viscosity (mean) $\eta \times 10^5$.
7.3 cm.	54.6 sec.	156	{ 552 551 550 549 }	550.5
11 "	41.6 "	205	{ 553 551 }	552.5
15 "	33.9 "	252	{ 552 551 }	551.5
20 "	26.5 "	323	{ 554 554 }	554
30 "	19.9 "	430	{ 557 555 558 556 559 }	557 *
35 "	17.9 "	478	{ 558 562 565 }	562 *
40 "	17.0 "	500	{ 615 607 617 613 604 609 611 }	612 *

Mean .00552

For Tube III. the figures are, pressure between 20.4 cms. and 30.4 cms., time of flow less than 26 secs., second term

60 per cent. of the viscosity; and for Tube IV. when the pressure is 27 cms. and the second term is nearly 60 per cent. of the viscosity, the deduced value is satisfactory, but an increase of pressure of 1 cm. brings about some decided change. The highest velocity reached in both cases, before

Results at 50° C. with Tube IV. in Glischrometer.

Pressure (<i>h</i>) (Approx.).	Time of Flow (T) (Approx.).	Second Term $\times 10^5$.	Viscosity (double readings) $\eta \times 10^5$.	Viscosity (mean) $\eta \times 10^5$.
7.3 cm.	61.5 sec.	123	{ 546 552 }	549
11 "	46.3 "	162	{ 547 547 }	547
15 "	37.5 "	201	{ 544 545 543 547 546 }	545
20 "	28.1 "	268	{ 546 546 546 }	546
24 "	25.3 "	297	...	554
25 "	24.4 "	308	{ 546 546 }	546
26 "	23.8 "	316	...	551
27 "	23.2 "	325	{ 550 548 }	549
28 "	22.7 "	332	...	556 *
29 "	22.6 "	333	...	588 *
30 "	22.3 "	339	...	593 *
31 "	21.9 "	343	...	598 *
32 "	21.9 "	344	...	634 *
34 "	21.5 "	350	...	668 *
36 "	21.2 "	355	...	709 *
38 "	21.0 "	359	...	751 *
40 "	21.0 "	359	{ 822 806 }	814 *
44 "	20.2 "	373	{ 835 832 }	837 *

Mean .00543

the change, was about 340 cms. per sec. The curves in fig. 5 (Pl. XIII.) indicate that when the change takes place, there is a large increase in the value of *m*, if the formula still holds; but the individual results do not agree sufficiently well to enable one to draw definite conclusions from them.

Viscosity Values ($\times 10^5$) at 50° C. collected.

Pressure.	Tube I.	Tube II.	Tube III.	Tube IV.	Average.
7.3 cm.	553	550	551	549	551
11.0 "	553	549	552	547	550
14.7 "	551	548	552	545	549
20 "	550	549	554	546	550
24 "	554	554
25 "	551	546	549
26 "	551	551
27 "	550	549	550
29 "	551	(588)	551
30 "	550	553	(557)	(593)	551
31 "	547	(598)	547
33 "	549	549
34 "	551	(668)	551
35 "	554	...	(562)	...	554
36 "	(574)	(709)	
37 "	(558)	...	(571)	...	
38 "	(578)	(751)	
39 "	(573)	...	(599)	...	
40 "	(591)	(559)	(612)	(814)	
42 "	...	(565)	
43 "	(837)	
Average =	·00551	·00550	·00552	·00548	

Mean Value = ·00550

The values for the viscosity of water at 50° C., obtained with the various capillary tubes in the glischrometer, are collected in the foregoing table. The mean value is ·00550.

Results.

(1) The constants in the reduction formula were all determined with the greatest possible degree of accuracy, including R , n , and m .

(2) For each capillary in the glischrometer—four were used separately—two values for m were found, one for each of the ends. These values were in every case greater than the theoretical value 1.12.

(3) For the series of capillary tubes used, experiments at temperatures 0° C., 25° C., and 50° C. gave in each case zero values for n .

(4) Absolute values for the viscosity of water at 0° C., 26° C., and 50° C. were obtained, namely, ·01793, ·00893, and ·00550; which are probably correct to 0.1 per cent.

(5) The values obtained for m were constant over a big range of pressure; and at a very high pressure there was an indication of an abrupt change in the value of m , or in the nature of the flow. The velocities at this pressure were much below the critical velocities for the various tubes, but were all above the lower limit of critical velocity.

(6) Consistent values for the viscosity of water at 50° C. were obtained in cases where the kinetic energy correction was as high as 60 per cent. of the viscosity.

I have much pleasure in acknowledging my indebtedness to Professor Sir J. J. Thomson, Cavendish Laboratory, Cambridge; Professor Lyle, Melbourne University; Professor Pollock, Sydney University; and Mr. G. H. Knibbs, F.R.A.S., Federal Statistician, formerly Director of Technical Education, N.S.W., and Lecturer in Surveying, University of Sydney, for valuable assistance during the progress of this research, which was commenced at the Cavendish Laboratory, Cambridge, and completed at the Sydney University.

Note on the Viscosity of Solutions.—The viscosity of certain lithium chloride solutions was determined with the glischrometer described in the previous paper. The only novel feature of the measurements was the automatic recording of the time of flow. The inner platinum wires at b and c (fig. 1, p. 503) were connected by insulated wires, also the inner wires at a and d . Wires were fastened to the outer wires at a , b , c , and d , and were connected to four plugs on a double reversing key. The two remaining plugs were joined by wires through a battery and one of the electromagnets already described. With the key in one position, there was electrical communication between the battery and electromagnet and the outer c on the one side, and the outer b on the other side. The circuit was complete only when the solution filled the spaces at both c and b . With the key reversed, the battery was connected to the outer d and the outer a , and the circuit was complete when the solution filled the spaces at a and d . By regulating the amount of solution in the glischrometer the signals could be made as short as necessary, at the beginning and end of the flow from R to L, or in the opposite direction, and the time of flow could be read off accurately on the tape.

The following set of readings will be sufficient to illustrate

the accuracy with which determinations of the viscosity of solutions can be made with this arrangement:—

Lithium Chloride Solution, Temperature 20°·75 C.

Pressure (<i>h</i>).	Time (<i>T</i>).	Correction. × 10 ³ .	η .	η (mean)
{ 198·0 cm.	57·2 sec.	150	·01201	·01200
{ 198·2 "	56·4 "	151	·01198	
{ 179·3 "	62·8 "	136	·01204	·01200
{ 179·1 "	62·8 "	138	·01197	
{ 151·0 "	73·1 "	118	·01200	·01200
{ 151·2 "	73·0 "	118	·01200	
{ 127·1 "	85·6 "	101	·01197	·01198
{ 127·0 "	85·1 "	101	·01198	
{ 100·5 "	106·6 "	81	·01200	·01201
{ 100·0 "	106·8 "	81	·01203	

Average ·01200

XLVII. *The Doppler Effect in Positive Rays.*

By JOHN TROWBRIDGE *.

[Plate XIV.]

THE discovery of canal rays by Goldstein, and that of the Doppler effect in these rays, marks an epoch in the study of the discharge of electricity through gases; for before these discoveries the multitude of confusing effects which arise in the space between the anode and the cathode made it difficult to observe any translation movements. The space, however, behind the cathode is comparatively free for the passage of the positive ions.

We now recognize, in addition to the positive rays behind the cathode—the canal rays—retrograde positive rays which are directed to the anode, or rather away from the cathode in the direction of the anode †.

This later discovery leads one to expect that the Doppler effect should be found also between the anode and the cathode. The result of my study shows that the effect does exist in this region and indicates a movement away from the cathode and toward the anode.

* Communicated by the Author.

† Wehnelt, *Wied. Ann. A.* 1899, p. 421; Runge & Paschen, *Wied. Ann.* lxi. 1897, p. 644; Paschen, *Wied. Ann.* xxiii. 1907, p. 247; Villard, *Comptes Rendus*, cxliii. 1906, p. 673; Goldstein, *Phil. Mag.* March 1908, p. 372; Jacob Kunz, *Phil. Mag.* July 1908, p. 161; J. J. Thomson, *Phil. Mag.* Oct. 1908, p. 657.

The form of tube I employed is represented in fig. 1. The slit of the Rowland grating was at X for the retrograde rays and at Y for the canal rays, A being the anode and C the cathode.

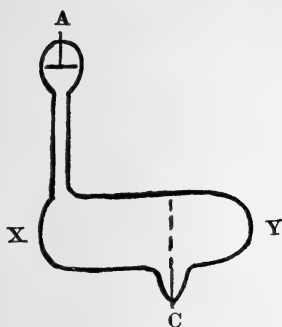


Fig. 1.

The Rowland grating gave, in the order of spectrum I employed, six Ångström units to .9 mm. The effect was observed with respect to the hydrogen line 4861.5, and the change in refrangibility was measured by comparison with the solar spectrum, which was photographed immediately beneath the gaseous line without changing the jaws of the slit.

The amount of the change in refrangibility was sensibly the same as in the canal rays.

The difference of potential between the anode and the cathode varied between 5000 and 10,000 volts; and the current from ten milliamperes to five, furnished by a storage-battery of 10,000 cells. A current of running water provided a large and steady resistance.

The appearance of the discharge at the cathode has often been described. The cathode appears to be the base of two rose-coloured cones of light, the apex of one directed to the anode and the apex of the other toward the canal region. The body of the luminous cone in the space between the anode and the cathode is, so to speak, a solid, while that in the canal region, or back of the cathode, is made up of a collection of tubes which in a short region come together at the apex of the cone, and in a more extended region spread out in a diffused manner.

When the cathode is unperforated the rosy glow which, in the case of hydrogen, characterizes the canal rays, emanates from the central portion of the aluminium cathode; it is no longer conical in form, or rather resembles a frustrum of a cone, the base directed to the anode. It does not extend as far toward the anode as the conical discharge from the perforated cathode, and is not so bright.

The form of tube I employed is represented in fig. 1. The anode was placed in a side tube which was at right angles to the tube containing the cathode C. There was thus the possibility of placing the slit of the spectroscope either at X or at Y.

When an unperforated cathode was employed and the slit

was placed at X, no Doppler effect was seen. When, however, the cathode was perforated the effect was very evident, and indicated a movement towards X which was equal in amount to that observed in the canal region toward Y.

Moreover, the photographs showed, when the light was observed at X, a line on each side of the ordinary stationary hydrogen line—my observations were confined to $\lambda = 4861.5$. There was evidently a movement toward the anode, and a movement away from it at the cathode.

When the observations were conducted at Y the same phenomenon was observed—a stationary hydrogen line and a diffuse line separated from the stationary line by a blank space on each side of the stationary line—indicating a movement toward the cathode and away from it. In fig. 2 (Pl. XIV.) *a* and *b* are photographs, *c* is a drawing which represents the effect too feeble to be strongly reproduced from the photographs—effects, however, which are very evident on the negatives *a* and *b*. The slit was a broad one to show differences of illumination. The light was strongest at the orifices.

When the observations were conducted by placing the slit of the spectroscope so that the light at the perforations did not enter the slit—in other words, placing it obliquely to the band of light, the companion of the stationary line which indicated a movement away in each case, from the slit, was not discernible. The effect took place at the orifices. The positive particles jostling through these orifices and mutually repellent transmit movements—like those resulting from elastic particles in impact—in opposite directions, and those driven in the direction of the anode meet others coming toward the cathode. There results a maximum of radiation of greater refrangibility, which is separated from the refrangibility of the stationary hydrogen line by a less luminous space.

When glass tubes are inserted in the orifices through which the canal rays pass, and the back of the cathode is protected by glass connected to these tubes and to the wall of the discharge-tube, the canal rays are still obtained. This proves that these rays are produced immediately in front of the cathode—that is, on the side toward the anode—or in the orifices. I incline to the belief, as I have stated, that the jostling in the narrow orifices accounts for the change in refrangibility.

In all discharge-tubes striæ are seen opposite to the edge

of the cathode on the glass. These striæ can be localized at a definite point by bringing the edge of the disk of the cathode nearer the wall of the tube; and we then have a source of positive rays which is analogous to that formed by pushing a glass tube surrounding the anode into the Crookes' space*.

The heating and oxidizing effect from the positive particles of these striæ is very marked, especially when the canal region is small, and is much greater than any effect produced by the canal rays in a possible rebounding from the end of the canal region. The photographs of cathode discs exhibit this effect of the positive rays coming from the striæ.

Fig. 3*a* (Pl. XIV.) is a photograph of the back of a cathode made of aluminium from as pure clay as could be obtained. The disk was .75 mm. thick, and the hard surface formed by the iron rolls was left upon it. The heat from the positive rays coming from the striæ caused the occluded gases of the aluminium to form blisters on its surface.

Fig. 3*b* is a photograph of the back of a cathode formed from the same quality of aluminium as in fig. 3*a*, except that the plate of aluminium was treated with nitric acid to remove any trace of iron coming from the rolls. The surface thus lost its polish and hardness. A very black deposit formed under the effect of the positive rays of the striæ, which was probably carbon from the aluminium.

Fig. 3*c* was a cathode formed from ordinary commercial aluminium 1.5 mm. thick. It represents the blackening which results from long running of the discharge. At increased exhaustions and during this long use of the tube the striæ shift their position, and the back of the cathode shows a general discoloration.

Fig. 3*d* represents the front of the cathode of fig. 3*c*. The centre is bright, while the edges are discoloured. Goldstein has noticed that positive rays remove deposits.

Fig. 4 is a photograph of the canal rays which shows also the luminosity on the back of the cathode produced mainly by the striæ.

The order of spectrum produced by the Rowland grating gave an interval of .9 millimetre to six Ångström units. The approximate interval between the Doppler effect on both sides of the stationary hydrogen line 4861.5 was three Ångström units; the difference of potential between the

* E. Wiedemann, *Wied. Ann.* lxiii. 1897, p. 242.

anode and the cathode varied from 6000 volts to 10,000. The current ran from 10 milliamperes to 5. A storage-battery of 10,000 cells was employed. The internal diameter of the discharge-tubes was 3 cm. The distance between X and C (fig. 1) was 6 cm. The distance between C and Y varied from 4 cm. to 10.

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XLVIII. *The Effective Resistance and Inductance of a Concentric Main, and Methods of Computing the Ber and Bei and Allied Functions.* By ALEXANDER RUSSELL, M.A., D.Sc.*

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I. INTRODUCTION.

IN connexion with the investigation of certain phenomena which occur when alternating currents of high frequency flow in thick wires, a knowledge of how the effective resistances and inductances of these wires vary with the frequency is most important. It is of little use to have "standard" inductances in high frequency circuits when we do not know how their values alter with the frequency, and therefore also,

* Communicated by the Physical Society: read January 22, 1909.

with the wave shape of the alternating currents. The "measurement" of inductances by means of alternating currents of unknown wave shape often leads to waste of time. Even when the wave shape is similar to a sine curve, yet if we do not know how the inductance will vary with the frequency the results have only a very limited application.

It would obviously be extremely useful to have formulæ which would take into account the appreciable variation of the density of the current which occurs over the cross section of the wires with high frequency currents. Even to fix inferior and superior limits to the possible values of the inductance would be a great help in many cases. Unfortunately the mathematical difficulties in the way of arriving at a solution in the case of a helical coil are very great. The author, therefore, has made a study of the simplest problem of all, namely that of a concentric main, as a preliminary to attacking the more difficult problems. A study of this problem is also of importance at the present time* in connexion with the discussion that is taking place amongst electrical engineers as to the magnitude of the skin losses in power transmission cables.

The problem was first discussed by Maxwell†. He obtains a few of the terms of a series by means of which the effective resistance of the inner conductor can be computed at low frequencies. Apparently, however, he did not fully appreciate the importance of the results given by the formula. In May ‡ 1884, Oliver Heaviside discussed the "throttling" effect in a core, that is, the increased resistance, the reduced inductance, and the tendency to surface concentration. He uses two functions M and N in his solution, which Kelvin subsequently called the ber and bei functions. In January § 1885, he described clearly the true nature of the current flow in a wire, laying particular stress on the initial surface effects and subsequent penetration. Lord Rayleigh ||, adopting Maxwell's method, next discussed the problem and gave a formula for the effective resistance of the inner conductor at very high frequencies. Oliver Heaviside ¶ subsequently made

* See the report of the evidence on the London Electric Power Bills given before Sir Luke White's Committee in the House of Lords (Nov. 1908).

† 'Electricity and Magnetism,' vol. ii. § 690.

‡ 'The Electrician,' p. 583, May 3, 1884, or 'Electrical Papers,' vol. i. p. 353.

§ 'The Electrician,' Jan. 3, 1885, or 'Electrical Papers,' vol. i. p. 429.

|| Phil. Mag. xxii. pp. 381-394 (1886), or 'Scientific Papers,' vol. ii. p. 486.

¶ 'Electrical Papers,' vol. ii. p. 64 *et seq.*

most important contributions to our knowledge of the subject. He was the first, for example, to give the approximate formula* for the effective resistance of a hollow cylindrical conductor carrying a low frequency current. This formula is a particular case of the general formula given in this paper. He also gives general descriptions of how the current-density varies in the conductors. He omits so many steps, however, in some places that it is very laborious to follow his reasoning. Three years afterwards Lord Kelvin† gave a practical solution for the effective resistance of a solid inner conductor. It is virtually the same as that given by Heaviside. He gave a table, which we shall examine later, of the numerical values in important practical cases. Sir Joseph Thomson‡ also gives practical formulæ for the effective resistance and inductance of a concentric main having a solid inner conductor when traversed by very high frequency currents.

The complete solution given in this paper is obtained from elementary electrical considerations, a knowledge of Ohm's law and of Faraday's law of induction being all that is assumed. The author proves the mathematical formulæ at length, as most of them are new and some of them will be helpful in other physical problems. It will also enable any slips he may have made in the algebraical work to be readily detected and easily rectified. He shows, however, that from his solutions all the previous solutions can be readily deduced, and as most of them are complex functions of the electrical and geometrical data of the main, the errors, if any, must be very minor ones.

II. MATHEMATICAL FORMULÆ.

1. *The Differential Equation.*

The differential equation § to which our problem leads is

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r} \frac{\partial i}{\partial r} = \frac{m^2}{\omega} \frac{\partial i}{\partial t} \quad . \quad . \quad . \quad . \quad (A)$$

where m and ω are constants and i is a periodic function. If we assume that i varies according to the harmonic law, and that its frequency is $\omega/2\pi$, we may write $i = u\epsilon^{\omega t \iota}$, where u is a function of r but not of t , and ι stands for $\sqrt{-1}$.

* *L. c. ante*, p. 192, formula (72).

† Journ. of the Inst. of El. Eng. vol. xviii. p. 4 (1889) or Math. and Phys. Papers, vol. iii. p. 491.

‡ 'Recent Researches,' p. 295.

§ This equation was first discussed by Joseph Fourier [*Mémoires de l'Académie*, Tome iv. (for the year 1819)].

The equation now becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - m^2 u = 0. \quad . \quad . \quad . \quad (B)$$

The solution of this equation is known * to be

$$u = A \cdot I_0(mr\sqrt{i}) + B \cdot K_0(mr\sqrt{i}),$$

where A and B are constants, and $I_0(x)$ and $K_0(x)$ can be computed by means of the following series:—

When x is small,

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \dots \quad . \quad . \quad . \quad (1)$$

and

$$K_0(x) = \alpha \cdot I_0(x) - \log x \cdot I_0(x) + \frac{x^2}{2^2} + \left(1 + \frac{1}{2}\right) \frac{x^4}{2^2 \cdot 4^2} + \dots \quad (2)$$

where $\alpha = \log 2 - \gamma$ and γ is Euler's constant, and so

$$\alpha = 0.1159315.$$

When x is large,

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 + \frac{1^2}{8x} + \frac{1^2 \cdot 3^2}{2(8x)^2} + \dots \right\}, \quad . \quad . \quad (3)$$

and

$$K_0(x) = \sqrt{\frac{\pi}{2x}} \cdot e^{-x} \left\{ 1 - \frac{1^2}{8x} + \frac{1^2 \cdot 3^2}{2(8x)^2} - \dots \right\}. \quad (4)$$

The values of the series (2) and (4), for many values of x , have been computed to a high degree of accuracy by W. S. Aldis †, and tables of $I_0(x)$ from 0 to 5.1 are given in the British Assoc. Reports, 1896, the interval of the argument being 0.001.

2. Kelvin's ber and bei functions.

Kelvin ‡ showed that the effective resistance of the inner conductor of a concentric main may be conveniently expressed in terms of two functions which he called the ber and the bei functions§. He published tables of the values of these functions for a few values of the argument, and from these tables

* See Gray and Matthews, 'Bessel's Functions.'

† Proc. Roy. Soc. vol. lxiv. p. 203.

‡ *L. c. ante*.

§ Noticing that in Heaviside's notation, ber=M and bei=N, Kelvin's formula follows at once from the formula (36) given in Heaviside's 'Electrical Papers,' vol. ii. p. 183.

he computed the numerical value of the effective resistance in various cases.

We may define Kelvin's functions by means of the equation

$$I_0(mr \sqrt{i}) = \text{ber } mr + i \text{ bei } mr.$$

From (1) we deduce at once that

$$\text{ber } mr = 1 - \frac{m^4 r^4}{2^2 \cdot 4^2} + \frac{m^8 r^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \dots \quad (5)$$

and

$$\text{bei } mr = \frac{m^2 r^2}{2^2} - \frac{m^6 r^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots, \quad (6)$$

which are the definitions of them given by Kelvin.

In Mascart and Joubert's *l'Electricité et le Magnetisme*, vol. i. p. 718, several of the values of the functions given in Kelvin's paper have been recomputed and certain corrections made. As the author uses these functions in the solutions given below, it was necessary therefore to recheck the calculations. He at first attempted to do this by direct calculation, but the work proved so laborious that he was led to devise shorter methods of calculating the functions. He found that this was easy and that comparatively simple formulæ can be obtained for them in those cases where the direct computation by (5) and (6) would be laborious.

The formulæ given below can also be usefully employed in simplifying the formulæ ordinarily given for computing the eddy-current losses* in a metallic cylinder, the inductance and resistance of two parallel cylindrical conductors†, the impedance of a solenoid with a cylindrical metal core‡, &c.

3. *Approximate formulæ for the ber and bei functions.*

When the argument is small the functions can be readily computed from the formulæ (5) and (6). These functions generally occur associated together in one or other of the following ways:—

$$X(x) = \text{ber}^2 x + \text{bei}^2 x, \quad Y(x) = \text{ber}'^2 x + \text{bei}'^2 x,$$

$$Z(x) = \text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x,$$

$$\text{and } W(x) = \text{ber } x \text{ bei}' x - \text{bei } x \text{ ber}' x.$$

In these definitions of X, Y, Z and W, $\text{ber}' x$ and $\text{bei}' x$ stand for the differential coefficients of $\text{ber } x$ and $\text{bei } x$ with

* A. Russell, 'Alternating Currents,' vol. i. p. 374.

† J. W. Nicholson, *Phil. Mag.* [6] xvii. p. 255, 1909.

‡ O. Heaviside, 'Electrical Papers,' vols. i. and ii., or R. T. Wells, *Phys. Rev.* xxvi. p. 357 (1908).

respect to x . The combinations Y/X , Z/X , W/X , Z/Y , and W/Y also occur in this and allied problems, and so we shall give formulæ for these functions as well.

By squaring the series for $\text{ber } x$ and $\text{bei } x$ and adding them together, we get

$$X = 1 + \frac{1}{2} \left(\frac{x}{2}\right)^4 + \frac{1}{4 \cdot 4} \left(\frac{x}{2}\right)^8 + \frac{1}{6^2 \cdot 6} \left(\frac{x}{2}\right)^{12} + \frac{1}{8^2 \cdot 9} \left(\frac{x}{2}\right)^{16}, \quad (7)$$

approximately.

When x is not greater than 4 this formula may be used. For instance, when x is 4, (7) gives $X(4) = 11.8275$. From the tables given in Gray and Matthews' 'Bessel's Functions', where the values of $\text{ber } x$ and $\text{bei } x$ are tabulated for values of x up to 6, the interval of the argument being 0.2, we find that $\text{ber } 4 = -2.56342$ and $\text{bei } 4 = 2.29269$, and thus $X(4) = \text{ber}^2 4 + \text{bei}^2 4 = 11.8275$.

Similarly we get the approximate formulæ :—

$$Y = \frac{x^2}{4} \left\{ 1 + \frac{3}{(3)^2} \left(\frac{x}{2}\right)^4 + \frac{10}{(5)^2} \left(\frac{x}{2}\right)^8 + \frac{35}{(7)^2} \left(\frac{x}{2}\right)^{12} \right\}, \quad (8)$$

$$Z = \frac{x^3}{16} \left\{ 1 + \frac{1}{4} \left(\frac{x}{2}\right)^4 + \frac{1}{6 \cdot 6} \left(\frac{x}{2}\right)^8 + \frac{1}{4 \cdot 12^2 \cdot 7} \left(\frac{x}{2}\right)^{12} \right\}, \quad (9)$$

and

$$W = \frac{x}{2} \left\{ 1 + \frac{6}{(3)^2} \left(\frac{x}{2}\right)^4 + \frac{30}{(5)^2} \left(\frac{x}{2}\right)^8 + \frac{140}{(7)^2} \left(\frac{x}{2}\right)^{12} \right\}. \quad (10)$$

When x is not greater than 2 the following formulæ can be employed :—

$$\frac{Y}{X} = \frac{x^2}{4} \left\{ 1 - \frac{5}{12} \left(\frac{x}{2}\right)^4 + \frac{143}{720} \left(\frac{x}{2}\right)^8 - \frac{7661}{4^2 \cdot 7} \left(\frac{x}{2}\right)^{12} \right\}, \quad (11)$$

$$\frac{Z}{X} = \frac{x^3}{16} \left\{ 1 - \frac{11}{24} \left(\frac{x}{2}\right)^4 + \frac{473}{3 \cdot 16} \left(\frac{x}{2}\right)^8 - \frac{304107}{4 \cdot 12^2 \cdot 7} \left(\frac{x}{2}\right)^{12} \right\}, \quad (12)$$

$$\frac{W}{X} = \frac{x}{2} \left\{ 1 - \frac{1}{3} \left(\frac{x}{2}\right)^4 + \frac{19}{15} \left(\frac{x}{2}\right)^8 - \frac{687}{7 \cdot 6^4} \left(\frac{x}{2}\right)^{12} \right\}, \quad (13)$$

$$\frac{Z}{Y} = \frac{x}{4} \left\{ 1 - \frac{1}{24} \left(\frac{x}{2}\right)^4 + \frac{13}{4320} \left(\frac{x}{2}\right)^8 - \frac{647}{12^2 \cdot 360 \cdot 56} \left(\frac{x}{2}\right)^{12} \right\}, \quad (14)$$

and

$$\frac{W}{Y} = \frac{2}{x} \left\{ 1 + \frac{1}{12} \left(\frac{x}{2}\right)^4 - \frac{1}{180} \left(\frac{x}{2}\right)^8 + \frac{11}{12 \cdot 28 \cdot 30} \left(\frac{x}{2}\right)^{12} \right\}. \quad (15)$$

Formula (15) agrees with that found by Lord Rayleigh* by another method. Heaviside† gives $73/(12^2 \cdot 28 \cdot 80)$, that is $657/(12^2 \cdot 360 \cdot 56)$ as the coefficient of $(x/2)^{12}$ in (14).

Formulae (7) to (10) should not be used if x is greater than 4, and formulae (11) to (15) should not be used if x is greater than 2. In practice, therefore, they have only a limited use.

We shall now find approximate formulae for $\text{ber } x$, $\text{bei } x$, X , Y , Z , and W , which can be used with sufficient accuracy for practical purposes when x is not less than 5, and can always be used for computing with a maximum inaccuracy of less than 1 in 10,000, when x is not less than 10, that is in those cases where the labour involved in the direct computation of the series becomes practically prohibitive.

If y denote $\text{ber } x + \iota \text{ bei } x$ we see from the definition we gave of these functions that

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{x} \frac{\partial y}{\partial x} = \iota y. \quad \dots \quad (16)$$

Putting $y = A' \epsilon^\theta / \sqrt{x}$, where A' is a constant, we get

$$\frac{\partial^2 \theta}{\partial x^2} + \left(\frac{\partial \theta}{\partial x} \right)^2 - \iota + \frac{1}{4x^2} = 0. \quad \dots \quad (17)$$

When x is large,

$$\theta = x \sqrt{\iota} + a_0 + b_0 \iota$$

is obviously an approximate solution of (17), where a_0 and b_0 are constants. Let us assume therefore that

$$\theta = x \sqrt{\iota} + a_0 + b_0 \iota + a_1/x + a_2/x^2 + \dots$$

is a solution of (17). Substituting this value for θ , equating the coefficients of x^{-2} , x^{-3} , ... to zero, and noticing that $\sqrt{\iota} = 1/\sqrt{2} + (1/\sqrt{2})\iota$, and $1/\sqrt{\iota} = 1/\sqrt{2} - (1/\sqrt{2})\iota$, we find that

$$\begin{aligned} a_1 &= \frac{1}{8\sqrt{2}} - \frac{\iota}{8\sqrt{2}}, & a_2 &= -\frac{\iota}{16}, \\ a_3 &= -\frac{25}{384\sqrt{2}} - \frac{25\iota}{384\sqrt{2}}, & a_4 &= -\frac{13}{128}, \end{aligned}$$

&c.

Hence we may write

$$\begin{aligned} y\sqrt{x} &= A' \epsilon^{a + a_0 + \beta \iota} \\ &= A \epsilon^a \cos \beta + \iota A \epsilon^a \sin \beta, \end{aligned}$$

* *L. c. ante.*

† 'Electrical Papers,' vol. ii. p. 64. [Dr. Heaviside has written me that he discovered this slip in 1894.]

where

$$\alpha = \frac{x}{\sqrt{2}} + \frac{1}{8\sqrt{2}x} - \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} - \dots \quad (18)$$

and

$$\beta = \frac{x}{\sqrt{2}} + b_0 - \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} - \frac{25}{384\sqrt{2}x^3} + \dots \quad (19)$$

Hence, since $y = \text{ber } x + i \text{ bei } x$, we have

$$\text{ber } x = (A/\sqrt{x}) \epsilon^{\alpha} \cos \beta, \quad \text{and} \quad \text{bei } x = (A/\sqrt{x}) \epsilon^{\alpha} \sin \beta.$$

To determine the values of A and b_0 we notice that

$$\text{ber } x + i \text{ bei } x = I_0(x\sqrt{i}),$$

and thus, from equation (3), we see that A is $1/\sqrt{2\pi}$, and b_0 is $-\pi/8$.

We thus find that when x is large

$$\text{ber } x = \frac{\epsilon^{\alpha}}{\sqrt{2\pi x}} \cos \beta, \quad \dots \quad (20)$$

and

$$\text{bei } x = \frac{\epsilon^{\alpha}}{\sqrt{2\pi x}} \sin \beta, \quad \dots \quad (21)$$

where α and β are given by (18) and (19), and b_0 is $-\pi/8$.

The series for α and β are semi-convergent and a rigorous mathematical justification of (20) and (21) is difficult. It is easy, however, to verify that, if we only include the terms of the series given above, (20) and (21) give the values of the functions with great accuracy when x is greater than 5. The values of α and β are easily computed by the formulæ

$$\alpha = 0.707105x + 0.08839/x - 0.046/x^3, \quad \dots \quad (22)$$

and

$$\beta = 0.707105x - 0.39270 - 0.08839/x - 0.0625/x^2 - 0.046/x^3. \quad (23)$$

Differentiating (20) and (21) we find that

$$\text{ber}' x = \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2} \right) \text{ber } x - \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3} \right) \text{bei } x, \quad (24)$$

and

$$\text{bei}' x = \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3} \right) \text{ber } x + \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2} \right) \text{bei } x. \quad (25)$$

Squaring equations (20) and (21), and adding, we get

$$X = \frac{\epsilon^{2\alpha}}{2\pi x}, \quad \dots \quad (26)$$

When the value of x is not less than 7, the inaccuracy of the formula

$$X = \frac{e^{x\sqrt{2} + 1/4\sqrt{2}x}}{2\pi x}$$

is less than 1 in 10,000.

Heaviside* has given the formula $X = e^{x\sqrt{2}}/2\pi x$, but in order to get a four-figure accuracy with this formula x would have to be greater than 1500.

In a similar manner, we find that

$$Y = X \left(1 - \frac{1}{x\sqrt{2}} + \frac{1}{4x^2} + \frac{3}{8\sqrt{2}x^3} \right), \quad \dots \quad (27)$$

$$Z = X \left(\frac{1}{\sqrt{2}} - \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2} \right), \quad \dots \quad (28)$$

$$W = X \left(\frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} + \frac{1}{8x^3} \right), \quad \dots \quad (29)$$

These formulæ also give the ratios Y/X , Z/X , and W/X . They correspond to (11), (12), and (13), and give a four-figure accuracy when x is not less than 8.

By the binomial theorem, we also readily deduce the following formulæ corresponding to (14) and (15):—

$$\frac{Z}{Y} = \frac{1}{\sqrt{2}} - \frac{3}{8\sqrt{2}x^2} - \frac{3}{8x^3} \quad \dots \quad (30)$$

and

$$\frac{W}{Y} = \frac{1}{\sqrt{2}} + \frac{1}{2x} + \frac{3}{8\sqrt{2}x^2} \quad \dots \quad (31)$$

To test these formulæ let us take the low value of 6 for x . We find from (20) and (21) that

$$\text{ber } 6 = -8.858 \quad \text{and} \quad \text{bei } 6 = -7.335.$$

These results are in exact agreement with their values found by direct computation from the series given in (5) and (6). The accuracy of the formulæ rapidly increases as x increases.

In the following table the numbers obtained by substituting 10 for x in formulæ (20), (21), (24), and (25) are compared with the numbers given by Kelvin† and by Mascart and Joubert‡.

* 'Electrical Papers,' vol. ii. p. 184.

† *L.c. ante*.

	ber 10.	bei 10.	ber' 10.	bei' 10.
Kelvin	138·8405	56·3704	51·373	135·23
Mascart & Joubert ...	138·840	56·370	51·207	135·31
Formulæ	138·840	56·368	51·202	135·29

Before the author worked out the approximate formulæ, he had verified Mascart and Joubert's corrections as far as four figures by direct calculation from the series given in (5) and (6) and the series obtained by differentiating them. As the corrections to Kelvin's table when the argument is 15 and when it is 20 are large, it will be interesting to calculate these values by our formulæ.

	ber 15.	bei 15.	ber' 15.	bei' 15.
Kelvin	-2969·79	-2952·33	86·648	-4089·2
Mascart & Joubert ...	-2967·26	-2952·72	91·061	-4088·5
Formulæ	-2967·26	-2952·66	91·010	-4087·7

	ber 20.	bei 20.	ber' 20.	bei' 20.
Kelvin	47583·7	11500·8	24325·1	41491·5
Mascart & Joubert ...	47489·2	114774·4	-43802·8	111853
Formulæ	47491·6	114770·1	-48797·9	111853

In evaluating the formulæ we have only used 7-figure logarithmic tables. To compute ber' 15 correctly we ought to have calculated ber 15 and bei 15 to eight significant figures, as ber 15 and bei 15 are nearly equal to one another and from (24) we see that ber' 15 is nearly equal to their difference. It will be seen that the accuracy of Mascart and Joubert's corrections is satisfactory. The values, however, they give for the functions when the argument is 30 are not correct.

	ber 30.	bei 30.	ber' 30.	bei' 30.
Mascart & Joubert ...	-4544×10^4	11024×10^4	-10933×10^4	4383×10^4
Formulae	-4611×10^4	10995×10^4	-10959×10^4	4330×10^4

Hence the ordinary tables need revision. A more useful set of tables, however, might be constructed of the functions X , Y , Z , W , Y/X , Z/X , W/X , Z/Y , and W/Y . The ber and bei functions never occur alone in any of the practical formulæ using these functions, with which the author is acquainted.

For instance, Heaviside* and Kelvin* have proved that the ratio of the effective resistance of the inner core of a concentric main with high frequency currents to its resistance with direct currents can be written down almost at once, when the ratio of W to Y is known. It was in fact in order to find the value of this ratio that Kelvin had a table of ber and bei functions computed.

The values of W/Y given in Kelvin's paper and found by (15) and also by direct computation are compared in the following table :—

x .	Values given in Kelvin's paper.	Values computed by (15).	True Values.
0.5	4.0000	4.0013	4.0013
1.0	2.00014	2.0104	2.0104
1.5	1.3678	1.3678	1.3678
2.0	1.0805	1.0782	1.0782

In Kelvin's table bei'1 is given as 0.4999, but its true value is 0.4974. Making this correction and using Kelvin's figures, we get 2.0104 for the value of W/Y when x is 1.

Including the next term in the expansion of W/Y given in (31), we find that

$$\frac{W}{Y} = 0.7071 + \frac{1}{2x} + \frac{0.265}{x^2} - \frac{0.35}{x^4} \dots \quad (32)$$

The values of W/Y given in the first column of the following table have been computed by this formula.

* *L. c. ante.*

x .	Kelvin.	Mascart & Joubert.	Formula (32).
5	0.8172	0.8172
5.5	0.8069	0.8064
6	0.7979	0.7976
8	0.7739	0.7737
10	0.7588	0.7596	0.7597
15	0.7431	0.7416	0.7416
20	0.7325	0.7328	0.7328
30	0.7251	0.7241
40	0.7196	0.7193
50	0.7172	0.7172
∞	0.7071	0.7071

As the approximate formulæ given above are very simple, it will be seen that tables need only be constructed for values of x lying between 2 and 5 or 6, although to have tables of other values would doubtless be a great convenience to those who have to use the formulæ.

4. Particular solution.

We have seen that a solution of the equation

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r} \frac{\partial i}{\partial r} = \frac{m^2}{\omega} \frac{\partial i}{\partial t}$$

is
$$i = I_0(mr \sqrt{\epsilon}) \epsilon^{\omega t}$$

$$= (\text{ber } mr + \epsilon \text{ bei } mr)(\cos \omega t + \epsilon \sin \omega t).$$

Hence, since both the real and imaginary parts of this solution must satisfy the differential equation, we see that a particular solution may be written in either of the following forms:

$$i = (A \text{ ber } mr + B \text{ bei } mr) \cos \omega t + (-A \text{ bei } mr + B \text{ ber } mr) \sin \omega t, \quad . \quad . \quad (33)$$

or
$$i = (A^2 + B^2)^{1/2} (\text{ber}^2 mr + \text{bei}^2 mr)^{1/2} \cos (\omega t - \epsilon), \quad . \quad (34)$$

where A and B are constants, and

$$\tan \epsilon = (-A \text{ bei } mr + B \text{ ber } mr) / (A \text{ ber } mr + B \text{ bei } mr).$$

For a solid core and an infinitely thin return conductor of infinite conductivity this solution suffices, and is the one

given by Kelvin. When, however, the core is hollow* or when we wish to take into account the effects of the return conductor, the complete solution has to be found, as the above solution cannot be made to satisfy all the boundary conditions.

5. The *ker* and *kei* functions.

Imitating Kelvin we shall write the second type of solution of (B), namely $K_0(mr\sqrt{i})$, in the form $\ker mr + i \operatorname{kei} mr$. Substituting $x\sqrt{i}$ for x in equation (2) and equating the coefficients of the real and imaginary terms on the sides of the equation, we find that

$$\ker x = (\alpha - \log x) \operatorname{ber} x + (\pi/4) \operatorname{bei} x \\ - (1 + \frac{1}{2}) \frac{x^4}{2^2 \cdot 4^2} + (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \dots, \quad (35)$$

and

$$\operatorname{kei} x = (\alpha - \log x) \operatorname{bei} x - (\pi/4) \operatorname{ber} x \\ + \frac{x^2}{2^2} - (1 + \frac{1}{2} + \frac{1}{3}) \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots, \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where $\alpha = 0.1159315\dots$

6. Approximate formulæ for the *ker* and *kei* functions.

When x is small, we may use the following approximate formulæ, which can easily be proved by (5) and (6):—

$$\ker mx = \alpha - \log mx + \frac{\pi}{4} \cdot \frac{m^2 x^2}{4} - (\alpha + \frac{3}{2} - \log mx) \frac{m^4 x^4}{64} - \frac{\pi}{4} \cdot \frac{m^6 x^6}{2^2 \cdot 4^2 \cdot 6^2}, \quad . \quad (37)$$

$$\operatorname{kei} mx = -\frac{\pi}{4} + (\alpha + 1 - \log mx) \frac{m^2 x^2}{4} + \frac{\pi}{4} \cdot \frac{m^4 x^4}{64} - (\alpha + \frac{11}{6} - \log mx) \frac{m^6 x^6}{2^2 \cdot 4^2 \cdot 6^2}, \quad (38)$$

$$\ker' mx = -\frac{1}{mx} + \frac{\pi}{4} \cdot \frac{mx}{2} - (\alpha + \frac{5}{4} - \log mx) \frac{m^3 x^3}{16} - \frac{\pi}{4} \cdot \frac{m^5 x^5}{2^2 \cdot 4^2 \cdot 6}, \quad . \quad . \quad . \quad (39)$$

and

$$\operatorname{kei}' mx = (\alpha + \frac{1}{2} - \log mx) \frac{mx}{2} + \frac{\pi}{4} \cdot \frac{m^3 x^3}{16} - (\alpha + \frac{5}{3} - \log mx) \frac{m^5 x^5}{2^2 \cdot 4^2 \cdot 6}. \quad . \quad (40)$$

When mx is small we see that

$$\ker mx = \alpha - \log mx \quad \text{and} \quad \operatorname{kei} mx = -\pi/4, \quad \text{approximately;}$$

and hence, when mx is very small, we may write

$$\ker^2 mx + \operatorname{kei}^2 mx = (\log mx)^2. \quad . \quad . \quad . \quad (41)$$

* The solution given in Mascart & Joubert, vol. i. p. 719 (1897), for the effective resistance of a hollow inner core is incorrect.

When x is not small we get suitable formulæ for calculating these functions by noticing that

$$\theta = -x\sqrt{i} + a_0' + b_0'\sqrt{i}$$

is also an approximate solution of (17). Hence, finding a series in descending powers of x for θ , and determining the constants by (4), we find that

$$\ker x = \sqrt{\frac{\pi}{2x}} \epsilon^{a'} \cos \beta', \quad . \quad . \quad . \quad (42)$$

and

$$\operatorname{kei} x = \sqrt{\frac{\pi}{2x}} \epsilon^{a'} \sin \beta', \quad . \quad . \quad . \quad (43)$$

where

$$\alpha' = -\frac{x}{\sqrt{2}} - \frac{1}{8\sqrt{2}x} + \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} + \dots \quad (44)$$

and

$$\beta' = -\frac{x}{\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} + \frac{25}{384\sqrt{2}x^3} - \dots \quad (45)$$

It will be seen that α' and β' can be deduced from the formulæ for α and β , (18) and (19), by merely changing the sign of x in the latter. In making calculations it is best to use the formulæ obtained by writing $-x$ for x on the right-hand side of the equations (22) and (23).

By differentiating (42) and (43) we find that

$$\begin{aligned} \ker' x = & -\ker x \left\{ \frac{1}{\sqrt{2}} + \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2} \right\} \\ & + \operatorname{kei} x \left\{ \frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} - \frac{1}{8x^3} \right\}, \quad . \quad . \quad . \quad (46) \end{aligned}$$

and

$$\begin{aligned} \operatorname{kei}' x = & -\ker x \left\{ \frac{1}{\sqrt{2}} + \frac{1}{8\sqrt{2}x^2} - \frac{1}{8x^3} \right\} \\ & - \operatorname{kei} x \left\{ \frac{1}{\sqrt{2}} + \frac{1}{2x} - \frac{1}{8\sqrt{2}x^2} \right\}. \quad . \quad . \quad . \quad (47) \end{aligned}$$

We also have

$$X_1(x) = \ker^2 x + \operatorname{kei}^2 x = \frac{\pi \epsilon^{2a'}}{2x} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and

$$Y_1(x) = \ker'^2 x + \operatorname{kei}'^2 x = X_1(x) \left\{ 1 + \frac{1}{x\sqrt{2}} + \frac{1}{4x^2} - \frac{3}{8\sqrt{2}x^3} \right\}. \quad (49)$$

When mx is very large we shall use the formulæ

$$\ker mx = \sqrt{\frac{\pi}{2mx}} e^{-\frac{mx}{\sqrt{2}}} \cos\left(\frac{mx}{\sqrt{2}} + \frac{\pi}{8}\right), \quad . \quad . \quad (50)$$

$$\keri mx = -\sqrt{\frac{\pi}{2mx}} e^{-\frac{mx}{\sqrt{2}}} \sin\left(\frac{mx}{\sqrt{2}} + \frac{\pi}{8}\right), \quad . \quad . \quad (51)$$

$$\ker' mx = -\sqrt{\frac{\pi}{2mx}} e^{-\frac{mx}{\sqrt{2}}} \cos\left(\frac{mx}{\sqrt{2}} - \frac{\pi}{8}\right), \quad . \quad . \quad (52)$$

and

$$\keri' mx = \sqrt{\frac{\pi}{2mx}} e^{-\frac{mx}{\sqrt{2}}} \sin\left(\frac{mx}{\sqrt{2}} - \frac{\pi}{8}\right). \quad . \quad . \quad (53)$$

7. *Formulæ containing both functions.*

We require the following formulæ, also, in our solutions. When mc is small,

$$\begin{aligned} S_c &= \text{ber}' mc \ker' mc + \text{bei}' mc \keri' mc \\ &= \frac{m^2 c^2}{4} \left(\alpha + \frac{3}{4} - \log mc\right) \\ &\quad + \frac{m^6 c^6}{12 \cdot 64} (\alpha - \log mc) + \frac{37 m^6 c^6}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8}, \quad . \quad (54) \end{aligned}$$

and

$$\begin{aligned} T_c &= \text{bei}' mc \ker' mc - \text{ber}' mc \keri' mc \\ &= -\frac{1}{2} + \frac{\pi}{4} \cdot \frac{m^2 c^2}{4} - \frac{m^4 c^4}{48} \quad . \quad . \quad . \quad . \quad . \quad (55) \end{aligned}$$

When mc is very great, we may write

$$S_c = -\frac{\cos mc \sqrt{2}}{2mc}, \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and

$$T_c = -\frac{\sin mc \sqrt{2}}{2mc}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

8. *The complete solution.*

On the assumption that i follows the harmonic law, we see that the complete solution of the equation (A) is

$$\begin{aligned} i &= (A \text{ber } mr + B \text{bei } mr + C \ker mr + D \keri mr) \cos \omega t \\ &\quad + (-A \text{bei } mr + B \text{ber } mr - C \keri mr + D \ker mr) \sin \omega t, \quad (58) \end{aligned}$$

where A, B, C, and D are constants.

III. THE FORMULÆ FOR THE EFFECTIVE RESISTANCE AND INDUCTANCE OF A CONCENTRIC MAIN WITH A SOLID INNER CONDUCTOR.

In order to simplify the problem, we shall first suppose that the inner conductor is a solid metal cylinder of radius a , and that the outer conductor is a coaxial hollow cylinder of inner and outer radii b and c respectively. Let μ be the value of the permeability of the metals forming the conductors, and let μ' be the permeability of the insulating material separating them. Let ρ be the volume resistivity of the conducting metal. We shall suppose that μ , μ' , and ρ are constants and, for the present, that both the capacity and leakage currents in the dielectric can be neglected. We can assume, therefore, that the flow of current in the conductors is parallel to their common axis, and hence, that the equipotential surfaces in each conductor are planes perpendicular to this axis.

Let us now consider the current in a cylindrical tube of unit length in the inner conductor, whose inner and outer radii are r and $r+dr$ respectively. If e_1 be the potential difference between the ends of this tube, the equation to determine the current-density i in it is, by Ohm's law and Faraday's law,

$$\begin{aligned} e_1 &= (\rho/2\pi r \partial r)(i . 2\pi r \partial r) + \partial \phi / \partial t \\ &= \rho i + \partial \phi / \partial t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59) \end{aligned}$$

where ϕ is the number of magnetic lines linked with the current in this cylindrical tube.

By hypothesis, the equipotential surfaces in the inner conductor are planes perpendicular to the axis. Hence e_1 is independent of the value of r , and thus

$$0 = \rho \frac{\partial i}{\partial r} + \frac{\partial}{\partial t} \frac{\partial \phi}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

From the symmetry of a concentric main, we see that the intensity of the current is the same at all points equidistant from the axis. Hence, since the magnetic force outside an infinite cylindrical tube, carrying a current flowing parallel to its axis, is the same as if all the current were concentrated at this axis, we have, at all points of the inner conductor,

$$\phi = \mu \int_a^x \frac{2I_x}{x} dx + 2\mu' I \log \frac{b}{a} + \mu \int_b^c \frac{2(I - I'_x)}{x} dx, \quad (61)$$

where I_x is the algebraical sum of the currents flowing through the cross section of a coaxial cylinder whose radius

is x , I is the total current flowing in the inner conductor, and I'_x is the sum of the currents flowing in the tube of the outer conductor whose inner radius is b and outer radius is x .

By differentiating ϕ with respect to r , we get

$$\frac{\partial \phi}{\partial r} = -\frac{2\mu}{r} \int_0^r 2\pi i r \partial r,$$

and hence, by (60),

$$\rho \frac{\partial i}{\partial r} = \frac{4\pi\mu}{r} \int_0^r r \frac{\partial i}{\partial t} \partial r, \quad . \quad . \quad . \quad (62)$$

and finally, by differentiating,

$$\frac{\partial^2 i}{\partial r^2} + \frac{1}{r} \frac{\partial i}{\partial r} = \frac{4\pi\mu}{\rho} \frac{\partial i}{\partial t} \quad . \quad . \quad . \quad (63)$$

Writing

$$m^2 = \frac{4\pi\mu\omega}{\rho} = \frac{8\pi^2\mu f^2}{\rho}, \quad . \quad . \quad . \quad (64)$$

where f is the frequency of the alternating currents, we see that the value of i is given by (58).

Let us suppose that the current-density along the axis of the cylinder is given by $i = i_0 \cos \omega t$. At the axis r is zero; and since $\ker 0$ is infinite both C and D must be zero, as otherwise the current-density would be infinite. Since also $\operatorname{ber} 0 = 1$ and $\operatorname{bei} 0 = 0$, we have $A = i_0$ and $B = 0$, and thus

$$i = i_0 \operatorname{ber} mr \cos \omega t - i_0 \operatorname{bei} mr \sin \omega t. \quad . \quad . \quad (65)$$

It is easy to verify by the help of the equations

$$\int r \operatorname{ber} mr \partial r = (r/m) \operatorname{bei}' mr \quad . \quad . \quad . \quad (66)$$

$$\text{and} \quad \int r \operatorname{bei} mr \partial r = -(r/m) \operatorname{ber}' mr \quad . \quad . \quad (67)$$

that this value of i , which is the same as that given by Kelvin, satisfies (62). It therefore gives the solution of the problem of finding the current-density at any point in the substance of the inner conductor.

From (65) we find that

$$\begin{aligned} I_r &= 2\pi \int_0^r i r \partial r \\ &= (2\pi/m) r \operatorname{bei}' mr \cdot i_0 \cos \omega t + (2\pi/m) r \operatorname{ber}' mr \cdot i_0 \sin \omega t. \end{aligned} \quad (68)$$

Putting r equal to a in this equation, we get for the total current,

$$I = (2\pi/m) a \operatorname{bei}' ma \cdot i_0 \cos \omega t + (2\pi/m) a \operatorname{ber}' ma \cdot i_0 \sin \omega t. \quad (69)$$

By differentiating this equation with regard to t , we find that

$$\frac{\partial I}{\partial t} = -(2\pi\omega/m) a \operatorname{bei}' ma \cdot i_0 \sin \omega t + (2\pi\omega/m) a \operatorname{ber}' ma \cdot i_0 \sin \omega t, \quad (70)$$

and hence, solving these equations for $i_0 \cos \omega t$ and $i_0 \sin \omega t$, we get

$$i_0 \cos \omega t = \frac{m \operatorname{bei}' ma}{2\pi a Y(ma)} I + \frac{m \operatorname{ber}' ma}{2\pi a \omega Y(ma)} \frac{\partial I}{\partial t}, \quad (71)$$

and
$$i_0 \sin \omega t = \frac{m \operatorname{ber}' ma}{2\pi a Y(ma)} I - \frac{m \operatorname{bei}' ma}{2\pi a \omega Y(ma)} \frac{\partial I}{\partial t}. \quad (72)$$

Let us now consider the currents in the outer conductor. Let i' be the current density in this cylinder at a distance r from the axis. Let e' be the potential-difference per unit length measured from the distributing station to the alternator, i' being considered positive when flowing in the same direction.

We have, therefore,

$$e' = (\rho/2\pi r \partial r)(2\pi r \partial r \cdot i') - \partial \phi' / \partial t \\ = \rho i' - \partial \phi' / \partial t, \quad (73)$$

where

$$\phi' = \mu \int_r^c \frac{2(I - I_r)}{x} \partial x. \quad (74)$$

Thus

$$r \frac{\partial \phi'}{\partial r} = -2\mu(I - I_r) \\ = -4\pi\mu \int_r^c r i' \partial r. \quad (75)$$

Also, since we have supposed that e' does not vary with r , we get from (73) and (75)

$$\rho \frac{\partial i'}{\partial t} = -\frac{4\pi\mu}{r} \int_r^c r \frac{\partial i'}{\partial t} \partial r; \quad (76)$$

and thus
$$\frac{\partial^2 i'}{\partial r^2} + \frac{1}{r} \frac{\partial i'}{\partial r} = \frac{m^2}{\omega} \frac{\partial i'}{\partial t}. \quad (77)$$

By (58), the solution of (77) may be written as follows:—

$$i' = (A \operatorname{ber} mr + B \operatorname{bei} mr + C \operatorname{ker} mr + D \operatorname{kei} mr) i_0 \cos \omega t \\ + (-A \operatorname{bei} mr + B \operatorname{ber} mr - C \operatorname{kei} mr + D \operatorname{ker} mr) i_0 \sin \omega t, \quad (78)$$

where A , B , C , and D are constants which have to be determined from the data of the problem.

Now noticing that

$$\int r \ker mr \partial r = (r/m) \ker' mr \quad . \quad . \quad . \quad (79)$$

$$\text{and} \quad \int r \ker' mr \partial r = -(r/m) \ker' mr, \quad . \quad . \quad (80)$$

we find, by substituting the value of i' given by (78) in (76), that

$$A \operatorname{bei}' mc - B \operatorname{ber}' mc + C \ker' mc - D \ker' mc = 0, \quad (81)$$

and

$$A \operatorname{ber}' mc + B \operatorname{bei}' mc + C \ker' mc + D \ker' mc = 0. \quad (82)$$

By equating also the integral value $\int_b^c 2\pi r i' \partial r$ of the current in the return conductor to the value of I given by (69), we get

$$A \operatorname{bei}' mb - B \operatorname{ber}' mb + C \ker' mb - D \ker' mb = -\frac{a}{b} \operatorname{bei}' ma, \quad (83)$$

and

$$A \operatorname{ber}' mb + B \operatorname{bei}' mb + C \ker' mb + D \ker' mb = -\frac{a}{b} \operatorname{ber}' ma. \quad (84)$$

The four equations (81)–(84) completely determine the four constants A , B , C , and D . Hence the current density at all points on the outer conductor is found.

From (73) and (74) we can see at once that

$$e' = \rho i'_b - \mu \frac{\partial}{\partial t} \int_b^c \frac{2(I - I'_x)}{x} \partial x, \quad . \quad . \quad . \quad (85)$$

where i'_b is the current density on the inner surface of the outer conductor. We also see from (59) and (61) that

$$e = \rho i_a + 2\mu' \log \frac{b}{a} \frac{\partial I}{\partial t} + \mu \frac{\partial}{\partial t} \int_b^c \frac{2(I - I'_x)}{x} \partial x, \quad . \quad (86)$$

where i_a is the current density on the outer surface of the inner conductor. Thus, by addition,

$$e + e' = \rho i_a + \rho i'_b + 2\mu' \log \frac{b}{a} \frac{\partial I}{\partial t}. \quad . \quad . \quad . \quad (87)$$

Hence writing for i_a and i'_b their values given by (65) and (68) respectively, and also writing for $i_0 \cos \omega t$ and $i_0 \sin \omega t$ their values in terms of I and $\partial I / \partial t$ from (71) and (72), we get, after a little reduction, that

$$e + e' = RI + L \frac{\partial I}{\partial t}, \quad . \quad . \quad . \quad . \quad (88)$$

where

$$\begin{aligned} R = & (\rho m / 2\pi a Y_a) (\text{ber } ma \text{ bei}' ma - \text{bei } ma \text{ ber}' ma) \\ & + \frac{\rho m \text{ bei}' ma}{2\pi a Y_a} \{A \text{ ber } mb + B \text{ bei } mb + C \text{ ker } mb + D \text{ kei } mb\} \\ & - \frac{\rho m \text{ ber}' ma}{2\pi a Y_a} \{A \text{ bei } mb - B \text{ ber } mb + C \text{ kei } mb - D \text{ ker } mb\}, \\ & \dots \dots \dots (89) \end{aligned}$$

and

$$\begin{aligned} L = & 2\mu' \log \frac{b}{a} + \frac{2\mu}{ma Y_a} (\text{ber } ma \text{ ber}' ma + \text{bei } ma \text{ bei}' ma) \\ & + \frac{2\mu \text{ ber}' ma}{ma Y_a} \{A \text{ ber } mb + B \text{ bei } mb + C \text{ ker } mb + D \text{ kei } mb\} \\ & + \frac{2\mu \text{ bei}' ma}{ma Y_a} \{A \text{ bei } mb - B \text{ ber } mb + C \text{ kei } mb - D \text{ ker } mb\}. \\ & \dots \dots \dots (90) \end{aligned}$$

In (89) and (90), Y_a stands for $\text{ber}'^2 ma + \text{bei}'^2 ma$, and A, B, C, and D can be found as follows from the equations (81)–(84), using the notation adopted in formulæ (54) and (55):—

$$A Y_c = -C S_c + D T_c \dots \dots \dots (91)$$

$$\text{and} \quad B Y_c = -C T_c - D S_c \dots \dots \dots (92)$$

Hence substituting the values of A and B, given by (91) and (94), in (83) and (84), we get

$$\begin{aligned} & C(Y_c S_b - Y_b S_c) + D(Y_b T_c - Y_c T_b) \\ & = -\frac{a}{b} Y_c (\text{ber}' ma \text{ ber}' mb + \text{bei}' ma \text{ bei}' mb), \quad (93) \end{aligned}$$

$$\begin{aligned} \text{and} \quad & -C(Y_b T_c - Y_c T_b) + D(Y_c S_b - Y_b S_c) \\ & = -\frac{a}{b} Y_c (\text{ber}' ma \text{ bei}' mb - \text{bei}' ma \text{ ber}' mb). \quad (94) \end{aligned}$$

From these equations C and D are easily found, and then A and B can be found from (91) and (92).

The above formulæ give the complete solution of the problem when the applied wave is sine-shaped and the capacity and leakage effects are neglected.

If the applied wave be not sine-shaped, but if it be a periodic function of the time, it may be expanded in a series of sines by Fourier's theorem; and hence we could write down the solution without difficulty, but it would be very cumbersome.

IV. SIMPLIFIED FORMULÆ FOR PARTICULAR CASES.

1. *With direct currents.*

From the next solution, by putting m equal to zero, we find that the resistance R_d and the inductance L_d with direct currents are given by

$$R_d = \frac{\rho}{\pi a^2} + \frac{\rho}{\pi(c^2 - b^2)}, \quad . \quad . \quad . \quad (95)$$

and

$$L_d = 2\mu' \log \frac{b}{a} + \frac{\mu}{2} + \frac{2\mu c^4}{(c^2 - b^2)^2} \log \frac{c}{b} - \mu \frac{3c^2 - b^2}{2(c^2 - b^2)}. \quad (96)$$

This value for the inductance agrees with that found by Lord Rayleigh*.

2. *With low frequency currents.*

Substituting the appropriate approximate formulæ (54) and (55) for S_c and T_c in the equations (91)–(94), and noticing that $Y_c = (m^2 c^2 / 4)(1 + m^4 c^4 / 192)$ approx., we find that

$$C = C_1 m^4 + C_2 m^8, \quad . \quad . \quad . \quad (97)$$

where

$$C_1 = \frac{a^2 c^2 (b^2 - a^2)}{16(c^2 - b^2)} - \frac{a^2 b^2 c^4}{4(c^2 - b^2)^2} \log \frac{c}{b},$$

and

$$C_2 = \frac{c^2 a^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8(c^2 - b^2)} \left\{ \frac{b^2 c^2 (61 b^2 + 19 c^2 - 42 a^2)}{(b^2 - a^2)(5 a^2 b^2 - a^4 - 7 b^4)} \right\} \\ - \frac{a^2 b^2 c^4 \log (c/b)}{2^2 \cdot 4^2 \cdot 12(c^2 - b^2)^2} \left\{ 14 c^2 b^2 - a^4 - 2 b^4 + 3 a^2 b^2 \right\} \\ - \frac{a^2 b^4 c^6 (b^2 - a^2)}{4^3 (c^2 - b^2)^3} \left\{ \log \frac{c}{b} \right\}^2 \\ + \frac{a^2 b^6 c^8}{4^2 (c^2 - b^2)^4} \left\{ \log \frac{c}{b} \right\}^3.$$

$$-D = D_1 m^2 + D_2 m^6, \quad . \quad . \quad . \quad (98)$$

* Phil. Mag. [5] vol. xxi. p. 381 (1886), or Russell's 'Alternating Currents,' vol. i. p. 53.

In power transmission cables $a^2 = c^2 - b^2$. In this case the greater the value of b the smaller the value of $\rho_1 + \rho_2 \xi + \rho_3 \xi^2$. In power transmission cables, when the frequency is not greater than 50, the increase in the effective resistance of the outer conductor due to the skin effect is negligibly small.

We find in a similar manner that

$$L = L_d - \frac{\mu}{2} \cdot \frac{a^4}{192} m^4 \\ - \mu (\lambda_1 + \lambda_2 \xi + \lambda_3 \xi^2 + \lambda_4 \xi^3) m^4, \quad . \quad . \quad (102)$$

where

$$\lambda_1 = \frac{19c^6 + 103c^4b^2 - 41b^4c^2 + 3b^6}{2^2 \cdot 4^2 \cdot 6^2 (c^2 - b^2)}$$

$$\lambda_2 = -\frac{14b^2c^4(2c^2 - b^2)}{2^2 \cdot 4^2 \cdot 3(c^2 - b^2)^2},$$

$$\lambda_3 = -\frac{b^4c^6}{4(c^2 - b^2)^3},$$

$$\lambda_4 = \frac{b^4c^8}{2(c^2 - b^2)^4}, \quad \text{and} \quad \xi = \log \frac{c}{b}.$$

Heaviside* gives the formula as if the coefficient of m^4 were zero. The term $-(\mu a^4/384)m^4$ can be deduced from the formula given in Gray and Matthews' 'Bessel's Functions,' p. 160, as the "self-inductance" of a cylindrical conductor.

Formula (102) shows that at low frequencies the inductance diminishes as the frequency increases, the effect being more pronounced the thicker the shell of the outer conductor.

3. *With high frequency currents.*

When ma is greater than 5 we can use the formulæ (50) to (53) for the ker functions and the corresponding formulæ for the ber functions. Substituting these values in (89) and (90) we get, after a little reduction,

$$R = \frac{\rho m}{2\pi a} \left\{ \frac{1}{\sqrt{2}} + \frac{1}{2ma} + \frac{3}{8\sqrt{2}m^2a^2} \right\} \\ + \frac{\rho m}{2\pi b \sqrt{2}} \cdot \frac{\sinh m(c-b) \sqrt{2} + \sin m(c-b) \sqrt{2}}{\cosh m(c-b) \sqrt{2} - \cos m(c-b) \sqrt{2}}, \quad (103)$$

* 'Electrical Papers,' vol. ii. p. 192. [Dr. Heaviside has pointed out to me that he was only giving the first term correction to $R + L\omega$, and thus it was unnecessary to give the coefficient of m^4 in (102).]

and

$$L = 2\mu' \log \frac{b}{a} + \frac{2\mu}{ma} \left(\frac{1}{\sqrt{2}} - \frac{3}{8\sqrt{2m^2a^2}} - \frac{3}{8m^3a^3} \right) \\ + \frac{2\mu}{mb\sqrt{2}} \cdot \frac{\sinh m(c-b) \sqrt{2} - \sin m(c-b) \sqrt{2}}{\cosh m(c-b) \sqrt{2} + \cos m(c-b) \sqrt{2}}. \quad (104)$$

It is to be noticed that R becomes infinite when $b=c$, but L becomes

$$2\mu' \log (b/a) + (2\mu/ma) (1/\sqrt{2} - 3/8\sqrt{2m^2a^2} - 3/8m^3a^3).$$

The first term in (103) gives the resistance of the inner conductor and the second that of the outer conductor. In (104) the first term gives the linkages of the magnetic flux in the dielectric with the current in the inner conductor. The second term gives the linkages of the magnetic flux in the substance of the inner conductor, and the third the linkages of the flux in the outer conductor.

4. With very high frequency currents.

When the frequency is very high, provided that c be not nearly equal to b , we may write

$$R = \frac{\rho m}{2\pi a\sqrt{2}} + \frac{\rho m}{2\pi b\sqrt{2}} \\ = \sqrt{\mu f \rho} \left(\frac{1}{a} + \frac{1}{b} \right). \quad \dots \dots (105)$$

Similarly when ma is very great we may write

$$L = 2\mu' \log \frac{b}{a} + \frac{\mu}{2\pi} \sqrt{\frac{\rho}{\mu f}} \left(\frac{1}{a} + \frac{1}{b} \right). \quad (106)$$

We see, therefore, that as f increases R continually increases, but L approaches the value $2\mu' \log (b/a)$ asymptotically. Whatever the frequency, we see that the value of L lies between the value given by (96) and $2\mu' \log (b/a)$. It is easy to see that it has the latter value when the currents are confined to an infinitely thin skin on the outer and inner surfaces of the inner and outer conductors respectively.

Formulæ (105) and (106) agree with those given by Sir Joseph Thomson (see 'Recent Researches,' p. 295). They are also given in Heaviside's 'Electrical Papers,' vol. ii. p. 193.

V. THE DENSITY OF THE CURRENT IN THE INNER AND OUTER CONDUCTORS.

1. *With low frequency currents.*

For the inner conductor we find from equation (65), that

$$i = i_0 \{X(mr)\}^{1/2} \cos(\omega t + \epsilon), \quad . \quad . \quad (107)$$

where $\tan \epsilon = \text{bei } mr / \text{ber } mr.$

It is known* that $X(mr)$ always increases as mr increases. Hence the amplitude of the current density is always greatest at the surface of the inner conductor and least along the axis. When the sixth and higher powers of mr can be neglected (107) becomes

$$i = i_0(1 + m^4 r^4 / 64) \cos(\omega t + \epsilon), \quad . \quad . \quad (108)$$

where $\tan \epsilon = m^2 r^2 / 4.$

If $mr=1$, (107) gives

$$i = 1.015 i_0 \cos(\omega t + 14^\circ 15'),$$

and (108) gives $i = 1.016 i_0 \cos(\omega t + 14^\circ 2').$

Hence when mr is not greater than 1, (108) may be used. Even when $mr=2$ the inaccuracy in the value of the amplitude of the current density given by (108) is less than 2 per cent. In this case the amplitude of the current-density at the surface of the conductor is about 23 per cent. greater than at its axis, and the current along the axis lags by about 52° behind the surface current.

Similarly when mc is small, we find by (78) that

$$\frac{i'}{i_0} = \frac{a^2}{c^2 - b^2} \left\{ 1 - \frac{m^2 c^2}{2} \log mr \right\} \cos \omega t. \quad . \quad (109)$$

Hence the amplitude of the current-density in the outer conductor diminishes as r increases.

3. *With high frequency currents.*

In this case, by (20) and (21), we get for the inner conductor

$$\frac{i}{i_0} = \frac{\epsilon^{mr/\sqrt{2}}}{\sqrt{2\pi mr}} \cos\left(\omega t + \frac{mr}{\sqrt{2}} - \frac{\pi}{8}\right). \quad . \quad (110)$$

* Russell's 'Alternating Currents,' vol. i. p. 373.

Similarly in the outer conductor we find that

$$\frac{i'}{i_0} = \frac{\epsilon^{ma/\sqrt{2}}}{\sqrt{2\pi mr}} \left(\frac{a}{b}\right)^{1/2} \left[\frac{\cosh m(c-r)\sqrt{2} + \cos m(c-r)\sqrt{2}}{\cosh m(c-b)\sqrt{2} - \cos m(c-b)\sqrt{2}} \right]^{1/2} \cos(\omega t + \theta). \quad (111)$$

where

$$\tan \theta = \frac{\epsilon \frac{-m(c-r)/\sqrt{2}}{\sin \epsilon_1} + \epsilon \frac{m(c-r)/\sqrt{2}}{\sin \epsilon_2}}{\epsilon \frac{-m(c-r)/\sqrt{2}}{\cos \epsilon_1} + \epsilon \frac{m(c-r)/\sqrt{2}}{\cos \epsilon_2}},$$

and

$$\epsilon_1 = \gamma - \frac{mr}{\sqrt{2}} + \frac{\pi}{8}; \quad \epsilon_2 = \delta + \frac{mr}{\sqrt{2}} + \frac{\pi}{8};$$

$$\tan \gamma = \frac{B}{A}; \quad \tan \delta = \frac{D}{C}, \quad \text{and } \gamma - \delta = mc\sqrt{2}.$$

Since $\{\cosh m(c-r)\sqrt{2} + \cos m(c-r)\sqrt{2}\}/r$ continually diminishes as r increases, we see that the amplitude of i always diminishes as r increases.

VI. CONCENTRIC MAIN WITH HOLLOW INNER CONDUCTOR.

Let us suppose that a_1 is the inner radius of the inner conductor. The solution (58) for the current-density i still applies. Assuming that $i = i_0 \cos \omega t$, when r is a_1 , we get two equations connecting the four constants A , B , C , and D . The equation corresponding to (62) is now

$$\rho \frac{\partial i}{\partial r} = \frac{4\pi\mu}{r} \int_{a_1}^r r \frac{\partial i}{\partial t} \partial r. \quad (112)$$

Substituting the value of i in this equation and equating the coefficients of $\cos \omega t$ and $\sin \omega t$ in the resulting equation to zero, we get two other equations connecting the four constants, and hence they can be found.

Adopting the method suggested by O. Heaviside*, however, the values of the effective resistance of each tube may be written down at once from the formulæ given above. From formula (101), for instance, the resistance R_0 of the outer conductor is given by

$$R_0 = \frac{\rho}{\pi(c^2 - b^2)} \left[1 + m^4 \left\{ \frac{(7c^2 - b^2)(c^2 - b^2)}{192} + \frac{b^2 c^4}{8(c^2 - b^2)} \log \frac{c}{b} - \frac{b^2 c^5}{4(c^2 - b^2)^2} \left(\log \frac{c}{b} \right)^2 \right\} \right]. \quad (113)$$

* 'Electrical Papers,' vol. ii. p. 192.

To deduce the resistance R_i of the inner tube we write a for b , and a_1 for c in the coefficient of m^4 , and noticing that the area of the cross section is now $\pi(a^2 - a_1^2)$, we get

$$R_i = \frac{\rho}{\pi(a^2 - a_1^2)} \left[1 + m^4 \left\{ \frac{(7a_1^2 - a^2)(a_1^2 - a^2)}{192} + \frac{a^2 a_1^4}{8(a^2 - a_1^2)} \log \frac{a}{a_1} - \frac{a^2 a_1^6}{4(a^2 - a_1^2)^2} \left(\log \frac{a}{a_1} \right)^2 \right\} \right]. \quad (114)$$

For a given cross sectional area $\pi(a^2 - a_1^2)$ we see from (114) that the coefficient of m^4 diminishes as a_1 increases. Hence, at low frequencies, making the inner conductor hollow diminishes the skin effect. Similarly for a given cross sectional area, the larger the outer tube the smaller the effective resistance of the tube.

From (103) we see that at high frequencies the resistance R_0 of the outer tube is given by

$$R_0 = \frac{\rho m}{2\pi b \sqrt{2}} \cdot \frac{\sinh m(c-b) \sqrt{2} + \sin m(c-b) \sqrt{2}}{\cosh m(c-b) \sqrt{2} - \cos m(c-b) \sqrt{2}}.$$

Hence the resistance R_i of the inner tube is given by

$$R_i = \frac{\rho m}{2\pi a \sqrt{2}} \cdot \frac{\sinh m(a-a_1) \sqrt{2} + \sin m(a-a_1) \sqrt{2}}{\cosh m(a-a_1) \sqrt{2} - \cos m(a-a_1) \sqrt{2}}. \quad (115)$$

The difference, therefore, between the effective resistance of a thin and a thick inner tube having equal outer radii is very small at high frequencies.

VII. THE IMPEDANCE OF A CONCENTRIC MAIN.

If the length of the main be very long compared with its diameter and the insulation resistance of the dielectric be very high, the assumption that the current flow is linear is permissible. In this case, if e be the potential-difference between the mains at a point at a distance x from the alternator, we have*

$$\begin{aligned} -\frac{\partial e}{\partial x} &= Ri + L \frac{\partial i}{\partial t}, \\ \text{and} \quad -\frac{\partial i}{\partial x} &= \frac{e}{S} + K \frac{\partial e}{\partial t}, \end{aligned}$$

where K and S are the capacity and the insulation resistance between the mains per unit length. In these equations R and L have the values found earlier in the paper. Hence we have to substitute these values in the well-known expressions for the impedance of the main.

* Russell, 'Alternating Currents,' vol. ii. p. 458 *et seq.*

VIII. NUMERICAL EXAMPLE.

In connexion with the transmission of electric power at low frequencies, the question of the magnitude of the skin effect in three-core cables has been recently discussed by engineers. As the question is one of considerable difficulty owing to the very complex nature of the magnetic field inside a three-core cable (see Russell, 'Alternating Currents,' vol. i. p. 321), it is important to know what are the corresponding losses in a concentric main.

Let us suppose that the inner core is solid and that mc is not greater than 2, so that we may use formula (101). Let the radius of the inner main be one centimetre. For a very high pressure cable a suitable value* of b would be 2.4 cms. Hence, since the section of the outer conductor is made equal to that of the inner, c is 2.6 cms. Substituting these numbers in the formula, we find that

$$\frac{R}{l} = \frac{\rho}{\pi} \left(1 + \frac{m^4}{192} \right) + \frac{\rho}{\pi} \left(1 + \frac{0.0072 m^4}{192} \right),$$

where l is the length of the main in centimetres.

With the frequencies used in practice m is not very different from 1. We see that the skin effect increases the resistance of the inner conductor by about the half of one per cent., and the increase in the effective resistance of the outer conductor is less than the hundredth part of this. For a low voltage cable we might have $c=5/3$ and $b=4/3$. In this case

$$\frac{R}{l} = \frac{\rho}{\pi} \left(1 + \frac{m^4}{192} \right) + \frac{\rho}{\pi} \left(1 + \frac{0.059 m^4}{192} \right),$$

and hence the increase in the loss of the outer conductor is only about the twentieth part of the corresponding quantity for the inner. If the return conductor were a tight-fitting tube so that $b=1$ and $c=\sqrt{2}$, the formula becomes

$$\frac{R}{l} = \frac{\rho}{\pi} \left(1 + \frac{m^4}{192} \right) + \frac{\rho}{\pi} \left(1 + \frac{0.148 m^4}{192} \right).$$

Even in this case the increase in the loss of the outer is less than the fifth part of the increase in the loss of the inner, although the losses for very high frequencies would be practically the same in each conductor.

* Russell, 'Electric Cables and Networks,' p. 203.

If the main consisted of a hollow inner conductor whose radii were 1 and $\sqrt{2}$ respectively and an outer conductor of radii 2.6 and 2.4, then we would have

$$\frac{R}{l} = \frac{\rho}{\pi} \left(1 + \frac{0.105 m^4}{192} \right) + \frac{\rho}{\pi} \left(1 + \frac{0.0072 m^4}{192} \right).$$

Hence, making the inner conductor hollow has appreciably reduced the losses due to skin resistance. It will also be noticed that the losses due to this cause in the $(1, \sqrt{2})$ cylindrical conductor are about 30 per cent. less when the return current is outside it than when it is inside it.

It is easy to see that if the current returns by a fine conductor in a minute hole along the axis of a cylindrical conductor, the increase in the losses for low-frequency currents will be nearly seven times greater than if it returned by a concentric tube outside the conductor.

IX. VALUES OF m FOR COPPER CONDUCTORS.

The volume resistivity of high conductivity annealed copper at 60° F. (15° C.) is 1696.5. Hence

$$m^2 = \frac{4\pi\mu\omega}{\rho} = \frac{8\pi^2\mu}{1696.5} f.$$

Assuming that $\mu=1$, we get the following table:—

f .	m .	f .	m .
5	0.4824	100	2.1573
10	0.6822	200	3.0509
15	0.8355	300	3.7366
20	0.9648	400	4.3147
25	1.0787	500	4.8240
30	1.1816	1000	6.8221
35	1.2763	5000	15.255
40	1.3644	10000	21.573
45	1.4472	100000	68.221
50	1.5255	1000000	215.734

XLIX. *Elastic Solid Æther, with two Moduli, satisfying MacCullagh's Crystalline Optical Conditions.* By Prof. ALEX. MCAULAY, M.A., *University of Tasmania* *.

1. Description of the solid.

THREE states of the medium require naming for our purposes. (1) The zero state ; this is a state (not of equilibrium) necessary for descriptive purposes. (2) The intrinsically strained state ; this is a state of stable equilibrium. (3) The actual disturbed state ; this is a state which is obtained from the intrinsically strained state by displacements small compared with those which produce the intrinsically strained state from the zero state.

Let an elastic solid of density

$$D=10^6 \text{ gm. per cub. cm.} \quad . \quad . \quad . \quad . \quad (1)$$

fill space. Let its potential energy per unit volume consist of two terms, a cubic rotational term and a hydrostatic quadratic term.

The cubic rotational term is

$$-efg \times C = -efg \times 1.8 \times 10^{38} \text{ ergs per cub. cm.} \quad . \quad (2)$$

where e, f, g are the small fractions expressing the *principal* elongations necessary to pass from the zero to the actual disturbed state.

It will be shown below that so long as we retain only the terms of lowest order in the stress, the cubic rotational stress is zero when the curl is zero, so that there is from the cubic term no elastic resistance to irrotational strain.

Let the medium be irrotationally intrinsically strained from the zero state according to the following specification. First impose everywhere and in all directions a uniform elongation of linear value

$$e_0 = f_0 = g_0 = 10^{-11}, \quad . \quad . \quad . \quad . \quad (3)$$

and call the state of the medium obtained free æther. Secondly, throughout certain volumes, to be called crystals, change the elongations from e_0 to values e_1, f_1, g_1 , given for three mutually perpendicular directions, the principal directions of the crystal; e_1, f_1, g_1 are each less than e_0 in the ratios of the squares of the corresponding velocities of light to the square of the velocity in free æther. Due to the difference

* Communicated by the Author. Read before the Australasian Association for the Advancement of Science, at the meeting at Brisbane, January 1909.

of the intrinsic strain throughout the crystalline volumes and throughout the remaining free æther there will be a misfit throughout thin interface regions; let the maximum elongation (of irrotational strain) necessary to account for this interface misfit be not greater than .05. [For a sphere of water, 100 metres in diameter, the interface thickness will thus be of the order 10^{-6} cm.] The e_0, f_0, g_0 of (3) may be looked upon as the particular free æther values of the general intrinsic principal elongations e_1, f_1, g_1 . It is permissible to change e_1, f_1, g_1 from positive to negative, if at the same time we change C from positive to negative, so that the cubic term retains a negative value.

Let the hydrostatic quadratic term be zero in the intrinsically strained state and have the value

$$b(e_2 + f_2 + g_2)^2, \dots \dots \dots (4)$$

where b is a positive constant and e_2, f_2, g_2 are elongations (either principal elongations or elongations in three given rectangular coordinate directions) of the actual disturbed state reckoned from the intrinsically strained state. The only condition for our purposes, it is necessary to impose on b , is that it is large enough to mask the effects of those higher order "cubic rotational" terms which have been neglected. For instance, b may be supposed practically infinite.

A medium thus endowed with two elastic moduli will oscillate about the intrinsically strained stable state according to MacCullagh's equations; in other words, it is MacCullagh's medium.

To obtain a light intensity as great as at the Sun's surface (2 ergs per cub. cm.) the maximum disturbed curl $< 10^{-13}$.

2. *Dynamical behaviour of the solid mathematically investigated.*

Without loss of generality we may divide by D and assume for the future that the density is unity; thus the C and b about to be used may be looked upon as our previous C/D and b/D . We will consider a more general intrinsic irrotational strain than that specified in § 1; that is to say, we will suppose the intrinsic strain to be a purely arbitrary continuous irrotational strain, but we continue to assume that the superposed disturbed strain is small compared with the intrinsic strain.

Let ρ_0 be the position vector of a point in the zero state of

the medium and η_0 be its arbitrary displacement (intrinsic + additionally disturbed); let χ_0 be the strain linity (that is linear vector function of a vector) and ψ_0 the pure strain linity, so that with ω for an arbitrary vector,

$$\chi_0 \omega = -S \omega \nabla \cdot \eta_0, \quad \psi_0 \omega = -\frac{1}{2} \eta_{01} S \omega \nabla_1 - \frac{1}{2} \nabla_1 S \omega \eta_{01}. \quad (5)$$

The cubic rotational term is

$$\begin{aligned} & -\frac{1}{6} CS \zeta_1 \zeta_2 \zeta_3 S \psi_0 \zeta_1 \psi_0 \zeta_2 \psi_0 \zeta_3 \\ & = -\frac{1}{6} CS \zeta_1 \zeta_2 \zeta_3 S \chi_0 \zeta_1 \chi_0 \zeta_2 \chi_0 \zeta_3 - \frac{1}{4} CS (V \zeta \chi_0 \zeta) \psi_0 (V \zeta \chi_0 \zeta) \\ & = -\frac{1}{6} CS \nabla_1 \nabla_2 \nabla_3 S \eta_{01} \eta_{02} \eta_{03} - \frac{1}{4} CS (V \nabla \eta_0) \psi_0 (V \nabla \eta_0). \end{aligned}$$

This is easily proved by expressing χ_0 in the form $\psi_0 + V \lambda_0 ()$, and verifying that the terms of $S \zeta_1 \zeta_2 \zeta_3 S \chi_0 \zeta_1 \chi_0 \zeta_2 \chi_0 \zeta_3$ which are linear and which are cubic in λ_0 are identically zero.

The volume integral of $S \nabla_1 \nabla_2 \nabla_3 S \eta_{01} \eta_{02} \eta_{03}$ for the whole of space reduces to a surface integral at infinity and is zero when η_0 converges properly. Putting it zero the cubic rotational part of the potential energy is the volume integral for all space of

$$-\frac{1}{4} CS (V \nabla \eta_0) \psi_0 (V \nabla \eta_0).$$

Since this is zero when the curl is zero, it follows that the cubic term gives rise to zero elastic resistance to irrotational strain. Since the intrinsic strain is irrotational, and the disturbed strain is small compared with the intrinsic strain; by neglecting all but the lowest order terms in the last expression we may therein suppose ψ_0 to stand for the *constant* (in time) intrinsic strain and $V \nabla \eta_0$ for the superposed curl, so that our original cubic term has taken a form which is quadratic in the (disturbed) curl.

Let ρ be the position vector of a point in the intrinsically strained position, η the disturbed displacement, χ the disturbed strain linity, ψ the disturbed pure-strain linity, all reckoned from the intrinsically strained state. Remembering the irrotational constant meaning of ψ_0 in the last form above of the cubic rotational term, put

$$\frac{1}{2} C \psi_0 \omega = h \omega = -S \omega \nabla \cdot \nabla g, \quad . \quad . \quad . \quad (6)$$

where g is a given scalar function of position whose first and second space derivatives are everywhere finite, and therefore h is a given self-conjugate linity.

The potential energy W for the whole of space is now given as the volume integral of w_m or of w thus

$$W = \iiint w_m ds = \iiint w ds, \quad . \quad . \quad . \quad . \quad (7)$$

where we have the following for w_m

$$(\text{MacCullagh form}) \quad w_m = -\frac{1}{2}S(V\nabla\eta)h(V\nabla\eta) + \frac{1}{2}b(S\nabla\eta)^2, \quad (8)$$

and the following for w

$$(\text{Green form}) \quad w = S(V\xi_1\xi_2)h(V\psi\xi_1\psi\xi_2) + \frac{1}{2}b(S\xi\xi\psi\xi)^2. \quad . \quad (9)$$

By transforming

$$S(V\xi_1\xi_2)h(V\chi\xi_1\chi\xi_2)$$

by putting $\chi = \psi + V\lambda$ () it is easy to show that

$$\begin{aligned} w - w_m &= S(V\xi_1\xi_2)h(V\chi\xi_1\chi\xi_2) \\ &= -S\nabla_1\nabla_2\nabla_3S\eta_1\eta_2\nabla_3g_3 \end{aligned} \quad . \quad . \quad . \quad (10)$$

and therefore that the volume integral of $w - w_m$ for the whole of space is zero.

It will be found that with h constant for a crystalline volume, (9) is Green's 1839 form*. When b is zero and h constant, (8) is obviously MacCullagh's 1839 form†.

To show that the equilibrium position of the solid is stable it is necessary only to show that w_m of (8) is positive. With the specification of § 1 we may, I think, regard this as obvious. We may take any point inside a crystal, say its centre of volume, and leave that point with the same intrinsic displacement as for free æther. The interface may be assumed to be of uniform thickness, say 10^{-6} cm., and in the three principal directions of the crystal e_1, f_1, g_1 may be supposed, through the interface, to change continuously to $e_0 = f_0 = g_0$.

The theory that we have been thus led to differs in form from MacCullagh's in that for stability we are not permitted to put $b=0$; it really differs from Green's in the interface conditions. It may be said further to differ in form from MacCullagh's in that h is large for the interface region as compared with the rest of space, and therefore perhaps we shall find that when we treat the interface as a surface of discontinuity we must reckon with a surface contribution to the potential energy.

* "On the Propagation of Light in Crystallized Media," Math. Papers, p. 293, Trans. Camb. Phil. Soc. 1839.

† An Essay towards a Dynamical Theory of Crystalline Reflexion and Refraction. Collected Works, p. 145, Trans. Roy. Ir. Ac. 1839.

Our bodily equation, by a straightforward application of the dynamical principle $\delta \int L dt = 0$ to the MacCullagh form (8), is found to be

$$\ddot{\eta} = -V\nabla(hV\nabla\eta) - \nabla(bS\nabla\eta). \quad (11)$$

Operating by $S\nabla(\)$ and by $V\nabla(\)$, we find (as we may anticipate from the usual treatment after Green) that the convergence and curl behave quite independently of one another. In the present theory, and this is where the present theory differs from Green's, this statement is true of the interface region as well as of the other two regions, the free æther and the crystal.

Thus in the present theory we are not confronted with the appearance of a wave of condensation when a wave of curl suffers reflexion.

Optical interpretations are concerned only with the curl $V\nabla\eta$. We may suppose initially, and therefore by (11) permanently, that $S\nabla\eta = 0 = S\nabla\dot{\eta}$; or, what is mathematically equivalent, we may henceforth suppose η to stand for that part only of the disturbed displacement which has curl and not convergence. The bodily equation becomes

$$\ddot{\eta} = -V\nabla(hV\nabla\eta), \quad (12)$$

and the potential energy is confined to the cubic term.

(12) is MacCullagh's equation and (11) is Green's. (12) is also Maxwell's bodily equation if we identify $\dot{\eta}$ with the magnetic force and h with the reciprocal of the permittivity. The condition that η is continuous at an interface is common to all four theories, MacCullagh's, Green's, Maxwell's, and the present one. MacCullagh's and Maxwell's second boundary condition is that $hV\nabla\eta$ is tangentially continuous, whereas Green's is different; and to-day it is well known that MacCullagh's and Maxwell's condition is in harmony and Green's condition is not in harmony with the optical facts of reflexion at the boundaries of transparent bodies. What does the present theory say on this second boundary condition?

The following appears to me to prove that the present theory gives the desired optical condition.

Examining the sense in which h is large at an interface the following at once appears. Let the axis of z be taken perpendicular to the interface. h is given by the six second space derivatives of g ,

$$g_{xx}, g_{yy}, g_{xy}; \quad g_{xz}, g_{yz}; \quad g_{zz}.$$

Of these, in the interface region, the first three g_{xx}, g_{yy}, g_{xy}

are of the same order of magnitude as in the rest of space; the next pair g_{xz} , g_{yz} are of the order of the reciprocal of the thickness of the interface; the sixth g_{zz} is of the order of the square of that reciprocal. On the other hand, attending to \dot{h}^{-1} we find that the first three scalars are again as usual; the next pair are small and of the order of the thickness; the sixth small and of the order of the square of the thickness.

This suggests that we should apply the principle $\delta \int L dt = 0$ in the following manner, in which there is zero contribution to L from the interface. Put

$$\dot{\eta} = \sigma, \quad h V \nabla \eta = \epsilon. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then

$$L = -\frac{1}{2} \iiint \sigma^2 d\mathbf{s} + \frac{1}{2} \iiint S \epsilon h^{-1} \epsilon d\mathbf{s}; \quad . \quad . \quad (14)$$

and we have to take account of the equation of condition

$$h^{-1} \dot{\epsilon} = V \nabla \sigma. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Working with (14) and (15) take account of (15) by indeterminate multipliers by adding to δL

$$\iiint S \alpha (h^{-1} \delta \dot{\epsilon} - V \nabla \delta \sigma) d\mathbf{s}.$$

We thus obtain the equations of the medium in the form

$$\left. \begin{aligned} \sigma &= -V \nabla \alpha, & \epsilon &= \dot{\alpha}, & h^{-1} \dot{\epsilon} &= V \nabla \sigma, \\ [V \alpha d\Sigma]_{a+b} &= 0, \end{aligned} \right\} \quad . \quad . \quad (16)$$

the last being the boundary condition and expressing that α is tangentially continuous. It follows that $\dot{\alpha}$ or ϵ or $h V \nabla \eta$ is tangentially continuous. This is the desired result, and is what we should expect as the limit of (12) applied to an element of the interface. The largeness of h at the interface prevents us from making the deduction directly from (12).

3. Origin and suggested modification of the above Theory.

The theory arose from an attempt to explain dynamically Maxwell's fundamental equations and not by a direct development from either MacCullagh or Green.

Maxwell's optical conditions may be thus briefly but fully enunciated. \mathbf{H} the magnetic force satisfies the bodily equation

$$\ddot{\mathbf{H}} = -V \nabla (c^{-1} V \nabla \mathbf{H}),$$

where c is the linity called permittivity, and the two boundary

conditions are that \mathbf{H} is continuous and $c^{-1}\mathbf{V}\nabla\mathbf{H}$ is tangentially continuous. It was sought to identify \mathbf{H} with the velocity $\dot{\eta}$ of a usual 21-constant elastic solid. Four quite definite steps were thence taken.

First step. In order that for an elastic solid with moduli unvarying from point to point, the evoked volume force β may, with arbitrary displacement η , satisfy the incompressible condition $\mathbf{S}\nabla\beta=0$, the twenty-one constants *must* reduce to six in the following definite manner. When the density is unity the potential energy w per unit volume must be of the form

$$w = \mathbf{S}(\mathbf{V}\xi_1\xi_2)h(\mathbf{V}\psi\xi_1\psi\xi_2),$$

where ψ is the pure strain linity and h is a constant self-conjugate linity.

Second step. In order that with h varying from point to point the same condition $\mathbf{S}\nabla\beta=0$ may hold (a condition imposed in the hope of attaining the boundary condition that $h\mathbf{V}\nabla\eta$ is continuous) h *must* have the definite form

$$h\omega = -\mathbf{S}\omega\nabla \cdot \nabla g,$$

where g is a scalar function of position.

Third step. We may add to the volume potential energy now attained, the hydrostatic quadratic term $\frac{1}{2}b(\mathbf{S}\xi\xi\psi\xi)^2$, because it is found that this generalized form, and this form *alone*, produces an equation of motion in which curl and convergence are propagated independently. [This step is essentially a step of Green's, but it was actually taken without reference to Green's work.]

Fourth step. We naturally seek a simple description of the volume energy attained. The very suggestive form of h in terms of g readily leads to the specification of § 1 above.

I think it may be claimed that the theory finally reached reconciles Green and MacCullagh, and especially that it provides what has been so persistently sought, an easily conceived dynamical explanation of MacCullagh's foundation. Taken in connexion with some such elaborate superstructure as Dr. Larmor's '*Æther and Matter*,' I think we may further say that the theory furnishes the fundamental dynamical explanation of modern electromagnetic theory.

The following modification of the details of the specification of § 1 seems worth considering. Suppose a large crystal to be built up of component crystals whose linear dimensions are small, in a sense, but are large compared with the wavelength of light. We can then make D as small as 10^{-8} gm. per cub. cm.

To do this let the large crystal consist of closely packed cubes each of whose edges is 3×10^{-3} cm. Let each cube consist of an outer shell of thickness $\frac{1}{2} \times 10^{-6}$ cm. and a kernel constituting the rest of the cube. Let the intrinsic strain change from the outside to the inside of the shell by the elongations gradually passing from the free æther values e_0, e_0, e_0 to the proper crystal values e_1, f_1, g_1 which are to be maintained uniform throughout the kernel. The double shells separating two neighbouring kernels will behave simply as thin transparent films separating two crystals of identical optical properties (that is like the dark spot of a soap-film), and will therefore in no way interfere with the propagation of light-waves. But with these new conditions we may change the constants of § 1 to the following values in c.g.s. measures

$$D = 10^{-8}, \quad C = 10^{18} \times 1.8, \quad e_0 = 10^{-5}.$$

The maximum disturbed curl of a plane-polarized wave in free æther with intensity two ergs per cub. cm. will then be $e_0/15$; in § 1 it was $e_0/150$.

*L. On Surface Separation from Solutions of Saponin, Peptone, and Albumin. By S. A. SHORTER, B.Sc.**

THE surface-layers of many solutions contain the dissolved substance in a degree of concentration much higher than that in the body of the liquid. In some cases the mechanical properties of the surface-layer indicate the separation of matter in the solid state†. The present paper deals with this phenomenon of surface separation in the case of solutions of saponin, peptone, and albumin. The primary object of the investigation was to study the way in which the thickness of the surface pellicle increases with the age of the surface in solutions of various concentrations. Incidentally it was found that the results threw much light upon the nature of the molecular processes involved in the formation of the surface-film. It was also found that the results cannot be explained on the ordinary theory of surface concentration.

In spite of the large amount of work which has been done in recent years on the superficial viscosity and rigidity of

* Communicated by the Author. Part of the expenses of this research has been defrayed by means of a Government Grant made through the Royal Society.

† See Ramsden, Proc. Roy. Soc. lxxii. p. 156 (1903).

solutions, such an obviously fundamental research as the above has never yet been carried out. The chief difficulty has been the lack of any suitable means of quantitative investigation of the excessively thin and fragile surface-film. Methods similar to that of Plateau*, involving a sweeping up of the surface-film, interfere too much with the process under investigation. The same objection applies to the torsion-balance method of measuring the shearing stress necessary to break the film†. The method of measuring the surface rigidity statically by means of the torsion-balance‡, though not open to this objection, is not suited to the solutions under investigation. In the case of fairly fresh surfaces, the elastic forces called into play by the shearing of the film exhibit the phenomenon of elastic relaxation, so that it is impossible to get any satisfactory balance between these forces and the elastic forces of the torsion-balance wire. In the case of older surfaces the method is unsuitable owing to the smallness of the elastic limit of the surface pellicle.

The method adopted in the present work is one which I devised in a previous work on the surface-elasticity of saponin solutions§. I found that the surface-film, apparently inelastic if tested in the ordinary way by means of the torsion-balance (*i.e.* by twisting the torsion-head so as to impress an initial torsion on the wire), exhibits an elastic effect if the motion is started by rotating the vessel containing the solution (*i.e.* by impressing an initial shear on the surface-film). It is possible in this way to make the disk of the torsion-balance execute torsional oscillations under the action of the surface-film. These oscillations can be obtained even when the surface is so fresh that the surface-film yields rapidly under the action of the wire. In this case the film is viscous, *i.e.* it shows elastic relaxation, but the relaxation time is sufficiently great for the film to exhibit an elastic effect when the vessel is rotated, though when the torsion-head is twisted (and an approximately constant stress applied) the film behaves like a viscous liquid, such as castor oil, which, according to Natanson||, has a relaxation time of about 0.01 sec., and whose behaviour, therefore, approximates to that of an ideal viscous medium for all ordinary motions.

* *Statique des Liquides*, Oberbeck, Wied. Ann. xi. p. 634 (1880).

† Schütt, *Ann. der Phys.* xiii. p. 714 (1904); Rohde, *Ann. der Phys.* xix. p. 935 (1906).

‡ Schütt, *loc. cit.*

§ Phil. Mag. Feb. 1906.

|| Phil. Mag. Nov. 1901, p. 469.

Theory of the Method.

Suppose that the solution is contained in a cylindrical vessel of radius R , and that a circular brass disk of radius r is suspended by means of a fine wire, so that the axes of the vessel, disk, and wire are coincident. If the solution fills the vessel to such a height that the disk is partially immersed, then, neglecting the capillary curvature at the edges, the surface pellicle will be in the form of a thin annular lamina. Let z be the thickness of this lamina and n the rigidity of its material. It is easily shown that a rotation θ of one of the circular boundaries with respect to the other about the common axis, will give rise to a couple of magnitude

$$\frac{4\pi n z R^2 r^2 \theta}{R^2 - r^2}.$$

Hence, if I is the moment of inertia of the suspended system and T the period of the oscillations set up by an initial rotation of the vessel, we have, neglecting the elastic forces of the wire and the inertia of the liquid set in motion,

$$nz = \frac{\pi(R^2 - r^2)I}{R^2 r^2 T^2}.$$

In the absence of any knowledge of the value of z we may write

$$\mu = nz,$$

where μ may be termed the surface-elasticity. We may also write

$$\mu = \frac{A}{T^2},$$

where A may be termed the "apparatus constant."

Apparatus.

Two forms of apparatus were used in the following experiments. The first was identical with that used in my previous experiments on old surfaces of saponin solutions. In the second form the suspended system was hung from supports attached to the wooden base which carried the turntable on which the vessel rested—the whole being placed in a wooden box. The bottom of the box was perforated so

as to allow the levelling-screws of the base to rest independently on the bench. The front of the box consisted of a glass door, which did not extend down to the bottom of the box, a small space being left through which projected the radial arm attached to the turntable. A few small vessels containing calcium chloride were placed on shelves inside the box in order to prevent the condensation of moisture on the surface. The oscillations were observed by means of a telescope placed just outside the box, and focussed on the reflexion of the filament of an incandescent electric lamp in the mirror attached to the suspended system.

The accuracy of the measurements was limited by the fact that the oscillations damped down rapidly. Within limits, the larger the initial amplitude, the more oscillations could be counted. If, however, the initial amplitude was too big the elasticity of the film was impaired and the disk would scarcely oscillate at all. Considerable practice was necessary before one could always obtain as large an initial amplitude as possible without injury to the surface-film. By having as large initial amplitudes as possible, much greater accuracy was obtained than in my previous experiments.

A reading from which a value of the surface-elasticity was calculated generally consisted of ten observations of as many oscillations as could be obtained, the times being added up on the stop-watch used (which was of the progressive type—starting again from the point at which it was previously stopped). In many cases the time of the first five oscillations was noted, at first as a check on the accuracy of the observations, and later because it was found that sometimes the period of oscillation increased during the taking of a reading. The significance of this decrease of surface-elasticity will be discussed later.

As each experiment lasted several weeks six sets of apparatus were used in order to save time. They were all made as nearly as possible similar in every respect—the approximate dimensions of the different parts being as follows :—

diameter of vessel = 15.5 cm.

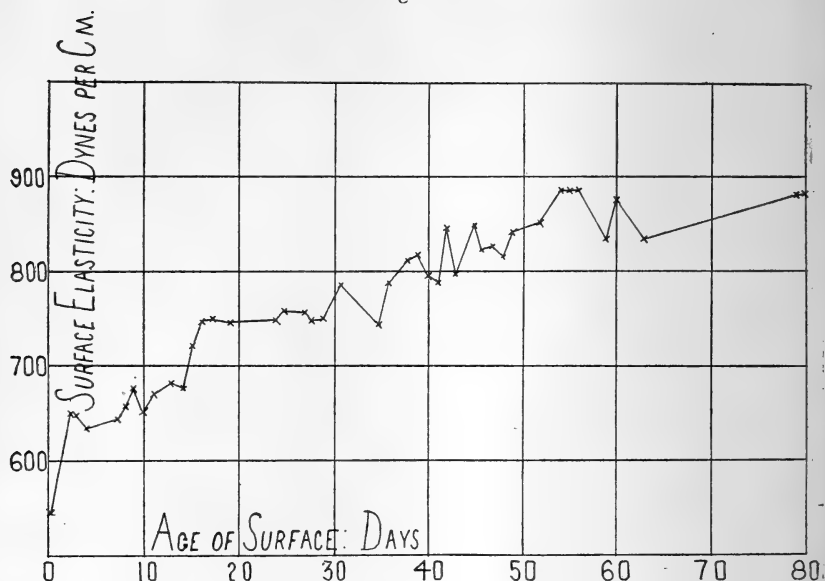
diameter of disk = 7.1 cm.

moment of inertia of suspended system = 1950 gr. cm.²

More accurate measurements gave for the different sets of apparatus values of the apparatus constant varying from 380 to 385 gr.

Fig. 1 shows the results of an experiment with a solution of saponin containing 1 part in 106,000 of water. The consecutive points have been joined together by straight lines.

Fig. 1.



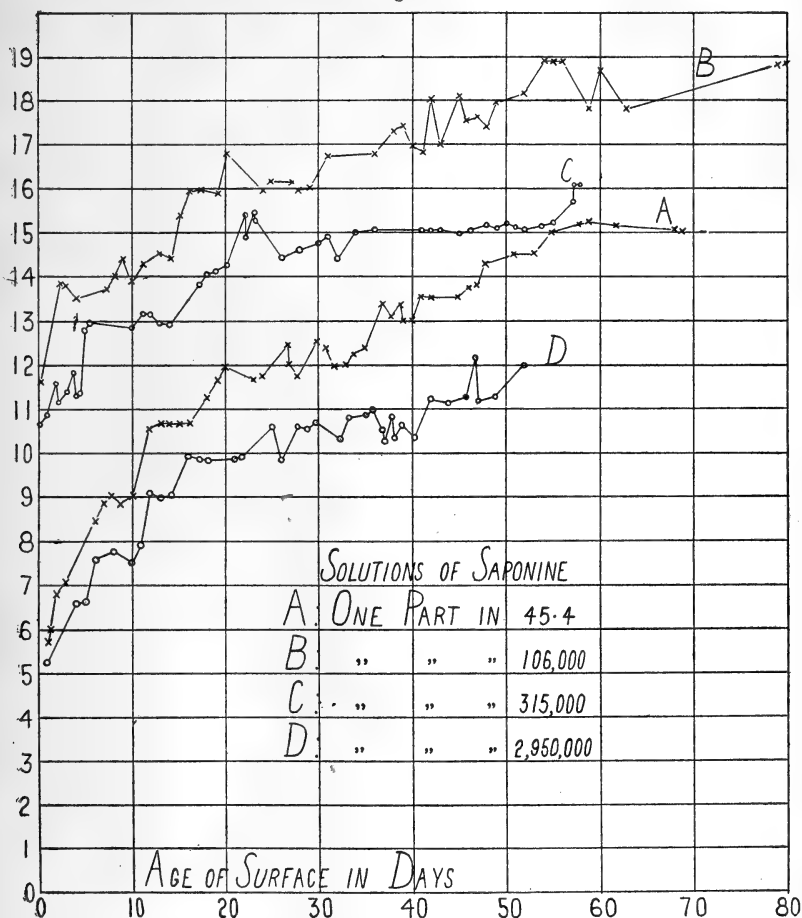
As will be seen later, a smooth curve drawn so as to give the closest fit to the points would have no physical meaning. Three important facts will be noticed from these results:—

- (1) The constancy of the surface-elasticity over a prolonged period does not necessarily indicate equilibrium between the surface pellicle and the body of the solution—the elasticity being liable to increase suddenly to a higher value after remaining constant for several days.
- (2) The surface-elasticity is very liable to sudden fluctuations of value, especially in the later stages of the experiment.
- (3) There is no sign of equilibrium even after several weeks.

Fig. 2 shows the results of experiments with solutions of saponin of different concentrations. The horizontal lines have been drawn to correspond to multiples of 46·8 dynes per cm. In the first instance the results were plotted in the

usual way. It was then noticed that many of the stationary values were the same for different solutions. Moreover, the

Fig. 2.



consecutive stationary values appeared to be about 50 units apart. This suggested the idea that these values might all be multiples of some unit value. The stationary values occurring twice were roughly

- 750 (B and C)
- 700 (A and C)
- 610 (A and C)
- 420 (A and D).

A simple calculation showed that these numbers were approximately multiples of 47. Horizontal lines were drawn to correspond to multiples of several values in the neighbourhood of 47, and finally 46.8 was chosen, as giving the best fit.

We can, therefore, state the following important conclusion:—

- (4) The stationary values are all multiples of a single unit value, viz. 46.8 dynes per cm.

A further inspection of the results shows that

- (5) The sudden fluctuations which occur are always between consecutive stationary values.

Physical Interpretation of the Results.

These results may be readily understood if we make the following two assumptions:—

- (1) that the stationary values correspond to uniform layers a definite number of molecules thick, *e. g.* the value 702 corresponds to a surface pellicle 15 molecules thick.
- (2) that a partially formed molecular layer does not contribute appreciably to the surface-elasticity till nearly complete.

Let us examine the consequences of these assumptions. A continuous process of surface-separation would give rise to a stepwise mode of increase of the surface-elasticity, such as, apart from certain irregularities, is observed in the diagram. Suppose, however, that owing to some cause molecules are sometimes reabsorbed into the portion of solution adjacent to the surface-layer, and at other times separate out again in the surface-layer, so that comparatively rapid fluctuations in the number of molecules in the surface-layer are superimposed on the slow general increase. It is easy to see that a small change of this kind will have very different effects at different times. If the latest added molecular layer is nearly complete, so that the surface-elasticity is practically the same as if the layer were complete, a small addition to the surface-layer will have no appreciable effect, while a decrease in the number of molecules in the surface-layer will have a very large effect on the surface-elasticity. If, however, a molecular layer is in the initial stages of formation, an increase will have a large effect and a decrease very little effect. Hence we see that *a small change in the number of molecules in the surface-layer may sometimes cause a large change in the value of the surface-elasticity, but this change will always occur between two molecular values and never across such a value.* It will be seen that the rapid fluctuations which occur always conform to this rule.

It has already been mentioned that the surface-elasticity sometimes decreased during the taking of a reading. At first this was attributed to the impairing of the elasticity by the motion set up. It was found, however, that this decrease occurred only in certain cases. In other cases the elasticity was unaffected by the taking of the reading. This point was further investigated in the light of the above theory. It was found that the decrease always occurred between consecutive molecular values, and that if the process of taking the observations was continued, the surface-elasticity generally decreased to the lower molecular value and then remained constant. This was not always the case. Occasionally an intermediate value was found to be stable, and in a few cases the elasticity fluctuated irregularly between two molecular values. In these latter cases there was a striking increase in the rate of damping of the oscillations. Thus at first it might be possible to count twenty oscillations, and the damping would increase till it was difficult to count ten.

The molecules forming part of an incomplete molecular layer are evidently easily detached from the surface pellicle, even when the layer is sufficiently complete to give an elastic effect nearly equal to that of a complete layer. The fact that the surface-layer rapidly recovers its initial value, seems to indicate that the detached molecules are not completely absorbed into the body of the solution, but remain in a kind of transition layer from which they easily separate out again in the surface-film. In the cases where the recovery is so very rapid that it occurs during the taking of the reading, there will obviously be a damping effect produced which will be absent when the cohesion between the pellicle and the molecules of the incomplete layer is for the time being totally destroyed.

Fig. 3.

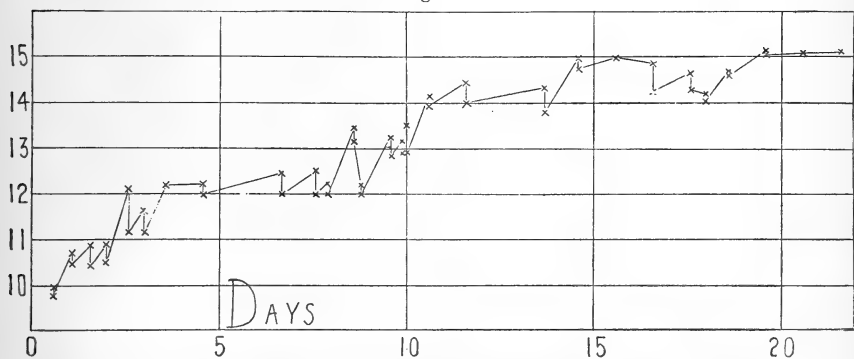


Fig. 3 shows the results of an experiment with a solution of saponin containing 1 part in 5568 of water. The horizontal

lines have been drawn to correspond to multiples of 46·8 dynes per cm. (the value deduced from the experiments shown in fig. 2). Each reading consisted of ten observations of as many oscillations as could be observed at once. Two values of the surface-elasticity were calculated, one from the first five, the other from the second five. The results confirm the above theory. It will be seen that whenever the two values differ appreciably, the second value is less than the first, and that this decrease always occurs between two consecutive molecular values.

Summary of Results: Solutions of Saponin.

In Table I. are summarized the results of a number of experiments, including those shown in figs. 2, 3, and 4.

TABLE I.
Solutions of Saponin.

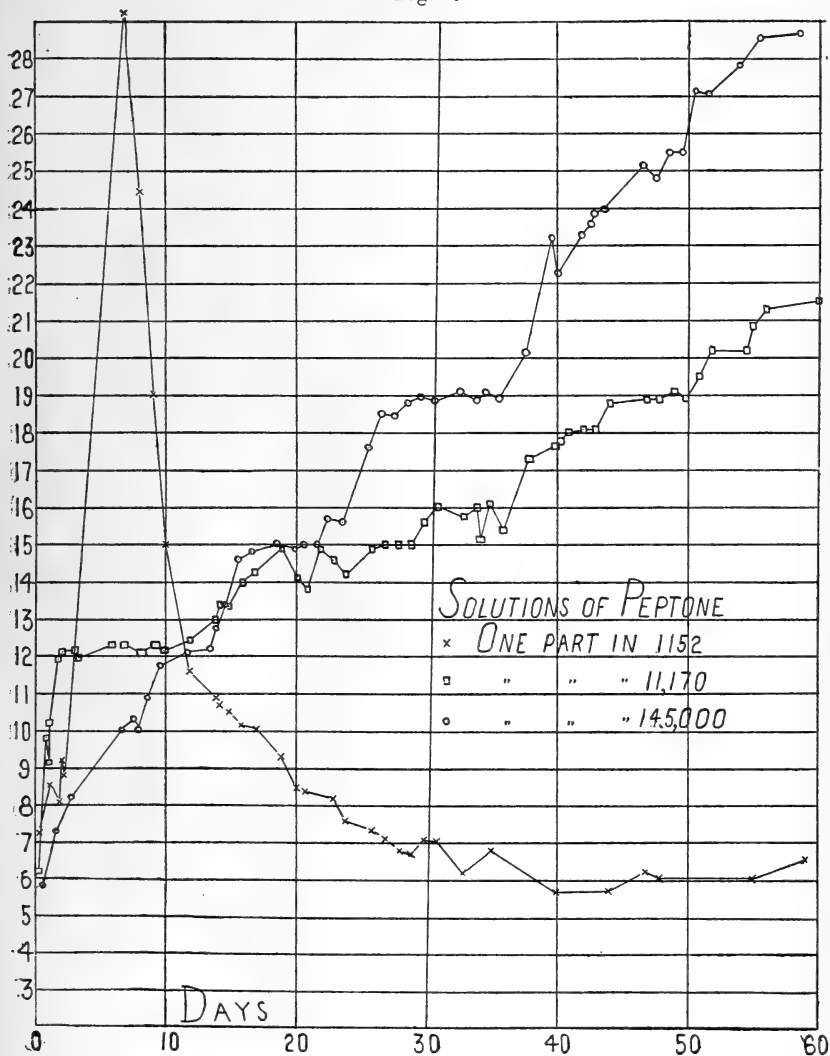
Age of Surface (days).	MOLECULAR THICKNESS OF SURFACE LAYER.									
	Concentration: 1 part in									
	45·4	100	5568	10,000	100,000	106,000	315,000	10 ⁶	2·95 × 10 ⁶	10 ⁷
1.....	5-6	7	10-11	11-12	8-9	11-12	11	7-8	5	1-2
2.....	7	...	10-11	...	10-11	13-14	11-12	9	...	2
3.....	7	8-9	11-12	12-13	12	13-14	11-12	10
4.....	...	8-9	13-14	11-12	...	6-7	...
5.....	...	10	12	13	13-14	...	13	10	6-7	...
6.....	8-9	10	...	13-14	13-14	10	7-8	...
7.....	9	10	12	13-14	13-14	14	...	10	...	4
8.....	9	10	12	14	13-14	14	7-8	4-5
9.....	9	...	13-14	14-15
10.....	9	...	13-14	14	13	...	7-8	...
15.....	11	11-12	15	14	...	15-16	...	11
20.....	12	13	15	16-17	14	...	10	...
25.....	12	16
30.....	12-13	13	14-15	...	10-11	...
40.....	13	17	15	...	10-11	...
50.....	14-15	18	15-16	...	11-12	...
60.....	15	18-19	16
70.....	15	15
80.....	15	16
90.....	20

It will be seen that the rate of growth of the surface pellicle does not vary much within wide limits of concentration. The rate of growth is less, not only in the case of very dilute solutions (as might be expected), but also in the case of concentrated solutions.

Experimental Results: Solutions of Peptone and Albumin.

Fig. 4 shows the results of a series of experiments with solutions of peptone. The molecular values have been taken

Fig. 4.



as multiples of 50 dynes per cm. The results show the same general characteristics as those shown in fig. 2. They show

in addition an important fact not shown very decidedly fig. 2, viz. that the stepwise mode of increase of the surface-elasticity is not entirely due to the molecular dimensions of the surface-layer—the actual rate of surface separation varies in an irregular manner. The history of the surface-layer of the 1 in 145,000 solution between the ages 28 and 40 days shows this very clearly. Another noteworthy point is that the increase of the surface-elasticity in the case of the two dilute solutions is, broadly considered, linear from near the beginning to the end of the experiment.

In the case of concentrated solutions of peptone and albumin there is a rapid initial thickening of the surface-film followed by a rapid thinning. In some cases this was followed by a second thickening much slower, however, than the first.

The results of a number of experiments are summarized in Tables II. and III.

TABLE II.
Solutions of Peptone.

Age of Surface (days).	MOLECULAR THICKNESS OF SURFACE LAYER. Concentration : 1 part in							
	89	183	530	1152	2900	11,170	27,000	145,000
1.....	2-3	7	5	8-9	...	9-10	4-5	...
2.....	4	...	9	8-9	8	12	7	7-8
3.....	7	...	11	...	9-10	12	...	8
4.....	9	9-10	10-11	...	9-10	12	9	...
5.....	10	9	11	...	8	12	9-10	...
6.....	6	9	10	...	9	12	10	...
7.....	2	9	10	29-30	8	12	...	10
8.....	1	...	10	24-25	8	12	...	10
9.....	1	...	10	19	...	12
10.....	1	7-8	10	15	...	12	15	12
11.....	1	12
12.....	11-12	...	12-13	...	12
13.....	11	12
14.....	2	11	...	13	...	13
15.....	2	10-11	...	14-15	...	14-15
20.....	8-9	...	14	20	15
25.....	7-8	...	15	...	17-18
30.....	7	...	16	28	19
40.....	6	...	17-18	...	22-23
50.....	6	...	19	...	27
60.....	6-7	...	21-22	...	29

TABLE III.
Solutions of Albumin.

Age of Surface (days).	MOLECULAR THICKNESS OF SURFACE LAYER. Concentration: 1 part in					
	133.7	280	1300	1808	10,750	282,800
1.....	2-3	5-6	1	4	4-5	1
2.....	5-6	6	2	5	5-6	2-3
3.....	7	6	3-4	6	7	3-4
4.....	8	6	4	6	7-8	4
5.....	6	6	4-5	7	8-9	5
6.....	6	6		7	10	6
7.....	6	6		...	10	...
8.....	...	6	
9.....	...	6	
10.....	...	6		8-9
11.....	9-10
12.....	6			10
13.....	6			...	15-16	10
14.....				10
15.....				...	16	10-11
20.....				10	17	12
25.....				...	22	11
30.....				12		12

*Conclusions regarding the nature of the process
of Surface Separation.*

The process of surface separation plays an important part in many physical, chemical, and physiological phenomena, and a more thorough knowledge of its nature is therefore very desirable. All the solids whose solutions exhibit this effect possess the property of lowering the surface tension of water. The concentration in the surface-layers of a solution of a substance possessing this property must be greater than that in the body of the solution, since this surface concentration tends to lower the potential energy of the system. There will be a contrary tendency towards equalization of the concentration owing to the osmotic pressure gradient in the neighbourhood of the surface, so that equilibrium will finally be established.

Hence, the most obvious explanation of the phenomenon of surface separation is that it is an extreme case of surface concentration. It is possible that if increase of concentration has a large effect on the surface tension, and only a small effect on the osmotic pressure, equilibrium is not attained

even when the surface-layers become saturated, so that the solute separates out in a solid form.

The results of these experiments do not support this view. In the first place, the rate of surface separation varies in an irregular manner quite inconsistent with the above theory. Secondly, there is no sign of equilibrium even after a prolonged period. Thirdly, this theory does not explain the initial thickening and subsequent thinning of the surface-layer, which occurs in the case of strong solutions of peptone and albumin.

The results indicate that the process is of a much more complex character than simple "capillary-osmotic" action. It is possible that the substance of the surface-layer differs chemically from the original solute. This view was put forward by Metcalf* as the result of experiments on surface-films formed by allowing a drop of peptone solution to spread over a water surface. At present I am engaged on a series of experiments on surface-layers formed at the interface between solutions of saponine, peptone, and albumin, and hydrocarbon oils. This case, though more difficult to investigate practically, possesses theoretical advantages due to the absence of evaporation and chemical action between the two media.

The University, Leeds.

Dec. 21st, 1908.

LI. *Magnetism of Basalt.* By G. E. ALLAN, D.Sc., Lecturer on Electricity, Pure and Applied, and Assistant to the Professor of Natural Philosophy, in the University of Glasgow†.

IN a previous paper‡ it was shown that bars of basalt heated in air undergo a magnetic change, partly temporary and partly permanent. In general the permanent effect consists in a loss of magnetic susceptibility. Some instances of an increase have, however, been observed. The temporary effect is, roughly, similar to that in magnetite; the bar reaches a state of maximum strength, which is followed by a state of minimum strength at a temperature which is not the same for all, but is, for the majority of cases examined, in the neighbourhood of 600° C.

In order to obtain further information as to the behaviour of such material, tests were carried out on new bars of

* *Zeit. für Phys. Chem.* vol. lii. p. 1 (1905).

† Communicated by Professor A. Gray, F.R.S.

‡ *Phil. Mag.* January 1904, p. 45.

basaltic rock, and on bars of magnetite, ilmenite, and hæmatite ore.

It is proposed to give here the particulars obtained for basalt. Five new bars were cut from specimens obtained in the following localities:—

1. Top of Sgurr nan Gillean, Skye.
2. The Storr Mountain, Skye.
3. Dunvegan district, Skye.
4. Stacks of Netherton, a basaltic * dyke in the sea near Stromness, Orkney.
5. Faule Ader, Dattenberg, a vein of non-columnar basalt which passes roughly across the middle of the Dattenberg quarry, near Linz, on the Rhine.

Method.

As in the previous research, the magnetometric method was employed, certain modifications having been introduced from time to time. The quartz-fibre suspension was replaced by a single silk fibre which has been found to work very satisfactorily. The length of the needle was reduced from 1 cm. to .25 cm. A platinum thermometer was employed in the measurement of temperature at first, but was replaced during the course of the experiments by a thermo-electric couple which was standardized in ice, steam, mercury-vapour, and molten aluminium.

Method of Compensation.

A few remarks may be made here on the compensation of the magnetizing coil. In making magnetometer observations with weakly magnetic material, such as basalt or Heusler's alloy, the specimen requires to be near the needle in order to cause a suitable deflexion. Erhard† has pointed out that with such a disposition, any slight deviation of the axes of the magnetizing and compensating coils from the magnetic east and west line gives rise to a component along the meridian which, for one direction of the magnetizing current, increases the directive force H , and for the opposite direction diminishes it. Erhard has also shown that the presence of this effect may be tested by first compensating for the zero position, and then deflecting the needle with a bar-magnet

* The geological nature of this rock is described by J. S. Flett in "The Trap Dykes of the Orkneys," Trans. Roy. Soc. Edin. vol. xxxix. p. 865.

† T. Erhard, *Ann. d. Physik* [4] vol. ix. p. 724 (1902).

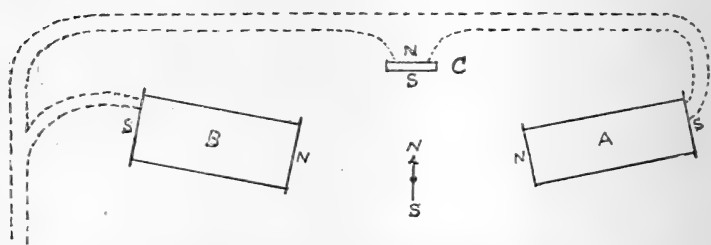
and testing the compensation in the new position of the needle.

The presence of these H-disturbing components may be more easily tested and their magnitude gauged by timing the period of oscillation of the needle with no current, and with direct and reversed currents in the magnetizing and compensating coils after compensation has been made in the usual way.

This method was used in the previous work, and indicated a change in the value of the horizontal field at the needle of over 5 per cent. when the magnetizing current was reversed.

The effect was neutralized in the later experiments by the addition to the magnetizing-coil circuit of a small auxiliary compensating coil. The latter consisted of seven turns of wire 9 cms. in diameter, and was placed to the north of the needle at about 20 cms. distance, with its axis in the meridian through the needle. The coil was in series with the magnetizing and compensating coils, and the current was passed through it in such a direction as to neutralize forces introduced by a want of alignment of the coils of the nature shown in fig. 1.

Fig. 1.



A is the magnetizing coil; B, the compensating coil; and C the small auxiliary compensating coil.

A position of the coil C (fig. 1) was found such that no change took place in the period of the needle when the magnetizing current was reversed. One coil was found to be sufficient to overcome the disturbing effect, and when adjusted the compensation was good at all parts of the scale, the variation of the field-intensity being reduced to less than 0.5 per cent.

The Basalt Bars.

A preliminary examination of the five new bars was made as to their permanent magnetism, and the bars, when placed

successively with near ends 26·75 cms. from the magnetometer needle, gave deflexions as noted below on reversal of their positions.

The relative susceptibility of the bars was also tested in a field of 40 c.g.s. units, the deflexions noted being those obtained by reversing the field.

TABLE I.

Bar.	Deflexion due to permanent magnetism.	Deflexion in reversed field of 40 c.g.s.
	mm.	mm.
Sgurr nan Gillean ...	61·3	20·1
Storr	8·0	27·1
Dunvegan	7·0	14·8
Stromness Dyke	0	45·8
Faule Ader	0	3·3

A comparison of the two columns shows that whilst the three Skye specimens are of medium susceptibility, the Sgurr nan Gillean bar is, for rock material, very highly magnetized; the Stromness bar is composed of a material of relatively high susceptibility, and the Faule Ader bar is almost non-magnetic.

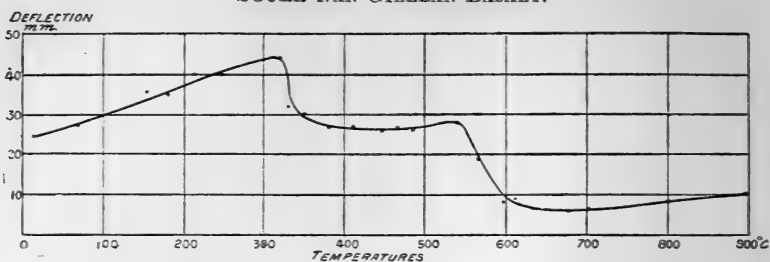
Rock Sections.

Thin sections of the rocks Sgurr nan Gillean, Stromness Dyke, and Faule Ader were examined, and all three were found to contain crystals of magnetite as their chief magnetic constituent uniformly dispersed throughout the sections; in the section of Faule Ader the linear dimensions of the magnetite crystals were about a sixth of those of the crystals in the other two sections. Further than this, an examination of the sections in the microscope gives no clue to the widely different magnetic behaviour of these three rock specimens.

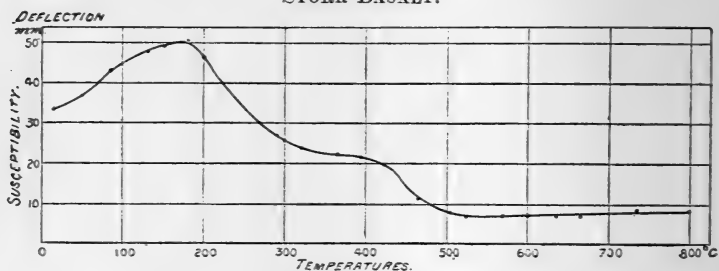
Temperature Effect.

The bars were next tested in the electric furnace in the manner previously described, and curves were obtained which, although exhibiting peculiarities, have a resemblance to the temperature-permeability curve for magnetite, and those for

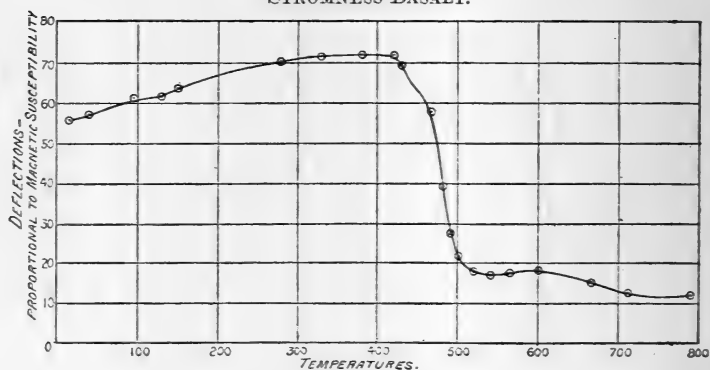
SGURR NAN GILLEAN BASALT.



STORR BASALT.

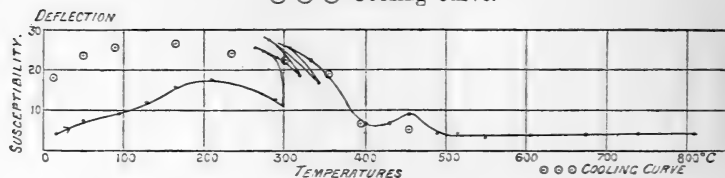


STROMNESS BASALT.



FAULE ADER No. 1. First heating.

⊙ ⊙ ⊙ Cooling curve.



the Sgurr nan Gillea and Storr specimens have a marked resemblance to the temperature curve found for magnetite by Barton and Williams*. Curves are given for all the bars except Dunvegan, which is similar to that found for the Storr bar. In character they appear to be modifications or combinations of two types, one the curve for magnetite†, to which that for the Stromness bar has the greatest similarity, and second, the curves IX., X., and XII.† for Dattenberg bars. In the one type there is a gradual increase of susceptibility up to about 500° C., followed by a rapid descent to a state of minimum susceptibility; in the second type the maximum susceptibility is reached at a comparatively low temperature, and this is followed by a gradual loss of power as the temperature rises. Whether the change exhibited in these curves between the temperatures of 15° and 550° is due entirely to the appearance of a new magnetic constituent as a result of chemical change, or to the effect of change of internal stress, is a point which cannot be decided at present. The temperature of minimum susceptibility had values for the five bars ranging from 430° for the Dunvegan bar to 710° for the Stromness bar, the average being about 565° C. The temperature curve for the Faule Ader bar may be noted as one in which a large increase of susceptibility took place during heating.

To confirm the effect found in this bar an irregular strip of rock from the same block was roughly ground to pass into the furnace, and was heated to 350° C. or thereby, and again cooled. Initially the bar had no observable magnetization, and gave a deflexion of 2 mm. in a reversed field of 35 c.g.s. units. On being cooled, the bar now gave a deflexion of 6.5 mm. in the same field as before, and was now magnetized so as to be capable of affecting a pocket compass. Further heatings appeared to increase the effect slightly.

After the heating of the bars, they were tested under the same conditions as those given in Table I., and the following deflexions were noted (p. 578):—

* B. A. Report, 1892.

† See previous paper.

TABLE II. (June 15-22, 1906).

Bar.	Deflexion caused by remanent magnetism.	Deflexion in reversed field of 40 c.g.s.
	mm.	mm.
Sgurr nan Gillean ...	9.5	16.0
Storr	17.5
Dunvegan	8.5
Stromness	26.5
Faule Ader*	15.0

On being tested about two and a half years after the date on which the above observations were made, it was found that all the bars had lost power, as shown in the next table.

TABLE III. (Nov. 11-12, 1908).

Bar.	Deflexion caused by remanent magnetism.	Deflexion in reversed field of 40 c.g.s.
	mm.	mm.
Sgurr nan Gillean ...	8.7	12.2
Storr	2.3	15.7
Dunvegan	1.2	8.1
Stromness	5.2	20.9
Faule Ader	2.9	11.6

The loss of susceptibility amounts to from 45 to 61 per cent. in the first four bars, and the increase of susceptibility in the fifth bar brings it almost to an equality with that of the first bar.

Effect of Baking and Cooling.

To test whether change of temperature alone could do anything to restore or increase the magnetic quality of basalt, the processes of baking in hydrogen and cooling in liquid air were tried on two bars.

The bar from Sgurr nan Gillean was heated at about 400° C. in a current of hydrogen for various periods, amounting in all to four hours. The total result was to further diminish the magnetic power of the bar, the deflexion caused by remanent magnetism being reduced from 9.5 to 8.7 mm. (see Tables II. and III.).

The temperature of one of the Dattenberg bars was reduced by means of liquid air, and the deflexions were noted which it produced when a field of 60 c.g.s. was reversed, no readings of temperature being taken. The low temperature diminished the susceptibility of the bar to one half its value at ordinary temperature, the bar slowly regaining its former condition as the temperature rose. This behaviour is different from that observed in other materials, *e. g.*, a bar of ilmenite became temporarily 1.2 times stronger in liquid air; a bar of magnetite exhibited a slight increase, also temporary in nature; and a bar of Heusler alloy became 1.2 times stronger than before, the change this time being permanent.

*Average Composition and Probable Average
Susceptibility of Basalt.*

From a series of eight basalts chosen for me by Professor W. W. Watts from Roth's 'Petrography' as giving average analyses, it is found that in these the average content of iron is 6.1 per cent. $\ddot{\text{Fe}}$, or iron in the form of Fe_2O_3 , and 5.9 per cent. Fe , or iron in the form FeO , these two forms varying in quantity in the eight basalts, the $\ddot{\text{Fe}}$ from 16.4 per cent. to zero, and the Fe from 11.6 per cent to zero.

Now if we assume that basalt contains, on an average, 12 per cent. by weight of magnetite, and the average weight of the basalt bars employed is 65 gms., the weight of magnetite contained in the bar is $\frac{12}{100} \times 65$ or 7.8 gms. The density of magnetite is 5.2, and therefore its volume is $\frac{7.8}{5.2}$ or 1.5 c.cs. Since the average volume of the bar is 22 c.cs. $\frac{1.5}{22}$ of that volume, or 7 per cent., is magnetite.

Now, if we consider the susceptibility of magnetite to vary from .15 to .75*, the susceptibility of basalt, if taken as

* Roughly the values found for two bars of magnetite. The former of the two values was erroneously stated in the previous paper as .016 instead of 0.16.

proportional to the quantity of magnetite contained, should lie between $\cdot 01$ and $\cdot 05$. But, since half the iron is present as Fe_2O_3 , the non-magnetic oxide, the lower limit should be reduced to $\cdot 005$, the upper limit being retained, since heat may change the ferric oxide into magnetic oxide. Hence, on the average, and if we neglect demagnetizing action, the susceptibility of basalt should lie between $\cdot 005$ and $\cdot 05$. The values found by Rücker and White* lie between $\cdot 0003$ and $\cdot 0139$, the average value for specimens of basalt found within the British Isles being $\cdot 0026$. The above average limits, therefore, seem to be rather high, but the average susceptibility of the nine bars mentioned in the previous paper is $\cdot 0046$, and that of the Stromness bar is $\cdot 033$.

Effect of Hæmatite.

In its main features the magnetic behaviour of basalt resembles that of magnetite; but there are cases in which another substance plays a part, as in that of the Faule Ader bar (Curve 4). Folgheraiter† has described another instance in which unbaked, ferruginous, non-magnetic clay was heated to a dull red heat, and became noticeably magnetized by the earth's field on cooling. These results seem to point to the sesquioxide of iron—otherwise known as ferric oxide, hæmatite, and Fe_2O_3 —as being a possible factor of disturbance in the cases mentioned. Moissan‡ has stated that the sesquioxide of iron, when heated in an atmosphere of hydrogen or carbonic acid at a temperature between 350° and 440° C., changes in a few hours into the magnetic oxide Fe_3O_4 , the new compound being an allotropic form of the ordinary magnetic oxide, and differing from it in density and in other particulars. It is distinctly probable that this change of non-magnetic sesquioxide into the magnetic oxide at about 400° C. may be the cause of the peculiar changes observed, and that this change constitutes an important factor, though an uncertain one, in questions of terrestrial magnetism. Further, it offers another means of explaining the presence of strong magnetization in surface rocks, other than that which attributes it solely to the magnetizing effect of a lightning flash. Pockels§ employs the observed intensity of magnetization in his basalt bars to calculate the magnetizing lightning current

* Rücker and White, *Proc. R. S.* lxiii. p. 460 (1898).

† Science Abstracts, No. 414, 1898.

‡ H. Moissan, *Comptes Rendus*, lxxvi. p. 600 (1878).

§ F. Pockels, *Ann. d. Physik*, lxiii. p. 195 (1897).

at 10,000 amperes, which he considers a low estimate. The effect of the lightning may be partly a heating one, the rock being heated so far as to transform a non-magnetic into a magnetic constituent, which is then magnetized by the earth's field. A second possibility is to attribute the chemical change to the sun's heat, and the magnetization to the lightning flash, since the experiments of Folgheraiter and Messrs. Platania* have proved that both the earth's field and lightning are operative in producing magnetization in rock material.

Experiments on hæmatite ores are in progress, but sufficient has been done to find that the second allotropic form of magnetite, into which they change on being heated at 400° , behaves similarly to the more common form as regards loss of magnetic power about 550° C.

The experiments described in this paper were made in the Natural Philosophy Buildings of Glasgow University, and I have to thank Professor Gray for valuable suggestions made during their progress.

Glasgow, 30 Nov. 1908.

LII. *The Secondary Spectrum of Hydrogen.*

By C. F. HOGLEY, *A.R.C.Sc., B.Sc.*†

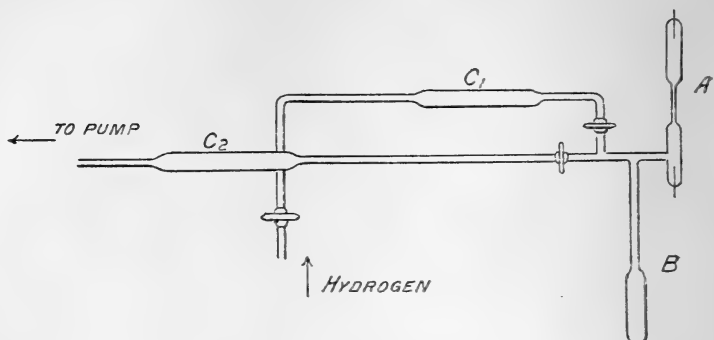
IT is well known that vacuum-tubes filled with hydrogen emit two spectra. One of these consists of the well-known lines which are seen reversed in the spectrum of the sun and in the spectra of some stars. The other spectrum is known as the secondary spectrum, and consists of a multitude of fine lines crowding the visible region, being especially conspicuous in the neighbourhood of the yellow. These lines have not been observed in stellar or solar spectra, and it has been regarded as doubtful whether they are due to hydrogen at all.

It seems almost certain that if this spectrum is **not** due to hydrogen, it is due to some more easily condensible substance. Sir James Dewar has shown that all gases except hydrogen, neon, and helium, are readily condensed by charcoal at the temperature of liquid air. This method of condensation seems to afford an opportunity of testing whether or not the secondary spectrum is in reality due to hydrogen.

* G. & G. Platania, *Comptes Rendus*, cxli. p. 974 (1905).

† Communicated by the Hon. R. J. Strutt, F.R.S.

To carry out the experiment the apparatus shown in the diagram was constructed.



A, vacuum tube; B, bulb containing charcoal; C_1 , C_2 , drying-tubes containing phosphoric oxide. The hydrogen was prepared from sulphuric acid and pure zinc.

The apparatus was exhausted and filled with hydrogen from the generator attached. This was repeated until the lines due to oxygen and nitrogen could no longer be seen in the spectrum of the gas. The charcoal had been heated previous to sealing on to the apparatus to drive off all moisture, and whilst the exhaustion was in progress it was again heated to expel any gases which might be condensed in its pores. The heating was not prolonged sufficiently to drive off all the carbon dioxide present in the charcoal. The amount of this gas was small, and would not interfere with the absorbing properties of the charcoal when cooled by liquid air. When a suitable vacuum had been obtained giving a spectrum of maximum brightness, the bulb B was immersed in liquid air. On observing the spectrum the intensity of the lines was seen to decrease, and the appearance of the discharge in the tube indicated a decrease in pressure of the gas. The relative intensities of the two spectra, however, remained unchanged up to the total disappearance of the whole spectrum, when a discharge would no longer take place through the gas owing to the high vacuum. A little more hydrogen was allowed to enter, and the spectrum reappeared with its original brightness, but after a time the same decrease in intensity as before was observed, though the condensation did not proceed sufficiently to stop the discharge.

Two attempts were made, the first using only a small bulb

containing about 1 gram, and the second with a larger bulb containing about 5 grams of charcoal. The change in the spectrum was only that due to the change in pressure of the gas, as was demonstrated by simply exhausting the apparatus and observing the spectrum.

So far, then, as this method can decide, the evidence points to the conclusion that hydrogen is really the source of the secondary spectrum.

I am indebted to Prof. Strutt, of the Royal College of Science, for his various suggestions during this research, and I take this opportunity of acknowledging my thanks to him.

LIII. *The Action of Electrolytes on Copper Colloidal Solutions.*
By E. F. BURTON, B. A., *Demonstrator in Physics, University of Toronto* *.

1. INTRODUCTION.

THE problem of the effect of the presence of very small traces of electrolytes on the stability of colloidal solutions is one of the most important unsolved questions relating to colloids. The recent paper by Jacques Duclaux † tends to throw doubt on the theory of the isoelectric point first suggested by Hardy ‡; while a later contribution by Lottermoser § dealing with Duclaux's results shows, at least, that we have yet much to learn regarding the interaction of the colloidal particles and the dissolved salts.

The experiments detailed in the present communication are a continuation of those performed by the writer || on the action of small traces of electrolytes on silver and gold colloidal solutions prepared by Bredig's method. In this former work it was found that, as minute quantities of aluminium sulphate were added in increasing amounts to either silver or gold solutions, the velocities with which the particles moved in a given electric field were at first reduced to zero and then reversed. As the particles in each case were negatively charged in the pure solution, according to Hardy's theory, the positively charged aluminium ions, and

* Communicated by Professor J. C. McLennan.

† *Jour. de Ch. phys.* vol. v. 1-2-3, p. 29.

‡ *Proc. Roy. Soc.* vol. lxvi. p. 110; *Jour. of Physiol.* vol. xxix. p. 26.

§ *Zeits. für phys. Chem.* vol. lx. 4, p. 451.

|| *Phil. Mag.* ser. 6, vol. xii. Nov. 1906.

not the negatively charged sulphate ions, are the active ones in bringing about the discharge of the particles. It was further found that, not only was the charge of the particles reversed, but at the point where the charge was zero, *i. e.* at the point when the particles did not move in the electric field (Hardy's isoelectric point), the solutions showed a minimum stability. For example, to refer to a table taken from the former paper, two samples of a gold solution to which

TABLE I.
Gold Colloidal Solution.

No.	Grms. Al added per 100 ccs. sol.	Specific Conduc- tivity at 18° C.	Velocity of the particles in cms./sec./volt/cm.
1.	0	3.6×10^{-6}	* -33×10^{-5}
2.	19×10^{-6}	5.2×10^{-6}	-17.1×10^{-5}
3.	38×10^{-6}	6.6×10^{-6}	$+1.7 \times 10^{-5}$
4.	63×10^{-6}	11.6×10^{-6}	$+13.5 \times 10^{-5}$

* In this table, as in all succeeding ones, the + sign denotes motion toward the cathode, the - sign toward the anode.

were added respectively 19×10^{-6} and 38×10^{-6} gm. of aluminium per 100 ccs. coagulated at the end of a few hours, while another sample of the same solution, to which 63×10^{-6} gm. per cc. were added, did not completely coagulate at the end of four days. As a result of this work we have strong evidence as to the validity of the assumption that, in these solutions at least, coagulation is brought about through the discharge of the colloidal particles by the ions of the electrolyte introduced into the colloidal solution, and that the deportment of the Bredig metallic colloidal solutions is similar to that which Hardy found displayed by his specially treated albumen solutions.

2. PURPOSE OF THE EXPERIMENTS.

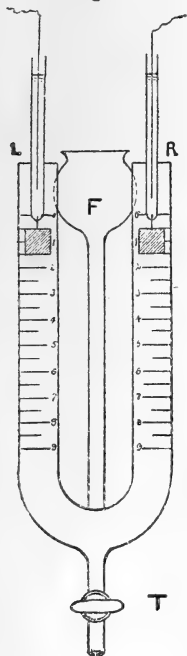
The results given below may be viewed as an instalment of a systematic investigation on the effect of various electrolytic solutions on the velocities of colloidal particles in an electric field, and consequently the effect on the size and sign of the charge possessed by those particles. Copper colloidal solution has been selected because its particles are positively charged,

i. e. the copper particles move to the cathode in the solution ; in other words, they are charged oppositely to those of the silver and gold solutions worked with formerly.

The major part of the paper will deal with, firstly, the effect of solutions of two salts having the same acid radical and, respectively, a monovalent and a trivalent metal, viz., potassium sulphate and aluminium sulphate ; and, secondly, with the effect of solutions of salts having the same metal ion and, respectively, monovalent, divalent, and trivalent acid radicals, viz., potassium chloride, potassium sulphate, potassium phosphate, and potassium ferricyanide. The results of these experiments fit in remarkably well with the theory first suggested by the work of Picton and Linder * and of Hardy †, and supported by the conclusions drawn from analogous experiments on silver and gold solutions.

3. EXPERIMENTAL WORK.

Fig. 1.



The solutions of copper were easily prepared according to Bredig's method by sparking with copper wires under the surface of conductivity water, the specific conductivity of which was usually about 2×10^{-6} . Using a current of from 5 to 6 amperes at a voltage of 110, 250 ccs. could be produced in a short time. The resulting solution was a very clear yellowish-brown liquid in which the particles remained suspended, apparently unchanged, for months. Several samples were made, the average copper content being .0088 grm. per 100 ccs.

The velocity with which the particles moved in an electric field was measured by the U-tube method (see fig. 1) described in a former paper ‡. Usually about 250 ccs. of the solution sufficed to carry out one series of experiments, and as each series was completed a fresh stock of solution was made from the same pieces of copper. In every case the velocity of the particles in the pure solution was found ; the results of these observations are collected in Table II.

* Jour. Chem. Soc. vol. lxi. p. 148 ; vol. lxvii. p. 63 ; vol. lxxi. p. 568.

† *Loc. cit.*

‡ Phil. Mag. ser. 6, vol. xi., April 1906.

TABLE II.
Pure Copper Colloidal Solutions.

No.	Date of preparation.	Date of measurement of velocity.	Specific Conductivity at 18° C.	Velocity at 18° C.
1.	June 19, 1907	June 20, 1907	7.7×10^{-6}	$+23.4 \times 10^{-5}$
2.	„ 20, 1907	„ 26, 1907	6.5×10^{-6}	$+25.4 \times 10^{-5}$
3.	„ 5, 1908	„ 5, 1908	8.2×10^{-6}	$+24.9 \times 10^{-5}$
4.	„ 16, 1908	„ 17, 1908	5.8×10^{-6}	$+25.4 \times 10^{-5}$
5.	„ 23, 1908	„ 24, 1908	4.3×10^{-6}	$+30.4 \times 10^{-5}$
6.	„ 27, 1908	„ 29, 1908	3.1×10^{-6}	$+33.0 \times 10^{-5}$

For solutions Nos. 5 and 6 it will be noticed that the velocities are rather larger than for the others, but it will be seen that the specific conductivities of these solutions are also much lower than those of the others; that is to say, these latter two solutions were freer from electrolytes than the others, and, as we shall see, the general effect of added electrolytes is to reduce the velocity of the particle.

Although the velocities are all given at 18° C., the experiments were not performed at precisely that temperature, but corrections have been introduced to bring each reading to that standard temperature. In the paper * referred to above the following relation has been deduced as holding between the velocity (v) and the coefficient of viscosity of the liquid medium (μ):

$$\mu \cdot v = \text{a constant,}$$

as long as one deals with the same material in the particle and the same liquid medium. That this is in accord with the actual facts is confirmed by the following series of velocity determinations carried out with a silver colloidal solution at the temperatures indicated in Table III. The whole velocity tube was lowered into a glass thermostat which could be kept at the required temperature.

A fundamental necessity for the validity of velocity results obtained in this way, is that the specific conductivity of the superincumbent layers of water should be the same as that of the colloidal solution at the bottom of the tube. As these conductivities in each case were made equal at the ordinary temperature of the room before the liquids were introduced into the

* Phil. Mag. ser. 6, vol. xi., April 1906.

TABLE III.
Silver Colloidal Solutions.

No.	Temperature Centigrade.	Velocity (v).	Viscosity of water (μ) at given temperature.	Product.
1.	3°	15.1×10^{-5}	·016214	24.5×10^{-7}
2.	9.9	18.6×10^{-5}	·013300	24.7×10^{-7}
3.	11	19.6×10^{-5}	·012822	25.1×10^{-7}
4.	21	25.3×10^{-6}	·009922	25.0×10^{-7}
5.	31	30.1×10^{-6}	·007972	24.0×10^{-7}
6.	40.5	37.2×10^{-6}	·006577	24.5×10^{-7}

velocity tube, evidently, in order that the results in Table III. may be of value, the temperature coefficients of the two liquids should be the same. These coefficients were compared for a sample of silver colloidal solution and a quantity of water of very nearly the same initial conductivity. The values of the resistances of these liquids at various temperatures are recorded in Table IV. and illustrated by the curves in fig. 2. From these we may safely conclude that the temperature coefficient of the resistance of colloidal silver solution is that of water itself.

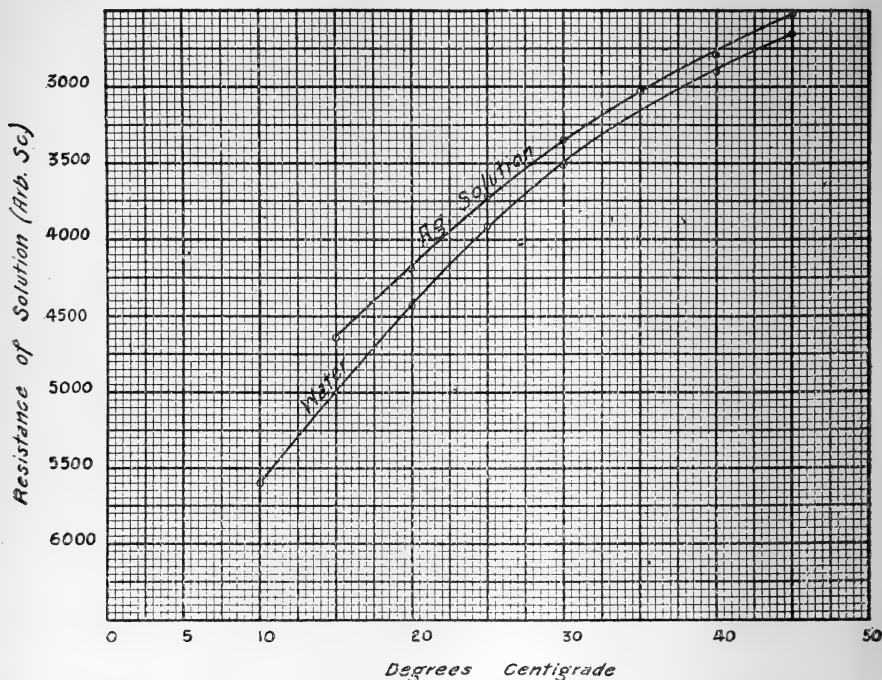
TABLE IV.
Temperature Coefficients of Resistance.

Temperature (Cent.).	Resistance of water (Arb. Sc.).	Resistance of Ag Sol. (Arb. Units.)
10°	5600	
15	4640
20	4420	4180
25	3910	3720
30	3510	3350
35	3150	3030
40	2900	2790
45	2660	2520

It is quite justifiable to apply these results for silver solutions to the manifestly analogous solutions of the other

metals in water, in particular to the copper solutions now under consideration.

Fig. 2.



With each of the salts named in the foregoing section the procedure was the same. Solutions of the salts of determined normality were prepared with the conductivity water, and the amount of such solution added to a given quantity of colloidal solution was measured by counting drops as they came from a fine burette. The tap of the burette was turned so that the liquid dropped at a convenient rate, the number of drops given by 2 ccs. of the liquid was counted, a required number of drops was allowed to fall into the chosen amount of colloidal solution, and finally the number of drops in 2 ccs. was counted again with the tap still turned as at the beginning; the two calibrations always agreed, the burettes giving from 24 to 30 drops per cc. The burettes were always washed with distilled water and, just before being used for any solution, were rinsed with the latter several times. One was thus enabled to introduce into a given amount of colloidal solution a very exact, small quantity of a given electrolytic solution.

4. RESULTS.

A typical series of experiments on the effect of each of the electrolytes was as follows :—The velocity of the particles in the pure copper solution was first taken ; gradually increasing quantities of the chosen electrolytic solution were then added to fresh samples of the stock copper solution and the corresponding velocity measured each time. The measurement of one velocity was completed usually within one hour of the addition of the electrolyte. All of the salts used were Kahlbaum's purest and were not specially re-tested.

a. *Potassium Chloride.*

In Table V. the velocities corresponding to the various amounts of this salt are given. Column 1 gives the number of grams of Cl (as KCl) per 50 ccs. of colloidal solution, a $\frac{N}{100}$ ($\frac{1}{100}$ gram.-mol.) solution of KCl being added drop by drop ; column 2 gives the specific conductivity of the mixture in each case. In the next column the velocities of the particles (in cms. per sec. per volt per cm.) are written, the positive sign indicating that the motion of the colloidal particle was toward the cathode. In each table are included particulars as to the coagulation caused in each case.

TABLE V.
KCl added to Copper Colloidal Solution.

No.	Grams of Cl per 50 ccs. Cu Sol.	Spec. Cond. at 18° C.	Velocity at 100° C.	Coagulation.
1.....	0	8.2×10^{-6}	$+24.9 \times 10^{-5}$	
2.....	3.0×10^{-5}	11.7×10^{-6}	$+25.7 \times 10^{-5}$	No sign.
3.....	6.8×10^{-5}	15.3×10^{-6}	$+26.2 \times 10^{-5}$	„ „
4.....	13.1×10^{-5}	22.0×10^{-6}	$+22.8 \times 10^{-5}$	„ „
5.....	27.3×10^{-5}	34.2×10^{-6}	$+18.7 \times 10^{-5}$	„ „

Anticipating the results which follow, we shall find that they confirm the fact that with colloidal solutions the particles of which are positively charged, the negatively charged ions are the ones to which the coagulative action is due ; consequently the amounts of electrolyte are entered always in terms of the weight of the acid radical.

It will be noticed from the table, that at first the addition of KCl produced a slight increase in the velocity, but as larger amounts were added the velocity reached a maximum and then slowly decreased. It is important to observe that in the above samples there was no sign of coagulation even at the expiration of several weeks. Two of the solutions did coagulate at the end of six weeks, but this interval is so long that this result may have been due to some ulterior cause.

b. *Potassium Sulphate.*

Similar results for this salt, a $\frac{N}{200}$ (gram-molecular) solution of which was used, are given in Table VI.

TABLE VI.
K₂SO₄ added to Copper Colloidal Solution.

No.	Grams of SO ₄ per 50 ccs. Cu Sol.	Spec. Cond. at 18° C.	Velocity at 18° C.	Coagulation.
1.....	0	5.8×10^{-6}	$+25.4 \times 10^{-5}$	
2.....	3.6×10^{-5}	9.3×10^{-6}	$+25.3 \times 10^{-5}$	No sign.
3.....	8.9×10^{-5}	12.4×10^{-6}	$+24.0 \times 10^{-5}$	„ „
4.....	17.8×10^{-5}	18.7×10^{-6}	$+21.8 \times 10^{-5}$	In two weeks.
5.....	44.5×10^{-5}	38.5×10^{-6}	$+14.4 \times 10^{-5}$	In two hours.
6.....	71.2×10^{-5}	58.3×10^{-6}	0	In one hour.

When more KCl and K₂SO₄ than the greatest amounts shown in the above tables was added to 50 ccs. of the colloidal solution, the conductivity became so large that the current through the liquid spoiled the velocity readings.

c. *Aluminium Sulphate.*

In similar experiments on colloidal gold and silver solutions from 18×10^{-5} to 25×10^{-5} gram of Al₂(SO₄)₃ per 100 ccs. of colloidal solution was sufficient to reduce the velocity of the gold and silver particles to zero. The very strong action of this salt was attributed to the trivalent aluminium ion. According to the theory then suggested, this salt should not have nearly so strong an action on the copper colloidal solution on account of the fact that the copper particle bears a positive charge, and therefore the active ion in the aluminium solution will be the SO₄. The results given in Table VII. confirm this view entirely.

TABLE VII.

 $\text{Al}_2(\text{SO}_4)_3$ added to Copper Colloidal Solution.

No.	Grams of SO_4 per 50 ccs. Cu Sol.	Spec. Cond. at 18°C .	Velocity at 18°C .	Coagulation.
1.....	0	7.7×10^{-6}	$+23.4 \times 10^{-5}$	No sign. * *
2.....	6.7×10^{-5}	12.9×10^{-6}	$+21.5 \times 10^{-5}$	
3.....	13.4×10^{-5}	14.9×10^{-6}	$+19.2 \times 10^{-5}$	
4.....	26.0×10^{-5}	20.2×10^{-6}	$+18.5 \times 10^{-5}$	

* The two samples were not retained for coagulation observations.

It will be seen from the table that, although about 62×10^{-5} gram of aluminium sulphate was added to 100 ccs. of copper colloidal solution, there is very slight diminution of the velocity, while in none of the samples was there any immediate sign of coagulation—results in marked contrast with those obtained when the same electrolyte was added to gold and silver colloidal solutions.

d. *Potassium Phosphate.*

This salt was chosen as one representative of those possessing a trivalent acid radical. That its action on the copper is very keen is shown by the numbers in Table VIII.

TABLE VIII.

 K_3PO_4 added to Copper Colloidal Solution.

No.	Grams of PO_4 per 50 ccs. Cu Sol.	Spec. Cond. at 18°C .	Velocity at 18°C .	Coagulation.
1.	0	6.5×10^{-6}	$+25.4 \times 10^{-5}$	No immed. sign. " " 3 to 4 hours. In 2 hours. In a few hours.
2.	1.7×10^{-5}	7.8×10^{-6}	$+21.5 \times 10^{-5}$	
3.	3.5×10^{-5}	8.3×10^{-6}	$+16.8 \times 10^{-5}$	
4.	6.8×10^{-5}	9.9×10^{-6}	$+3.4 \times 10^{-5}$	
5.	10.2×10^{-5}	11.7×10^{-6}	-4.8×10^{-5}	
6.	15.6×10^{-5}	15.7×10^{-6}	-7.9×10^{-5}	

e. Potassium Ferricyanide.

The results (Table IX.) for this electrolyte₂ which also has a trivalent acid radical, are in complete accord with those given by the phosphate.

TABLE IX.

$K_6(FeCy_6)_2$ added to Copper Colloidal Solution.

No.	Grams of $FeCy_6$ per 50 ccs. Cu Sol.	Spec. Cond. at 18° C.	Velocity at 18° C.	Coagulation.
1.	0	4.3×10^{-6}	$+30.4 \times 10^{-5}$	No. immed. sign.
2.	7.5×10^{-5}	7.6×10^{-6}	$+14.0 \times 10^{-5}$	
3.	15.0×10^{-5}	9.3×10^{-6}	$+3.8 \times 10^{-5}$	In one hour.
4.	22.5×10^{-5}	13.0×10^{-6}	$+1.0 \times 10^{-5}$	" "
5.	30.0×10^{-5}	18.8×10^{-6}	-1.5×10^{-5}	" "
6.	45.0×10^{-5}	28.6×10^{-6}	-9.1×10^{-5}	" "

We see that very small quantities of either potassium phosphate or potassium ferricyanide are sufficient to reduce the velocity of the copper particles to zero and even to reverse the direction of their motion in an electric field.

5. COLLOCATION OF RESULTS.

In order the better to illustrate the relation between the number of molecules of the various electrolytes added per cc. of the copper colloidal solution and the resulting velocity of the copper particles in a given field, all of the results recorded in Tables V. to IX. inclusive are brought together in slightly different form in Table X., and illustrated in the curves in figs. 3 and 4. Column 1 gives the numbers from which the solutions may be identified in the above tables. In column 2 is given the normality, in respect of the electrolyte, of the mixture of the colloid and the electrolyte; *i. e.* the number of gram-molecules of the salt per cc. of the mixture. According to the accepted dissociation theory, we may look upon the salts in these extremely dilute solutions as being completely ionized, and so the normality as defined above will be directly proportional to the number of ionized molecules of the particular salt per cc. However,

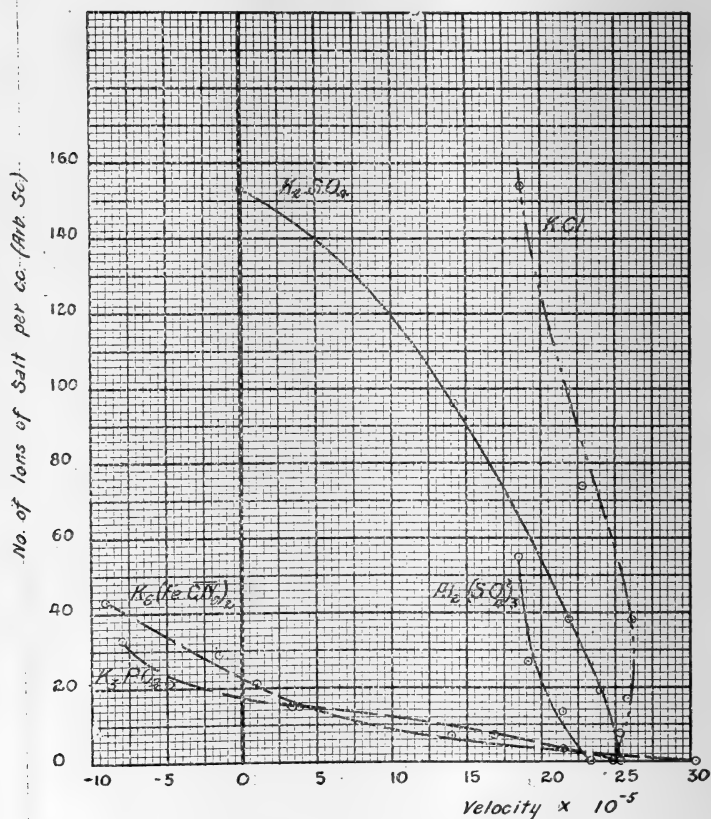
TABLE X.

Relation between the number of negative ions added per cc. to the copper colloidal solutions and the resulting velocities of the particles.

Solution.	Normality : gram-mols per cc.	Numbers proportional to the number of ions per cc. : Gram- ions/cc.	Velocity at 18° C.
KCl 1.	0	0	+24.9 × 10 ⁻⁵
2.	17.0 × 10 ⁻⁶	17.0 × 10 ⁻⁶	+25.7 × 10 ⁻⁵
3.	38.0 × 10 ⁻⁶	38.0 × 10 ⁻⁶	+26.2 × 10 ⁻⁵
4.	74.0 × 10 ⁻⁶	74.0 × 10 ⁻⁶	+22.8 × 10 ⁻⁵
5.	154.0 × 10 ⁻⁶	154.0 × 16 ⁻⁶	+18.7 × 10 ⁻⁵
K ₂ SO ₄ 1.	0	0	+25.4 × 10 ⁻⁵
2.	7.7 × 10 ⁻⁶	7.7 × 10 ⁻⁶	+25.3 × 10 ⁻⁵
3.	19.2 × 10 ⁻⁶	19.2 × 10 ⁻⁶	+24.0 × 10 ⁻⁵
4.	38.4 × 10 ⁻⁶	38.4 × 10 ⁻⁶	+21.8 × 10 ⁻⁵
5.	96.0 × 10 ⁻⁶	96.0 × 10 ⁻⁶	+14.4 × 10 ⁻⁵
6.	153.0 × 10 ⁻⁶	153.0 × 10 ⁻⁶	+ 0.0 × 10 ⁻⁵
Al ₂ (SO ₄) ₃ ... 1.	0	0	+23.4 × 10 ⁻⁵
2.	4.6 × 10 ⁻⁶	13.8 × 10 ⁻⁶	+21.5 × 10 ⁻⁵
3.	9.2 × 10 ⁻⁶	27.6 × 10 ⁻⁶	+19.2 × 10 ⁻⁵
4.	18.3 × 10 ⁻⁶	54.9 × 10 ⁻⁶	+18.5 × 10 ⁻⁵
K ₃ PO ₄ 1.	0	0	+25.4 × 10 ⁻⁵
2.	3.6 × 10 ⁻⁶	3.6 × 10 ⁻⁶	+21.5 × 10 ⁻⁵
3.	7.2 × 10 ⁻⁶	7.2 × 10 ⁻⁶	+16.8 × 10 ⁻⁵
4.	14.4 × 10 ⁻⁶	14.4 × 10 ⁻⁶	+ 3.4 × 10 ⁻⁵
5.	21.6 × 10 ⁻⁶	21.6 × 10 ⁻⁶	- 4.8 × 10 ⁻⁵
6.	32.8 × 10 ⁻⁶	32.8 × 10 ⁻⁶	- 7.9 × 10 ⁻⁵
K ₆ (FeCy ₆) ₂ ...1.	0	0	+30.4 × 10 ⁻⁵
2.	3.55 × 10 ⁻⁶	7.1 × 10 ⁻⁶	+14.0 × 10 ⁻⁵
3.	7.15 × 10 ⁻⁶	14.3 × 10 ⁻⁶	+ 3.8 × 10 ⁻⁵
4.	10.7 × 10 ⁻⁶	21.4 × 10 ⁻⁶	+ 1.0 × 10 ⁻⁵
5.	14.3 × 10 ⁻⁶	28.6 × 10 ⁻⁶	- 1.5 × 10 ⁻⁵
6.	21.4 × 10 ⁻⁶	42.8 × 10 ⁻⁶	- 9.1 × 10 ⁻⁵

if we are to compare the efficiency of various ions in discharging the colloidal particles, we must deduce numbers directly proportional to the number of such ions per cubic centimetre. For example, taking the above definition of normality, we may look upon a 3×10^{-6} normal solution of potassium phosphate as containing the same number of

Fig. 3.

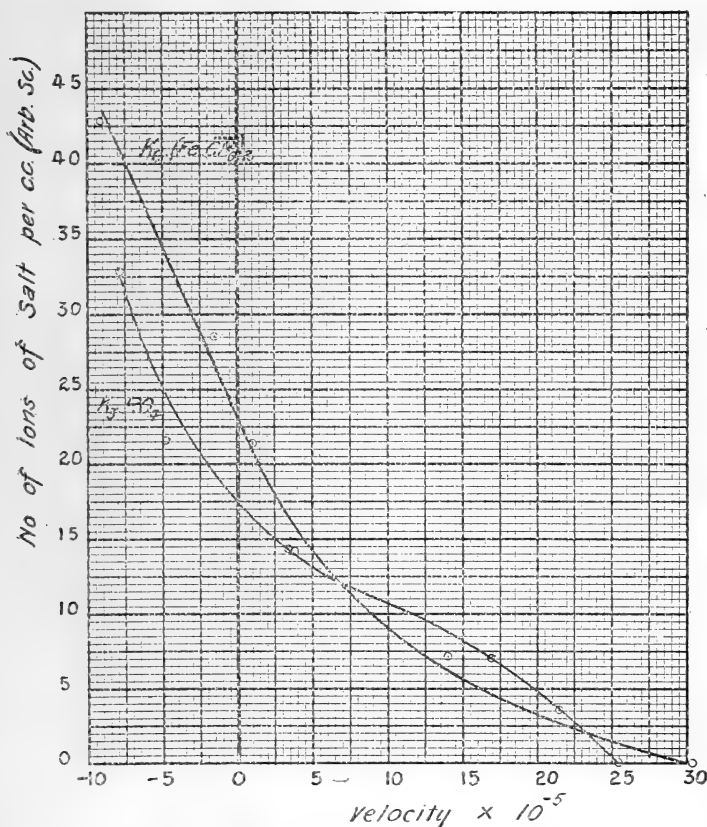


molecules as the number of molecules of $K_6(FeCy_6)_2$ in a 3×10^{-6} normal solution of potassium ferricyanide. But the latter solution will contain twice the number of $FeCy_6$ ions that the former contains of PO_4 ions. Consequently in column 3 are written the numbers directly proportional to the number of acid radical ions present per cubic centimetre; of course, those numbers opposite the aluminium sulphate and the potassium ferricyanide are the only ones which will

differ in the two columns. In the last column are copied the various velocities in cms./sec./volt/cm.

The curves in fig. 3 are drawn with the velocities as abscissæ and the numbers proportional to the number of ions per cc. as ordinates. The very marked overlapping of the curves for potassium phosphate and potassium ferricyanide, which are shown to larger scale in fig. 4, at once suggests

Fig. 4.



the fact that the two ions PO_4 and $FeCy_3$ have the same power of reducing the velocity of the copper particle, and, therefore, of producing the coagulation of the copper. The evidences of coagulation given in the last columns of Tables VIII. and IX. support the latter statement, as does the work of many other writers. Although the experiments

on aluminium sulphate were not carried out as far as those on potassium sulphate, nevertheless the corresponding curves show a remarkable coincidence in the region common to the two curves.

These latter two curves have an additional importance in that they show that the action of the SO_4 ion is practically independent of the metal ion. Since aluminium is trivalent and potassium monovalent, if the metal ions exerted any marked influence on the copper particle we should expect these two curves to be very far apart.

Again, comparing the five curves, one has the very strongest evidence of the great differences in the powers of monovalent, divalent, and trivalent acid ions to reduce the velocity of the positively charged copper, and consequently, to produce coagulation. Examination of the curves will show that the velocity results indicate that the ratios of the powers of various acid ions to reduce the velocity of the copper particles are not very far removed from the observed ratios of the powers of the same ions to produce coagulation. These latter ratios for monovalent, divalent, and trivalent ions as found by Picton and Linder* were 1 : 35 : 1023; the analogous ratios suggested in accordance with Whetham's theory† of coagulation were 1 : 32 : 1024.

Some time since Duclaux published results of his work on colloids synthetically prepared by a double decomposition, as, for example, the colloidal solution of copper ferrocyanide resulting from the action of cupric chloride (CuCl_2) on potassium ferrocyanide. In this case he finds that the colloidal particle retains varying quantities of the potassium, to which he gives the credit for the existence of the colloidal particle. This portion of the colloidal unit he calls the "partie active," and draws the conclusion that the coagulation of the colloid is brought about through the substitution of the "partie active" by some other ions. In the course of his argument, he expresses "most formal doubts as to the exactitude of Hardy's law according to which the coagulative power of an ion is the greater, the higher its valency." The law of Hardy referred to above is based on the experimental work of Picton and Linder and others, as well as that of Hardy himself, and is uniquely supported by the theoretical contribution of Whetham. The conclusions arrived at by Duclaux seem so diametrically opposed to all this work, as to suggest the possibility that Duclaux was dealing with colloidal solutions quite different in their constitution from those upon which the above law was based. This conclusion

* *Loc. cit.*

† 'Theory of Solution,' p. 396.

is supported by Lottermoser's criticism of Duclaux's work, in which the former constantly refers to a class of colloids which can be changed by the addition of certain electrolytes from hydrosols to hydrogels and *vice versa*; whereas in no case have the metallic colloids at present under discussion shown that they are thus reversible.

Exception must be taken, however, to the remark of Duclaux when he says that "it is by no means evident that the immobility of a micelle in an electric field implies that the micelle does not carry a charge." Historically, the only reason we have for attributing the possession of a charge to the colloidal particle is the fact that it does move in an electric field, and, if dependence is not to be placed on this assumption, there is no ground for speaking of the particles as being charged at all. If we mean anything by the statement that a colloidal particle is positively or negatively charged, it is surely a just conclusion to say that, if the particle does not move in an electric field, it has lost its charge.

6. CONCLUSIONS.

1. For a copper colloidal solution, the particles of which are positively charged (*i. e.* move to the cathode), the influence of added electrolytes on the velocity of the particle has been determined. The electrolytes used were potassium chloride, potassium phosphate, potassium sulphate, potassium ferri-cyanide, and aluminium sulphate.

2. Evidence has been produced to show that it is the ion bearing a negative charge that is active in reducing the velocity.

3. This power of the negative ion depends on the valency in a way analogous to the valency relations found by Picton and Linder, and by Hardy, for the coagulative power of ions.

4. The discharging power of two negative ions of the same valency is the same.

5. Observations on the coagulation produced in each case shows that the particles coagulate when they lose their charge.

In conclusion, I wish to thank Professor J. C. McLennan for his kind assistance and encouragement at all times; I also desire to express my indebtedness to Professor A. B. Macallum for his helpful suggestions on many occasions.

University of Toronto,
November 2nd, 1908.

LIV. *Contribution to the Osmotic Theory of Solutions.* By the Earl of BERKELEY, F.R.S., and C. V. BURTON, D.Sc. (London) *.

IN the following communication our aim has been to obtain exact relations between the various quantities involved in osmotics, without making any assumption about the properties of, or even the separate existence of the components forming the solution †. In other words, we have tried to express the osmotic phenomena in terms of the physical properties of the solution itself.

Although Rothmund ‡ postulates two osmotic pressures for liquid mixtures, yet it seems to have been overlooked that any solution whatever (necessarily containing two components) must have more than one osmotic pressure. As pointed out by one of us §, besides the ordinary osmotic pressure of an aqueous solution of sugar, realizable by means of a copper ferrocyanide membrane, there is another which would be realizable if we could place the solution in communication with liquid sugar (say at its melting-point) through a membrane permeable to the sugar only. It is probable that this complementary osmotic pressure has been ignored because it was assumed that the properties of the liquid solute were involved, and when the solute was unknown in the liquid state it may have been argued that this pressure was imaginary.

In this paper, however, we shall establish a relation between the two osmotic pressures which, as above stated, only involves the physical properties of the solution, thus proving that in so far as the ordinary osmotic pressure may be regarded as a real physical quantity, the complementary pressure can be similarly so regarded ||. We shall also obtain a relation applicable to the final stratification of any solution when subjected to gravity or other steady bodily force.

These relations will be established in two ways : first by the consideration of a column of solution in static osmotic equilibrium ; secondly, by a brief examination of the dynamic processes involved in the progress towards that equilibrium.

This latter method evidently touches on the theory of

* Communicated by the Authors.

† This is of some little importance, for, in many chemical reactions, it has been proved that a solution may contain substances which cannot be isolated.

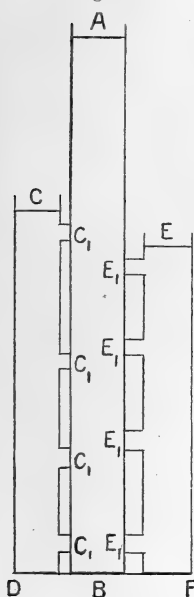
‡ *Zeit. Phys. Chem.* vol. xxvi. p. 433 (1898).

§ *Proc. Roy. Soc., Series A*, vol. lxxix. p. 129.

|| One of us (C. V. B.) would limit this statement to its differential aspect.

diffusion. In the ordinary theory of diffusion, as developed originally by Nernst*, the motion of the solute under the influence of osmotic pressure differences is considered. A more rigorous theory (especially when strong solutions are in question) must deal with the osmotic pressure differences (differences in the ordinary osmotic pressure and in its complement) which cause the solute and solvent particles to stream past one another; regarding the matter from this point of view we shall deduce thermodynamically an exact expression for the forces which act on the two sets of particles.

Fig. 1.



General case for a solution of two volatile (or involatile) components †.

We will distinguish the two components as solute and solvent.

Let the three columns, represented in fig. 1, be at a constant temperature throughout and be subjected to the influence of gravity. AB is a column of solution, CD and EF are columns of pure solvent and pure solute, respectively. The membranes C₁ are permeable to the solvent only, while E₁ are permeable to the solute only.

Let the relative positions of the upper surfaces of the three liquids be such that at any given level the vapour-pressure ‡ due to the solvent in the solution is the same as that due to the pure solvent, and similarly for the vapour of the solute. Evidently, under these conditions, the three liquids can be in equilibrium.

* *Zeit. Phys. Chem.* vol. ii. (1888) p. 613.

† The equations can be extended so as to include any number of components.

‡ The vapour-pressures are only introduced into the argument so as to give a clear mental picture of the equilibrium subsisting between the liquids. The calculation, however, is quite independent of vapour-pressures and is applicable to solutions neither of whose components is volatile; it is perfectly valid if one or more of the liquids are limited upwards by a piston, by means of which a pressure or even a tension could be imparted, provided always that the pressures are so arranged that the three liquids are in osmotic equilibrium with one another. It may be mentioned here that the final results apply to any solution extended upwards indefinitely; the extension downwards may or may not have a limit according to the physical "constants" of the solution. The results also apply to any mixture of molecules, whether of gases, vapours, or liquids.

Symbols used throughout this communication.

For the solution :—

w is the specific volume of the solution.

s_1 is the decrease in volume of a large mass of solution when one gram of solvent is withdrawn from it ;

s_2 is the corresponding quantity for the withdrawal of one gram of solute.

c_1 is the number of grams of solvent in one gram of solution ; c_2 is the corresponding quantity of solute.

P_1 is the osmotic pressure when the solution is in osmotic equilibrium with the pure solvent ; P_2 is the osmotic pressure when it is in equilibrium with the solute.

p is the pressure on the solution.

Ω is the total pressure of the mixed vapours in equilibrium with the solution.

G is the gravity potential.

For the solvent :—

u is the specific volume of the liquid.

q_1 is the pressure on the pure solvent.

π_{00} is the vapour-pressure when the solvent is under the pressure of its own vapour.

v is the specific volume of the vapour.

For the solute :—

y is the specific volume of the liquid solute.

q_2 is the pressure on the pure liquid.

ϕ_{00} is the vapour-pressure when the solute is under the pressure of its own vapour.

z is the specific volume of the vapour.

Where necessary, the particular vapour-pressure which is under discussion will be identified by the addition of a suffix denoting the pressure on the liquid ; the specific volume of the vapour will be identified by a suffix denoting the pressure on the vapour. For example :—

π_q is the vapour-pressure of the solvent when there is a pressure q_1 on the solvent.

π_p is the vapour-pressure of the solvent in the solution when there is a pressure p on the solution.

π_π is the vapour-pressure of the solvent in the solution when the solution is under pressure π .

z_ϕ is the specific volume of the solute vapour when the pressure on the vapour is ϕ .

Stratification of the solution by gravity.

We will first consider the stratification as referred to P_1, c_2, u, s_1 . Take as independent variables the concentration c_2 at any point in the column of solution, and p , the hydrostatic pressure at that point; w and s_1 are both functions of c_2 and p ; u is a function of $p - P_1 (= q_1)$, while P_1 in turn is a function of c_2 and p , so that u is a function of c_2 and p .

$\partial P_1 / \partial c_2$ is the rate of change of the osmotic pressure (measured against the pure solvent) with change of concentration, the hydrostatic pressure on the solution remaining constant.

$\partial P_1 / \partial p$ is the rate of change of the osmotic pressure with change of hydrostatic pressure on the solution, the concentration remaining constant.

dp/dG is the rate of change of hydrostatic pressure in the column as we pass to places of higher gravitational potential (*i. e.*, as we pass downwards through the column).

dc_2/dG is similarly the rate of change of concentration with change of gravitational potential.

When we pass from a level in the solution where the gravitational potential is G to a place where it is $G + dG$, the hydrostatic pressure is different by

$$dp = dp/dG \times dG \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, between the same levels, the increment in concentration is

$$dc_2 = dc_2/dG \times dG \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, the osmotic pressure, P_1 , when we pass from G to $G + dG$, is different, because (1) the hydrostatic pressure has changed by dp , and (2) the concentration has changed by dc_2 , thus

$$\begin{aligned} dP_1 &= \frac{\partial P_1}{\partial p} dp + \frac{\partial P_1}{\partial c_2} dc_2 \\ &= \left(\frac{\partial P_1}{\partial p} \cdot \frac{dp}{dG} + \frac{\partial P_1}{\partial c_2} \cdot \frac{dc_2}{dG} \right) dG \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

But, in the column AB

$$dG = w dp \quad ; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

similarly in the column CD we have

$$dG = u d(p - P_1) = u dp - u dP_1 \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore (u - w) dp = u dP_1 \quad . \quad . \quad . \quad . \quad (6)$$

By substituting in (3) we get

$$\frac{dc_2}{dG} = \left(u - w - u \frac{\partial P_1}{\partial p} \right) / uw \frac{\partial P_1}{\partial c_2} \quad \dots \quad (7)$$

Now, in Appendix I. (equation 29), it is proved that

$$\frac{\partial P_1}{\partial p} = \frac{u - s_1}{u};$$

substituting in (7) we get

$$\frac{dc_2}{dG} = (s_1 - w) / uw \frac{\partial P_1}{\partial c_2} \quad \dots \quad (8)$$

Prof. Callendar, using a different notation, states* that

$$s_1 - w = -c_2 \partial w / \partial c_2 \quad \dots \quad (9)$$

(this relation can be easily proved), hence (8) becomes

$$\frac{dc_2}{dG} = - \frac{c_2 \partial w / \partial c_2}{uw \partial P_1 / \partial c_2} \quad \dots \quad (10)$$

Now consider the stratification in relation to P_2 , c_1 , y , and s_2 ; that is, the relations subsisting between the solution and the pure solute in column EF.

An equation exactly analogous to (10) will result, where P_2 will replace P_1 when the solution at hydrostatic pressure p is in osmotic equilibrium with the solute at pressure $p - P_2$ and c_1 will replace c_2 ; the equation is

$$\frac{dc_1}{dG} = - \frac{c_1 \partial w / \partial c_1}{yw \partial P_2 / \partial c_1} \quad \dots \quad (11)$$

Since $c_1 = 1 - c_2$ we get

$$\frac{dc_2}{dG} = - \frac{dc_1}{dG}, \quad \frac{\partial w}{\partial c_2} = - \frac{\partial w}{\partial c_1}, \quad \text{and} \quad \frac{\partial P_1}{\partial c_1} = - \frac{\partial P_1}{\partial c_2};$$

and also from (10) and (11)

$$\frac{\partial P_1 / \partial c_2}{\partial P_2 / \partial c_2} = - \frac{c_2}{(1 - c_2)} \cdot \frac{y}{u} \quad \dots \quad (12)$$

Transformation.

The relations (8), (10), (11), and (12), have been obtained from a simple consideration of equilibrium conditions; but, though these equations are rigorously true, a difficulty sometimes arises in their exact interpretation. For u , y are

* Proc. Roy. Soc. Series A, vol. lxxx, p. 470.

the specific volumes of the pure solvent and the pure solute respectively, when in osmotic equilibrium with the solution at hydrostatic pressure p . If the pressure p is not great (one atmosphere for example), and if, at the same time, the concentration of (say) the solute is high, the corresponding pressure on the solvent will have a high negative value, that is to say there will be a considerable tension on the solvent; in many cases this tension will exceed the tension which the solvent can support and will certainly not be experimentally realizable. Moreover, when the solute is a solid substance, our conception of osmotically equilibrating columns cannot be realized for the pure solute.

We can, however, put our differential equations in such a form that they involve only the properties of the solution, with their differential coefficients, *at the point under consideration*; it is evident, *à priori*, that all criteria of the behaviour of the solution must be expressible in such terms.

The relations desired will first of all be deduced from (8), (10), (11), and (12), by means of a simple transformation; a more general investigation will then be given, starting from first principles, and without restriction to the case where equilibrium is supposed to have been attained. (See pp. 609-613.)

In the first place we must adopt another mode of differentiating the osmotic pressure with respect to concentration. Instead of supposing solutions of concentrations c_2 , and $c_2 + dc_2$, and both at pressure p , to be *separately* put in equilibrium with the *pure solvent* with resulting osmotic pressures P_1 and $P_1 + dP_1$; let us put the same two solutions in direct osmotic communication with one another through a membrane, permeable to the solvent only. The two solutions will be in equilibrium when the pressures on them differ by dp , where $dp = dP_1$ and dP_1 is the (infinitesimal) osmotic pressure of the solution $c_2 + dc_2$ relatively to solution c_2 . On dividing this dP_1 by dc_2 we obtain a differential coefficient which we shall write $\frac{dP_1}{dc_2}$.

This coefficient refers to two infinitesimally different solutions belonging to that (singly infinite) series, any member of which is in osmotic equilibrium with the solution of concentration c_2 and pressure p .

Now, since the solution c_2 and the solution $c_2 + dc_2$ are in direct osmotic equilibrium with one another, they will be separately in equilibrium with the pure solvent at some one

pressure q_1 ; and despite the fact that this pressure may, in some cases, have a high negative value, yet we may regard the differentiation as conditioned by $q_1 = \text{constant}$; so that

$$\frac{\mathfrak{d}P_1}{\mathfrak{d}c_2} = \left(\frac{\partial P_1}{\partial c_2} \right)_q \dots \dots \dots (13)$$

The transformation required is now easily obtained as follows:

$$dP_1 = \left(\frac{\partial P_1}{\partial c_2} \right)_p dc_2 + \frac{\partial P_1}{\partial p} dp = \left(\frac{\partial P_1}{\partial c_2} \right)_p dc_2 + \frac{\partial P_1}{\partial p} (dq + dP_1).$$

$$\therefore dP_1 \left(1 - \frac{\partial P_1}{\partial p} \right) = \left(\frac{\partial P_1}{\partial c_2} \right)_p dc_2 + \frac{\partial P_1}{\partial p} dq, \left\{ \right.$$

$$\text{also } dP_1 = \left(\frac{\partial P_1}{\partial c_2} \right)_q dc_2 + \frac{\partial P_1}{\partial q} dq. \left. \right\}$$

Dividing the former of these by $\left(1 - \frac{\partial P_1}{\partial p} \right)$, equating the coefficients of dc_2 and remembering that $1 - \frac{\partial P_1}{\partial p} = \frac{s}{u}$ (see Appendix I.), we get

$$\left(\frac{\partial P_1}{\partial c_2} \right)_p = \left(1 - \frac{\partial P_1}{\partial p} \right) \left(\frac{\partial P_1}{\partial c_2} \right)_q = \frac{s}{u} \frac{\mathfrak{d}P_1}{\mathfrak{d}c_2} \dots \dots (14)$$

The equations (8), (10), (11), and (12) can now be replaced by

$$\frac{dc_2}{dG} = \frac{s_1 - w}{s_1 w \frac{\mathfrak{d}P_1}{\mathfrak{d}c_2}} = - \frac{c_2 \frac{\partial w}{\partial c_2}}{s_1 w \frac{\mathfrak{d}P_1}{\mathfrak{d}c_2}} \dots \dots \dots (15)$$

$$\frac{dc_1}{dG} = \frac{s_2 - w}{s_2 w \frac{\mathfrak{d}P_2}{\mathfrak{d}c_1}} = - \frac{c_1 \frac{\partial w}{\partial c_1}}{s_2 w \frac{\mathfrak{d}P_2}{\mathfrak{d}c_1}} \dots \dots \dots (16)$$

$$\frac{\mathfrak{d}P_1}{\mathfrak{d}c_2} / \frac{\mathfrak{d}P_2}{\mathfrak{d}c_1} = - \frac{c_2}{1 - c_2} \cdot \frac{s_2}{s_1} \dots \dots \dots (17)$$

or

$$c_1 s_1 \mathfrak{d}P_1 / \mathfrak{d}c_2 = c_2 s_2 \mathfrak{d}P_2 / \mathfrak{d}c_1 \dots \dots \dots (18)$$

Comparison of Equation (18) with Experiment.

The only data suitable for verifying the equation are those derived from the experimental examination of the vapour-pressures of binary mixtures by Zawidski* and others; it is

* *Zeit. Phys. Chem.* vol. xxxv. pp. 129-203.

therefore necessary to deduce a relation connecting $\frac{\partial P_1}{\partial c_2}$ with the vapour-pressure, this is obtained as follows :—

Prof. Porter*, assuming the validity of the partial pressure law, has put forward an equation connecting the osmotic pressure and the vapour-pressure of a mixture when both components are volatile. This equation can be made exact, whether the partial pressure law holds or not†, by defining π_Ω as the pressure of the pure solvent vapour which is in osmotic equilibrium with the mixed vapours (Ω) through a membrane permeable to the solvent vapour only; an exactly complementary definition holding good for ϕ_Ω .

The equation in question, in its strict signification, may be regarded as expressing the osmotic pressures of the volatile mixture in terms of the pressure of the saturated mixed vapour, and its osmotic pressures $\Omega - \pi_\Omega$ and $\Omega - \phi_\Omega$, measured with respect to solvent and solute vapour respectively.

In the new notation the equation becomes

$$\int_{\Omega}^p s_1 dp = \int_{\pi_{00}}^{q_1} u dp + \int_{\pi_{00}}^{\pi_{00}} v dp \dots \dots (19)$$

Differentiating (19) with respect to c_2 and keeping q_1 constant, we get

$$s_1 \frac{\partial P_1}{\partial c_2} = \int_{\Omega}^p \frac{\partial s_1}{\partial c_2} dp + s_1 \frac{d\Omega}{dc_2} - v \frac{d\pi_{00}}{dc_2} \dots \dots (20)$$

(This equation is exact and can be derived from first principles without reference to the pure solvent.)

But
$$\frac{d\pi_{00}}{dc_2} = \frac{\partial \pi}{\partial c_2} + \frac{\partial \pi}{\partial p} \cdot \frac{d\Omega}{dc_2},$$

and Prof. Porter‡ has shown that $\frac{\partial \pi}{\partial p} = \frac{s_1}{v}$;

therefore (20) becomes

$$s_1 \frac{\partial P_1}{\partial c_2} = \int_{\Omega}^p \frac{\partial s_1}{\partial c_2} dp - v \frac{\partial \pi}{\partial c_2} \dots \dots (21)$$

* Proc. Roy. Soc. Series A, vol. lxxx. p. 460.

† It would seem from the determinations of the osmotic pressures derived from the vapour-pressures of solutions of calcium ferrocyanide set forth in the Phil. Trans. Roy. Soc. Series A. vol. ccix. pp. 199-203, that for water-vapour in air at 0° C., the assumption of the validity of the partial pressure law may not be warranted.

‡ Proc. Roy. Soc. A. vol. lxxix. p. 525.

Multiplying by c_1 and subtracting the complementary equation (multiplied by c_2) we get*

$$c_1 s_1 \frac{\partial P_1}{\partial c_2} - c_2 s_2 \frac{\partial P_2}{\partial c_1} = -c_1 v \frac{\partial \pi}{\partial c_2} + c_2 z \frac{\partial \phi}{\partial c_1}. \quad (22)$$

By assuming Boyle's law, the right-hand member of (22) reduces to Margule's equation

$$d \log \pi / d \log c_2 = d \log \phi / d \log c_1.$$

This equation was experimentally verified by Zawidski †, and consequently (18) is also experimentally verified.

Concentration of a solution by centrifugalization.

It is evident that the exact equation (15)

$$\frac{dc_2}{dC} = - \frac{c_2 \cdot \partial w / \partial c_2}{s_1 w \cdot \partial P_1 / \partial c_2},$$

can be applied to cases other than that of a stationary mass of solution under gravity; for instance, in the case of a vessel containing a solution and whirled at a uniform rate round a vertical axis the equation becomes ‡

$$\frac{dc_2}{dC} = \frac{c_2 \partial w / \partial c_2}{s_1 w \partial P_1 / \partial c_2} \quad \dots \quad (23)$$

where C is the potential of the "centrifugal force."

So far as we know, the only experimental work on this method of stratifying a solution, is that of van Calcar and de Bruyn §, but their results, as calculated from the data they give and other known quantities, seem incompatible with (23).

As an example take the following experiment recorded by these authors:—

A drum, 6 cm. in radius, was filled with a 12 per cent. solution of cane-sugar, which gave a rotation of $46^{\circ} \cdot 8$ in a polariscope. The drum was rotated about its vertical axis at a uniform rate of 40 revolutions a second for four hours. Without stopping the apparatus, samples were then taken in four capsules placed at different distances from the axis and were tested in the polariscope. Sample 1 (central) was

* Note that $c_1 \frac{\partial s_1}{\partial c_2} = c_2 \frac{\partial s_2}{\partial c_1}$.

† *Loc. cit.*

‡ Neglecting the effect of gravity on the horizontally rotating liquid.

§ *Rec. Trav. Chem. Leiden*, vol. xxiii, pp. 218–223.

empty, 2, 3, and 4 (peripheral) gave rotations of $41^{\circ}0$, $51^{\circ}5$, and $47^{\circ}0$ respectively.

The circumstance that the third and not the fourth sample showed the greatest concentration was attributed to an irregular stirring action produced by the opening of the capsules; clearly the results can only be regarded as qualitative. But the changes in concentration recorded are, as we will show, many times greater than theory indicates, and the discrepancy is the more difficult to account for in that any errors due to manipulation would lie in the contrary direction.

The calculation is as follows:—

In a system which is rotating uniformly about a fixed axis, with angular velocity ω (radians a second), any particle of mass m at a distance r from the axis requires an inwardly directed force $m\omega^2 r$ to maintain its position unaltered in the system. Thus the system may be regarded as pervaded by a field of force, everywhere outwardly directed, and derivable from a potential function

$$C = \frac{1}{2}\omega^2 r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Therefore, in the drum under consideration, the difference of potential between the axis and the periphery is

$$\Delta C = \frac{1}{2}(2\pi \times 40)^2 \times 6^2 = 1140000 \text{ approximately.}$$

Now for the limiting equilibrium condition, when c_2 , w , u , and $\bar{v}P_1/\bar{v}c_2$ are nearly uniform throughout the space, we have from (23)

$$\frac{\Delta c_2}{c_2} = - \frac{\Delta C}{s_1 w} \frac{\partial w / \partial c_2}{\bar{v}P_1 / \bar{v}c_2} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Here s_1 is practically unity, w is obtainable from Landolt and Börnstein's tables; $\bar{v}P_1/\bar{v}c_2$, for so dilute a solution, may be taken as following the gas law, and therefore we can put (where 339 is the molecular weight of cane-sugar)

$$\bar{v}P_1/\bar{v}c_2 = 22.3(1 + 15/273) \times 1000/339.$$

Multiplying this value by 1.01×10^6 to bring it to dynes per square cm., and substituting the numerical values in (25) we get

$$\Delta c_2/c_2 = .007.$$

That is to say, the proportional difference in concentration between the central and peripheral parts of the rotating solution should only be about 7 parts in 1000 in the final

equilibrium state, while the authors report a difference of 20 per cent. in four hours. The discrepancy is so great that we cannot but conclude that there has been some important oversight in the conduct of the experiments or in the presentation of the results. This conclusion is confirmed when we consider that the forces required to cause such a rapid migration of the solute through the solution, even with no opposing forces except those arising from internal viscosity, would be enormously greater than the available driving forces due to the rotation.

Similar remarks apply to the other cases described by Messrs. van Calcar and de Bruyn.

We have also endeavoured to examine experimentally the centrifugalization of solutions. Numerous trials have been made, but the difficulties have proved considerable, though they now appear to be largely overcome, and we hope shortly to obtain more reliable results with an improved centrifuge.

The solutions we have chiefly examined are potassium iodide, iodine in potassium iodide, and a solution of about 51 per cent. by weight of caesium chloride.

A brief description of the best experiment so far made may not be out of place.

The solution of the caesium chloride was contained in six sealed glass tubes arranged radially in a massive flat circular box of gun-metal; the length of the solution column in each tube was 9.3 cm. and the run was continued (at about 57.6 revs. per sec.) for 243.3 hours, a time long enough for a fair proportion of the maximum degree of separation to be effected. We had satisfied ourselves by previous tests that the internal diameter of the tubes (0.34 cm.) was sufficiently small to prevent any sensible re-mixing of the liquid strata during the very slow stopping of the centrifuge, or in the succeeding few seconds which elapsed before the tubes could be stood vertical. On the other hand, it was impossible to pipette the contents of each tube into three samples without some slight mixing taking place; and it is partly on this account and partly because the centrifuge did not revolve perfectly smoothly (thus causing some slight stirring action), that we regard our results as only qualitative.

The following table gives the results:—

	% CsCl (found).	difference	% CsCl (theoretical)	difference (theoretical)
Sample 1.....	51.648	.061	51.645	.064
" 2.....	51.709	.033	51.709	.103
" 3.....	51.742		51.812	

The numbers under the heading “% CsCl (theoretical)” are only approximate—they are calculated on the assumption that equal molecular concentrations of CsCl and NaCl solutions have the same osmotic pressures, and the value found for the concentration of the middle sample is taken as a basis for comparison.

It will be noticed that considerably less difference was found between the second and third samples than between the first and second, while theory indicates that the inequality should be markedly in the other direction; the differences of concentration are, however, of the order of number indicated by theory.

General Investigation.

Although the following argument can be readily applied to the more general case of heterogeneous liquid or gaseous mixtures, *i. e.* mixtures that differ in properties from point to point of their mass in any manner, yet to enable a clearer view to be obtained, we will limit the discussion to a column of solution (subject to gravity) whose properties are uniform at any given level. We shall not assume that equilibrium has been attained, so that in general there will be relative motion of solvent and solute throughout the liquid mass; at any given level one of the components will be moving downwards and the other upwards. This relative motion is opposed by “internal viscosity,” and fixing our attention on an infinitesimal volume element of the solution, we realize that the solvent molecules in the aggregate must be acted upon by a force in one direction (say downwards) and the solute molecules by a force in the contrary direction.

These two aggregate forces must be almost exactly equal and opposite, for their resultant is measured by the acceleration of the volume element as a whole multiplied by its mass; and the acceleration in all practical cases will be excessively minute.

The same holds good for solutions contained in a uniformly rotating centrifuge, provided we understand that the axes of reference are rotating with the centrifuge, and avoid the explicit appearance of accelerations by introducing the fiction of centrifugal force; so that here again we may always consider the solvent and solute within any very small element of volume to be acted on by sensibly equal and opposite forces.

Let $d\tau$ be the volume of the element under consideration,
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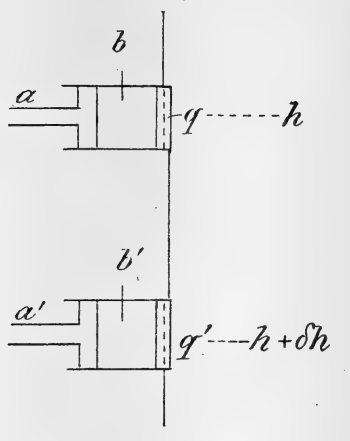
$F_1 d\tau$ the aggregate downwardly-measured force on the solvent molecules contained in $d\tau$, and $F_2 d\tau$ the downwardly-measured force on the solute molecules; then $F_1 + F_2 = 0$.

The forces, F_1 and F_2 , reckoned per unit volume of solution, we shall call the "*driving forces*" on solvent and solute respectively at the level under consideration.

Now imagine the downwardly-measured driving forces F_1 and $-F_1$ to be balanced by an ideal field of force, whose action at each level amounts to a force $-F_1$ on the solvent molecules per unit volume of solution and $+F_1$ on the solute molecules per unit volume. The *total* driving force is thus reduced to zero throughout the solution, and the concentration retains its distribution unaltered, although the mobility of individual molecules remains unimpaired.

Consider now the small portion of the column represented by figure 2. At the level h let the concentration be c_2 grams of solvent per gram of solution, and the hydrostatic

Fig. 2.



pressure be p . At the neighbouring level $h + \delta h$ the values of these quantities will be, to a first order,

$$c_2 + (dc_2/dh) \times \delta h \quad \text{and} \quad p + (g/w) \times \delta h \quad \text{respectively,}$$

where w is the specific volume of the solution at the level h .

At h , in the wall of the containing vessel, let q be a membrane permeable to the solvent only, and q' a like membrane at level $h + \delta h$. Let the cylinders b and b' (furnished with

pistons a and a' , and containing solutions of the same concentration and pressure as at the corresponding levels in the column) be fixed behind q and q' as shown in the figure.

In the thermodynamic cycle now to be considered, the final pressures and concentrations are the same as the initial ones, and all operations are performed isothermally; hence, although the changes of pressure cause changes in the volume of the contents of the cylinders and thus give rise to terms in the expression for the work done, these, in so far as they relate to definite masses of solution of definite concentration, must eliminate. Accordingly these terms will not be taken into account when writing down the work done in the cycle.

(I.) Let the piston a , under a pressure which never differs more than infinitesimally from p , be pushed in until a mass dm of solvent has passed through q into the column, and simultaneously let the piston a' be withdrawn, so that at each instant the mass of solvent which has left the column by q' is precisely equal to that which has entered by q . Moreover, suppose the operation to be performed so slowly that the column of solution is everywhere sensibly in a state of equilibrium; the solvent thus streams slowly from q to q' without affecting the concentration at any level, and without producing, on the whole, any perceptible displacement of the solute.

Adding up the work done by the external forces, we have for the two pistons

$$p s_1 dm - \left(p + \frac{g}{w} \delta h \right) \left\{ s_1 + \left(\frac{\partial s_1}{\partial c_2} \frac{dc_2}{dh} + \frac{\partial s_1}{\partial p} \frac{g}{w} \right) \delta h \right\} dm \\ = \left\{ -p \left(\frac{\partial s_1}{\partial c_2} \frac{dc_2}{dh} + \frac{\partial s_1}{\partial p} \frac{g}{w} \right) - s_1 \frac{g}{w} \right\} \delta h dm,$$

and the work done by our ideal field of force is $-F_1 \frac{c_1}{w} \delta h dm$, since $-F_1 c_1/w$ is the force acting *per unit mass of solvent*.

(II.) Close the membranes q and q' by shutters, and change the pressure on the solution in b' so that it would be in osmotic equilibrium with that in b , the pressure in b' has now become $p + \frac{\partial P_1}{\partial c_2} \frac{dc_2}{dh} \delta h$, but the work thus done may be disregarded as above explained.

(III.) Withdraw the cylinders (carrying the membranes with them) * from the column and place the solution in b' in

* The use of shutters, where necessary, is to be understood.

osmotic contact with that in b at the level of the latter. These two solutions are now in osmotic equilibrium through the membranes q and q' . Push in the piston a' and simultaneously withdraw a until a mass dm of solvent has passed, under osmotic equilibrium conditions, from b' to b , the final pressure in b being exactly p .

The work done on the two pistons is

$$-ps_1dm + \left\{ p + \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc}{dh} \delta h \right\} \left\{ s_1 + \left(\frac{\partial s}{\partial c_2} \frac{dc_2}{dh} + \frac{\partial s}{\partial p} \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc_2}{dh} \right) \delta h \right\} dm$$

$$= \left\{ p \frac{\partial s_1}{\partial c_2} \frac{dc_2}{dh} + p \frac{\partial s_1}{\partial p} \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc_2}{dh} + s_1 \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc_2}{dh} \right\} \delta h dm.$$

(IV.) Reshutter the two membranes q and q' , separate the cylinders, change the pressure in b' back to its former value $p + (g/w) \times \delta h$, restore the cylinders to their original places and put their contents in osmotic communication with the solution-column. The initial conditions are now restored.

The balance of work done against gravity is $g\delta h dm$.

Regarding stages (II.) and (IV.), when the changes of pressure between $p + (g/w)\delta h$ and $p + \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc_2}{dh} \delta h$ are being made, the volume dealt with in (II.) is greater by $s dm$ than in the reverse operation (IV.); hence the change in volume is greater by

$$\frac{\partial s_1}{\partial p} \left\{ \frac{g}{w} - \left(\frac{\partial P_1}{\partial c_2} \right) \right\} \delta h dm,$$

and the corresponding work-term is

$$p \frac{\partial s_1}{\partial p} \left\{ \frac{g}{w} - \left(\frac{\partial P_1}{\partial c_2} \right) \right\} \delta h dm.$$

Collecting the various terms, and remembering that $c_1 = 1 - c_2$, we obtain, on division by $\delta h dm$,

$$\frac{F_1 w}{1 - c_2} = \frac{s_1 - w}{w} g - s_1 \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc_2}{dh} \quad \dots \quad (26)$$

or

$$F_1 = \frac{s_1 - w}{w^2} (1 - c_2) g - \frac{s_1}{w} (1 - c_2) \left(\frac{\partial P_1}{\partial c_2} \right) \frac{dc_2}{dh} \quad \dots \quad (27)$$

Equation (26) gives the downward force acting on the solvent molecules per unit mass of the solvent, while

equation (27) gives the downward force acting on the solvent molecules per unit volume of solution, in other words the driving force. The driving force F_2 on the solute molecules is equal and opposite to F_1 as already pointed out.

When we require the driving forces in a uniformly rotating mass of solution, we have only to replace dh by dr , and g by dC/dr , where r represents radial distances from the axis of rotation, and C the potential of the centrifugal force. To pass to equilibrium conditions, we have to put $F_1 = 0 = F_2$; and (27) then becomes identical with (15) above, and (16) and (17) follow as before.

Thus all these equations can be obtained without taking vapour-pressures into consideration and without assuming it possible for the pure liquid solvent and solute to exist at such a pressure as to be in osmotic equilibrium with the solution, or even to be capable of separate existence at all.

Note on the Application of the Preceding General Discussion to a Diffusing Column of Solution.*

As before, consider a volume element $d\tau$ at any level in the column. The solvent molecules in this element collectively experience a force $F_1 d\tau$, and the solute molecules an equal and opposite force $F_2 d\tau$; these forces tend to produce opposite displacements of the solvent and solute *relatively to the mass centre of the volume $d\tau$* , without causing any bodily acceleration of the solution contained in $d\tau$. Draw through the mass centre of the solution in $d\tau$ an element dS of horizontal surface; *let this element of surface be moving with the mass centre in question*, and let the rates of displacement of solute and solvent be measured with respect to it. These rates we shall denote by m_1 and m_2 , and define as the algebraic total of grams of solvent (or solute) passing downwards through dS per unit of area per unit of time.

Let V_1 be the mean downward velocity, relative to dS , of the solvent molecules in $d\tau$, and V_2 the corresponding velocity (downward) of the solute molecules, then, ρ being the density of the solution, $V_1 \rho c_1 = m_1$ and $V_2 \rho c_2 = m_2$.

But $V_1 \rho c_1 d\tau$, $V_2 \rho c_2 d\tau$ are the respective momenta of the solvent and solute, both reckoned relatively to the mass centre of the element; their sum is therefore zero, and hence

$$m_1 + m_2 = 0.$$

* A fuller discussion is reserved for a subsequent communication.

For any values of m_1 and m_2 attained in diffusion experiments (with a reservation to be explained immediately), we may regard F_1 as proportional to m_1 (and therefore F_2 to m_2) to a high order of accuracy, and may write

$$F_1 = km_1, \quad F_2 = -km_1 = km_2,$$

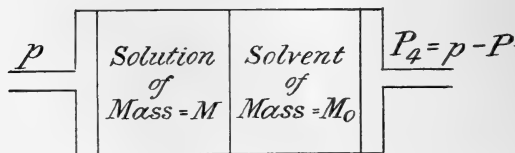
where k is independent of m , but depends on the nature of solute and solvent and on the temperature, as well as on the concentration and hydrostatic pressure at the point considered. We may call k the "internal viscosity," and for very dilute solutions it becomes proportional to the molecular viscosity as ordinarily defined.

The reservation made in the last paragraph refers to the case where two portions of solution differing finitely in concentration are in immediate contact. Our elementary theory would indicate, initially, an infinite rate of diffusion across the interface; but the circumstances are not completely realizable even in imagination, when the discrete structure of liquids is taken into account; and the nearest imaginable approach to discontinuity of concentration would take us far from our original assumption, namely, that a volume element anywhere in the system may be taken small enough for its content to be treated as of uniform concentration, without reduction to dimensions so minute that the aggregate of molecules can no longer be treated statistically.

APPENDIX 1.

Consider the annexed diagram, which represents a solution under an external pressure p separated from the solvent which is under an external pressure $q = p - P$, by a semi-permeable membrane, and let there be osmotic equilibrium.

Fig. 3.



Let there be M grams of solution containing c_2 grams of solute per gram. Let there be M_0 grams of solvent, and let the specific volumes of solution and solvent be w and u respectively.

Carry out the following isothermal thermodynamic cycle.

1st operation.—Keeping the pressure p constant, slightly withdraw the solution piston until a mass dm of solvent has passed from solvent to solution. Let this be done so slowly that there is all the time osmotic equilibrium.

2nd operation.—Increase pressure p to p' and simultaneously increase the pressure on the solvent so as to maintain the osmotic equilibrium without any solvent passing across the membrane.

3rd operation.—Keeping the pressure p' constant, push the solvent piston in very slowly until a mass dm of solvent has been driven into the solution.

4th operation.—Decrease the pressure p' to p , and simultaneously decrease the pressure on the solvent in such a manner as to maintain the osmotic equilibrium.

The initial state has now been restored, and the work done by external forces on the system must vanish for the whole cycle.

In operation (I.) the work done on the solution is $p s dm$. The work done on the solvent is due to two causes :—

(a) The concentration of the solution has increased by

$$dc = \frac{c}{M} dm,$$

and the osmotic pressure by

$$dP = \frac{\partial P}{\partial c} \frac{c}{M} dm,$$

which expresses the decrease of pressure in the mass M_0 of solvent, hence the work done is

$$-(p-P) \frac{M_0 c}{M} \frac{\partial u}{\partial q_0} \frac{\partial P}{\partial c} dm.$$

(b) The mass of solvent has increased by dm , so that we get

$$-(p-P) ndm.$$

In the 2nd operation the work done on the solution is

$$-(M-dm) \int_p^{p'} p \frac{\partial}{\partial p} \left(w + \frac{c}{M} \frac{\partial w}{\partial c} dm \right) dp ;$$

while if P' represent the osmotic pressure corresponding to a concentration c of the solution, and a pressure p' upon it

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and we put $q = p' - P'$, then the pressure on the solvent
changes from

$$p - P - \frac{\partial P}{\partial c} \frac{c}{M} dm$$

to

$$p' - P - \left(\frac{\partial P}{\partial c} \right)' \frac{c}{M} dm,$$

and so the work done on the solvent is

$$-(M_0 + dm) \int q \frac{\partial u}{\partial q} dp,$$

with these two pressures as the limits; that is

$$-(M_0 + dm) \left\{ \int_{p-P}^{p'-P'} q \frac{\partial u}{\partial q} dq + q' \left(\frac{\partial u}{\partial q} \right)' \left(\frac{\partial P}{\partial c} \right)' \frac{c}{M} dm - q \frac{\partial u}{\partial q} \frac{\partial P}{\partial c} \frac{c}{M} dm \right\}.$$

In the third operation, the work done on the solution is

$$-p' s' dm,$$

and that on the solvent is

$$+(p' - P') \left\{ u' + \frac{M_0 c}{M} \left(\frac{\partial u}{\partial q} \right)' \left(\frac{\partial P'}{\partial c} \right) \right\} dm.$$

In the fourth operation, the work done on the solution is

$$+ M \int_p^{p'} p \frac{\partial w}{\partial p} dp,$$

and that on the solvent is

$$+ M_0 \int_{p-P}^{p'-P'} q \frac{\partial u}{\partial q} dq.$$

On summing up and dividing by dm , and equating to zero, we get

$$0 = \int_p^{p'} s dp - \int_{p-P}^{p'-P'} u dq. \quad . \quad . \quad . \quad (28)$$

This, on taking the upper limits p' and $p' - P'$ equal respectively to $p + dp$ and $p - P + dp - \frac{\partial P}{\partial p} dp$, becomes

$$s dp = u \left(dp - \frac{\partial P}{\partial p} dp \right)$$

or

$$\frac{\partial P}{\partial p} = \frac{u - s}{u}. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This result is obtained by Prof. Porter by differentiating equation (1) of his first paper; but by the method followed the relation is obtained as an exact equation whatever assumption is made as to the volatility of solute and solvent.

APPENDIX 2.

In his recent paper on vapour-pressure and osmotic pressure, Prof. Callendar* has discussed the differences of concentration to be found in a column of cane-sugar solution (at 0° C.), which has attained a state of final equilibrium under gravity as shown in fig. 1 by column AB (we must here assume that B is a semi-permeable membrane in contact with the free surface of water, and that the vapour-pressure of the solution at A is in equilibrium with that of the water at the same level). Prof. Callendar assumes that the concentration and osmotic pressure at A is given by Berkeleyy and Hartley's results†, and calculates the concentration just above B by means of the approximate relation

$$dc_2/c_2 = -P(dw/dP_2)/w, \quad . \quad . \quad . \quad (30)$$

in which P is the mean osmotic pressure in the column.

In revising these estimates, we have taken the concentration and osmotic pressure just above B as given by the experiments (this is more correct than Prof. Callendar's procedure in that the conditions in the experiments only differ from those indicated above by the fact that there is a pressure on the water of one atmosphere instead of that of its vapour only), and have calculated the concentration at the top of the column at A, by using a relation which, in the first instance, can be written down exactly, and allows our calculation to be carried to any degree of precision for which the accuracy of the data suffices.

In (26), put $F=0$, since we are dealing with equilibrium conditions; then remembering that $dp = gdh/w$, we get the exact relation

$$dp = s_1/(s_1 - w) \cdot dP_1/dc_2 \cdot dc_2. \quad . \quad . \quad . \quad (31)$$

Neglecting the density of the vapour in comparison with that of the solution, and the pressure of the vapour in

* Proc. Roy. Soc., Series A, vol. lxxx. p. 473.

† Phil. Trans., Series A, vol. ccvi. p. 503.

comparison with the osmotic pressure, an integration from A to B gives,

osmotic pressure at B = pressure of solution at B

$$= \int s_1/(s_1 - w) \cdot \partial P_1/\partial c_2 \cdot dc_2, \quad (32)$$

in which the upper limit of the integral is the given concentration at B, and the lower limit is the unknown to be found from the equation.

The values of $\partial P_1/\partial c_2$ (subject as below to a small correction) are obtained on differentiating the curve which expresses the relation between concentration (in grams per gram of solution) and the corresponding equilibrium osmotic pressures; for the equilibrium pressures are all measured with the same pressure (one atmosphere) on the solvent, and accordingly they fulfil the condition which we imply by the notation $\partial P_1/\partial c_2$. But each $\partial P_1/\partial c_2$ thus derived from the curve corresponds to a definite hydrostatic pressure on a solution of given concentration, and in the column the pressure associated with that concentration will be different, except just at the level in contact with B.

To apply the correction we may first find p approximately in terms of c_2 by integrating (31) without taking into account the effect of pressure on the factors $s_1/(s_1 - w)$ and $\partial P_1/\partial c_2$ *. The values of p thus obtained are amply accurate for computing the corrections required, since the whole pressure-correction for the strongest solution here considered only amounts to 0.8 per cent.

For the more dilute solutions a simple approximate formula suffices: namely

osmotic pressure at B = pressure of solution at B

$$\begin{aligned} &= [s_1]/([s_1] - [w]) \times [P_1/c_2] \times \Delta c_2 \\ &= [s_1]/([s_1] - [w]) \times \Delta P_1, \quad (33) \end{aligned}$$

where the brackets $[\]$ indicate mean values and Δc_2 and ΔP_1 are the total changes in value in c_1 and P_1 , respectively, as we pass from the top to the bottom of the column.

* The correction to the factor $\partial P_1/\partial c_2$ is given by

$$\partial(\partial P_1/\partial c_2)/\partial p = -1/s_1 \times (\partial P_1/\partial c_2) \times \partial s_1/\partial p - 1/s_1 \times \partial s_1/\partial c_2,$$

an exact relation, which is not difficult to verify. The correction to $s_1/(s_1 - w)$ involves extrapolation from Tait's results for the compressibility of sugar solutions (see Landolt and Börnstein's Tables, p. 62, 3rd Ed.).

Equation (33) like the more exact form (32) has to be solved by trial and error.

The final results are given in the first four columns of the following table, the last four columns are from Prof. Callendar's paper :—

Our results.				Prof. Callendar's results.			
Concentration in grams per litre.		Diff.	per cent.	Concentration in grams per litre.		Diff.	per cent.
At bottom.	At top.			At top.	At bottom.		
750.6	676.7	73.9	9.85	760*	900	140	18.4
660.5	598.6	61.9	9.06	660	741	81	12.3
540.4	490.9	49.5	9.16	540	600	60	11.1
420.3	385.2	35.1	8.35	420	460	40	9.5
300.2	278.9	21.3	7.10	300	325	25	8.3
180.1	170.0	10.1	5.61	180	193	13	7.3

* It is probable that this number is copied from a misprint in Lord Berkeley's paper on van der Waals' equation (Proc. Roy. Soc. Series A, vol. lxxix. p. 126).

Prof. Callendar deduces from his results that "the numbers in the last column appear to indicate a systematic error in the experimental numbers for strong solutions." On comparing the two sets of figures it will be seen that those calculated from the more accurate formula do not bear out Prof. Callendar's deduction.

LV. *On the Wilson-Gerdien Theory of Thunderstorm Electricity.* By GEORGE C. SIMPSON, D.Sc., Meteorological Department, Government of India*.

IN 1900 Mr. C. T. R. Wilson put forward a theory to account for the electrical phenomena connected with thunderstorms based on his well-known experiments on the condensation of water vapour from supersaturated air upon negative ions. Judging from a number of references to this theory in recent articles in scientific journals it would appear that it has obtained considerable acceptance, both in England and on the continent. It therefore appeared to me desirable that the theory should be subjected to a critical examination, and I propose in this paper to undertake such an examination

* Communicated by the Author

in order to show that in six important particulars the theory is not capable of accounting for the observed facts.

At the end of the paper in which the results of the experiments already referred to were published, Wilson added the following paragraph*:

“In order that the results of these investigations should have a direct bearing on the subject of atmospheric electricity, it is necessary to assume that condensation in the atmosphere frequently takes place from the supersaturated condition. There is very little direct evidence of the existence of supersaturation in the atmosphere; but there is at least an equal lack of evidence against its existence above the lower cloud layers, even as a normal accompaniment of precipitation. That supersaturation occurs in connexion with thunderstorms is held by V. Bezold and others (*Sitzab. Akad. d. Wiss. zu Berlin*, 1892).”

This idea, that the electrical effects observed with precipitation might owe their origin to the greater facility with which water vapour is deposited on negative than on positive ions, was eagerly taken up on the continent, and Elster and Geitel considered it so promising that they gave to it a prominent place in their lecture before the Brunswick Society of Science in April 1900†.

The only serious objection to the suggestion came from Mr. Aitken‡, who was led to doubt, from his classical researches on the dust of the atmosphere, that “there is such a thing as dust-free air in our atmosphere.” To answer Mr. Aitken’s objections Wilson wrote a letter to ‘Nature’§ in which he developed his idea into a detailed theory, showing how he considered the dust might be entirely removed from air during thunderstorms, and how the separation of the electricity could be supposed to take place. The following summary of this letter, written as far as possible in the original words, may be taken as giving the essential points of Wilson’s theory.

If an ascending current of air contains drops of water in the form of cloud-particles these drops will lag behind the air-current to a greater or less extent, according to their sizes. Hence, the air which enters the base of a cumulus cloud will pass amongst the water-drops, and ultimately reach the top. During the passage of the air through the

* Wilson, *Phil. Trans. A.* 193. p. 307 (1900).

† *Met. Zeit.* xvii. pp. 226–231 (1900).

‡ Aitken, ‘Nature,’ xli. pp. 514–515 (1900).

§ Wilson, ‘Nature,’ xlii. pp. 149–151 (1900).

cloud the innumerable water-drops will act as a kind of filter and remove all the dust-particles from the air. The drops at the top of the cloud will also lag behind the ascending air, and the air between them is likely to be dust-free. "Under these conditions a dust-free layer will be formed above the cloud, and will continually increase in vertical thickness. This layer will be saturated with moisture at its lower edge, above this it will be supersaturated; the amount of supersaturation being greater near its upper limit, and depending on the vertical distance through which the air has risen since escaping from the cloud. Now to produce in air initially saturated the supersaturation (approximately four-fold) necessary to cause water to condense on negative ions, it is sufficient to let the volume of the air increase adiabatically to 1.25 times its initial value; an expansion which will result from an ascent of the air through a vertical distance of 2,500 metres, if we suppose the air on escaping from the cloud to be at a temperature of 10° C. (at lower temperatures a smaller elevation would suffice). Thus, when the air in the uppermost layers of the supersaturated stratum has reached a height of about 2,500 metres above the level at which it escaped from the cloud, a sudden change will result; condensation will there take place on the negative ions. The thickness of the supersaturated stratum (*i. e.* the vertical distance which the upper surface of this cloud has lagged behind the air) when the condensation on the negative ions begins, may vary greatly; it may be very small if the drops are small and the ascent of the air rapid; it may amount to nearly the whole 2,500 metres in the case where the drops grow large enough to acquire a velocity relative to the air as great as the upward velocity of the air, so that the upper surface of the cloud has ceased to ascend." The drops which condense on the negative ions at the top of the supersaturated layer will either fall at once as rain or remain in suspension till they have travelled into regions where the ascending current is insufficient to support them. "In either case, if the drops fall through a supersaturated layer of some thickness, they are likely to reach the ground as negatively charged rain." "The positive ions after being carried up out of reach of the drops formed on the negative ions, will, under the action of the electrical field produced by this separation, tend to travel downward relatively to the air with a velocity of the order of one centimetre per second for a field of 100 volts per metre, as the measurements of Rutherford and others have shown. After being carried beyond the region of ascending air-currents, they will travel downwards towards

the earth's surface; but long before reaching it they will become attached to cloud-particles or to the dust-particles of the lower layers of the atmosphere, where the positive charge will accumulate."

At the time when this theory was published it was generally held that Elster and Geitel had satisfactorily explained the normal electrical condition of the atmosphere by their theory of ion absorption; but in 1903 it was shown* that this theory could not be substantiated by experiment. Thus meteorologists were left without any satisfactory theory for the normal electrical phenomena of the atmosphere. It is easy to understand that under these circumstances an attempt should be made to press Wilson's theory into explaining both normal and abnormal electrical phenomena.

This was done with considerable thoroughness by Dr. H. Gerdien in his article on "Der Electrizaritätshaushalt der Erde und der unteren Schichten der Atmosphäre," in the *Physikalische Zeitschrift* †. Gerdien introduced no essential differences into the theory, but developed it into greater detail than Wilson had done, and added numerical calculations to show that it was capable of accounting for all the electrical phenomena associated with thunderstorms.

Turning now to our criticism, the theory will be examined first from the electrical side of the problem and then from the meteorological. In the first part of the discussion we shall accept the meteorological conditions supposed by the theory; *i. e.* we will assume the possibility of the formation of a supersaturated stratum above a lower cumulus cloud, and of a condensation layer above in which water vapour is condensed around the negative ions; and then examine whether the electrical effects observed can be supposed to follow from these conditions. Afterwards we will examine the meteorological conditions themselves to find if they are likely to be present during thunderstorms. (1) The first question we will ask is:—

Could a sufficiently rapid separation of electricity take place, under the conditions supposed by the theory, to account for the numerous lightning discharges observed during thunderstorms?

In order to obtain simplicity of treatment we will follow Gerdien in assuming that the processes we are investigating take place in horizontal layers sufficiently near to one another in comparison with their lateral extent to be considered as infinite planes. Under such conditions we may assume that

* Simpson, *Phil. Mag.* [6] vi. p. 589 (1903).

† H. Gerdien, *Phys. Zeit.* vi. pp. 647–66 (1905).

the air which enters any given cross-section at the base of the cloud will rise vertically and leave the region under discussion through a similar area vertically over that through which it entered.

We will now consider the electrical condition of the air at the different stages of its ascent. Before entering the cloud the air may be supposed to contain the mean number of ions per cubic metre found in air under normal conditions. The air enters the cloud with this amount of free electricity, and commences its upward journey through the cloud-particles. We have already agreed that a cloud acts as a filter for dust-particles, and it will be quite as efficient a filter for ions; in fact, direct observations have shown that within a cloud (fog) practically no free ions can be measured. Thus the air which enters the cloud with a large number of free ions will pass out at the top with none, and will commence to travel through the supersaturated stratum with neither dust nor ions. On account of what is generally called natural ionization new ions will at once be formed, and as now there are no water-particles to catch them they will continue to exist as free ions. But the rate of formation is not rapid. Measurements have been made of the rate at which new ions are being formed in the atmosphere near the earth, and numbers varying between 10 and 50 ions per c.c. per second found*.

In the case under consideration it is not likely that the ions would be formed at anything like the rate that they are formed near the earth, if for no other reason than that the radioactive particles which are supposed to be the cause of a great deal of the natural ionization of the air will also have been filtered out in the cloud. Gerdien† has estimated that the rate at which ions form in the region which we are considering is about 10 of each kind per c.c. per second. As, however, I do not wish to err on the side of underestimation, I will assume that the ions are generated at the rate of 25 per c.c. per second.

Now consider an area of 1 sq. cm. at any position on the upper surface of the lower cloud; the air which passes out of this area will rise vertically and enter the condensation layer through another area of 1 sq. cm. Let us assume that every ion which forms in a cubic cm. of air while passing from the lower cloud to the condensation layer above continues to exist as a free ion; *i. e.* we will entirely neglect the recombination of ions; and let us further assume that

* Schuster, Proc. Man. Lit. and Phil. Soc. vol. xlviii. part ii. p. 1 (1904).

† Gerdien, *l. c.* p. 662.

every negative ion has water deposited on it when it reaches the condensation layer, and is retained there, while every positive ion passes through. In this way we shall find the absolute maximum rate at which the charge at the condensation layer can increase with naturally ionized air.

Let the upward velocity of the air

$$= V \text{ cm./sec.}$$

Then the vol. of air which crosses the sq. cm. under consideration in one second

$$= V \text{ c.c.}$$

The time taken for a given small vol. of air to pass from the cloud to the condensation layer, 2,500 metres above

$$= \frac{2.5 \times 10^5}{V} \text{ sec.}$$

The number of ions of each sign generated in a c.c. in this time

$$= \frac{25 \times 2.5 \times 10^5}{V} = \frac{6.2 \times 10^6}{V}.$$

The total number of negative ions which are retained at each sq. cm. of the condensation layer each second

$$= \frac{6.2 \times 10^6}{V} \times V = 6.2 \times 10^6.$$

Now the charge on 3×10^9 ions is equal to 1 electrostatic unit of electricity.

Hence the maximum rate at which the charge would grow on the condensation layer

$$= \frac{6.2 \times 10^6}{3 \times 10^9} = 2 \times 10^{-3} \text{ els. unit per sq. cm. per sec.}$$

It is interesting to note that this maximum charge possible is quite independent of the upward velocity of the air, and therefore, from ordinary considerations, independent of the rate of rainfall. This does not mean, however, that if the process as here sketched actually takes place in nature the rate of separation of the electricity would be independent of these factors; for we have neglected the rate of recombination of the ions, which would be considerable if the rate of rise were so small that it took an appreciable time for the air to pass through the supersaturated stratum; all factors, however, which we have neglected could only reduce, and not increase, the charge separated at the condensation layer.

We will now consider how long it will take for electricity accumulating at such a rate at a certain layer in the atmosphere to give rise to a lightning discharge.

We have assumed that the whole of the negative ions are separated at the condensation layer, and that the positive ones are carried up above. Thus we have got a charged layer with the corresponding quantity of electricity in the space above. Let σ = the charge on each sq. cm. of this layer at any time t seconds after condensation first started, then the field above the layer will be

$$\frac{dV}{dh} = 4\pi\sigma \frac{\text{els. units}}{\text{cm.}}$$

or if V is measured in volts

$$\begin{aligned} \frac{dV}{dh} &= 300 \times 4\pi\sigma \frac{\text{volts}}{\text{cm.}} \\ &= 3.8 \times 10^3 \sigma \frac{\text{volts}}{\text{cm.}} \text{ approx.} \end{aligned}$$

Now it was shown above that

$$\sigma = 2 \times 10^{-3} t \text{ els. unit.}$$

Therefore the potential gradient produced would be

$$\begin{aligned} \frac{dV}{dh} &= 3.8 \times 10^3 \times 2 \times 10^{-3} t \frac{\text{volts}}{\text{cm.}} \\ &= 8 t \frac{\text{volts}}{\text{cm.}}. \end{aligned}$$

That is, the potential gradient above the condensation layer would grow at the rate of 8 volts/cm. per second. Now air at atmospheric pressure can resist an electric stress of about 30,000 volts per cm.; hence it would take $\frac{30,000}{8}$ sec. = 53 minutes for the potential gradient to grow sufficiently great for a lightning discharge to take place. Thus the first lightning discharge could not, under the favourable conditions we have assumed, occur until nearly an hour after the supersaturated stratum had extended from the cloud to the condensation layer. This first lightning discharge would travel between the condensation layer and the positive charge which had been separated and carried upwards, and would more or less completely conduct away the negative electricity from the portion of the layer affected. In other words, the discharge would neutralize the negative electricity contained in a portion of the layer and destroy the field above it. Thus

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a second discharge could not take place until the stress of 30,000 volts per cm. had again been set up by a continuation of the process under discussion, and that would take, as we have seen, nearly another hour. Thus, if the process were the simple one we have considered, lightning flashes could only take place from any given area of the condensation layer at intervals of about an hour each.

But we must consider if it would not be possible, after the first discharge had taken place, for any other action to come into play which would increase the rate at which the potential gradient could grow, and so make it possible for more rapid discharges to take place.

Gerdien has suggested a "far stronger and more rapidly acting source" of ions than natural ionization. He says*:—

"Such a source must in fact come into action as soon as a single negative ion, within the strong field caused by the separation of the positive and negative ions, is able freely to traverse the 'ionization difference of potential,' and so introduce a disruptive discharge within the field†. In this case ionization through ion impact takes place, and in a very short time a degree of ionization is caused which exceeds the ordinary ionization several million times. A part of these newly formed ions will be drawn by the field into the discharge current, and so neutralize a part of the charge separated by the condensation; the greater part will, however, disappear through recombination. Nevertheless, for a considerable time after the current has passed, a degree of ionization must remain within the region traversed by the discharge, which by far exceeds that present under normal conditions. Thus each disruptive discharge will create a large number of ions which the condensation process can immediately separate so long as the ascending current is present to provide the necessary energy."

But this reasoning is not so conclusive as it would appear on first sight. The discharge considered must take place either upwards or downwards from the condensation layer. In the former case the production of an infinite number of ions would be of no use, for in the region above the condensation layer the air is not supersaturated to a fourfold degree, and so there would be no tendency for water to be deposited on negative ions, no matter how numerous they were. On the other hand, if the discharge took place downwards from

* Gerdien, *l. c.* p. 662.

† "An irgendeiner Stelle ein negatives Ion die Ionisierungsspannung frei durchlaufen kann und damit die selbständige Stromung innerhalb des Feldes einleitet."

the condensation layer it would of necessity traverse the supersaturated stratum. Now Barus* has shown that whenever ions are formed in dust-free air there is produced at the same time a large number of nuclei, which exist for a long time after the ions have disappeared. As water vapour is very readily deposited on these nuclei fourfold supersaturation is prevented in any air in which they are present, unless the supersaturation is caused by exceedingly rapid rarefaction. Each lightning discharge which passes through the supersaturated layer will produce a large number of these efficient nuclei; and any considerable amount of electrical discharge through the stratum would be the cause of the total destruction of the stratum. Thus we see that lightning discharges would not act in the way Gerdien describes; but, on the contrary, would be a hindrance rather than a help to the process on which the theory is based.

We are therefore forced to the conclusion that the lightning itself does not provide a source of ions for condensation to take place upon, and that we must depend upon the ascending currents of air to provide all the free ions for the separation of the electricity at the condensation layer.

Now we have already seen that under these circumstances we could not get lightning discharges from any given area of the storm more frequently than one during each hour. It must, however, be pointed out that even this frequency is far too high, for in order to discuss the most favourable circumstances we have assumed conditions which could not possibly take place in nature. We have assumed that every ion formed in the air as it passes from the lower cloud through the supersaturated stratum continues to exist, and that complete separation of the ions takes place at the condensation layer, neither of which conditions could be fulfilled. Also we have neglected the whole loss of electricity due to any precipitation from the condensation layer, and further we have not taken into account the loss of electricity from the condensation layer which would take place through conduction in the great fields necessary to produce a lightning discharge. It would not perhaps be estimating these losses too high if we assumed that they would increase the intervals between two flashes ten times, and so make it only possible to have one flash in ten hours.

It thus appears that the theory does not account for the rapid electrical discharge observed in thunderstorms, and therefore in this respect it is unsatisfactory.

* C. Barus, *Amer. Journ. Sci.* xix. pp. 349-356, May 1905; and numerous later papers.

(ii.) We will now consider the charge which rain formed in the condensation layer could carry to the earth with it. In order to treat again of the maximum effect possible we will assume that the whole water deposited in the condensation layer is deposited on the negative ions, and that every negative ion becomes the nucleus of a drop.

We will consider a cubic metre of air rising from the lower cloud to the condensation layer, where it deposits its water vapour on the negative ions which it contains.

We can accurately calculate the amount of water deposited under these conditions from a cubic metre of air. Let us assume that the temperature of the air on the ground is 25°C ., and that the height of the top of the lower cloud is 2,000 metres. Under these conditions the temperature of the air when it leaves the cloud will be about 15°C ., and the pressure approximately 600 mm. From these initial conditions, according to the method described on pp. 122-124 of Professor J. J. Thomson's "Conduction of Electricity through Gases," we find that when condensation takes place at four-fold supersaturation 6.5 grams of water will be deposited from each cubic metre of air. Now if we know the number of ions on which this amount of water is deposited we can determine the quantity of electricity contained in each c.c. of water.

So far we have not found it necessary to determine how many ions are actually contained in the air when it reaches the condensation layer; we have only discussed the rate of formation of the ions, and this will not help us, for we do not know the upward velocity of the air, and so the time taken in traversing the supersaturated stratum. We must find the probable state of ionization from other considerations. Now at the surface of the earth numerous measurements of the ionization of the air have shown that under normal conditions a cubic metre of air contains about a quarter of an electrostatic unit of free electricity of each kind. From balloon ascents it has been found that this amount tends to increase as one rises from the surface, but for reasons given above it is very unlikely that the air in the supersaturation stratum would ever contain so many ions as have been found at the corresponding heights under normal conditions*. We will, however, for the sake of a round number, and in order to take the data used by Gerdien, assume that the air just before entering the condensation layer contains one electrostatic unit of each kind of electricity in the form of free ions. We shall then have approximately 3×10^9 ions of each

* *L. c.* pp. 13 and 15.

kind in a cubic metre of air. Now, if every negative ion in the cubic metre of air receives some water, 6.5 grams of water will be deposited on 1 electrostatic unit of electricity, *i. e.* the charge per cubic centimetre of water will be $1/6.5 = .15$ electrostatic unit.

It is important to notice that this is the maximum charge per c.c. which it is possible for the water to obtain according to the theory, for we have assumed that every negative ion, in what must be granted to be highly ionized air, has been caught, and that only the water which has been deposited at the instant of condensation from four-fold supersaturation has condensed around the ions. We can even go further than this and say that not only is this the maximum charge possible, but that it would be a practical impossibility for the rain which reaches the earth to have so large a charge. The water which receives this maximum charge in the condensation layer has to fall through the whole of the lower cloud before it can reach the earth, and in the process it is bound to become mixed with a certain amount of uncharged water. Further, the lower cloud will itself be raining, so that the rain which reaches the earth will be a mixture of the charged precipitation from the condensation layer, and uncharged rain from the lower cloud. Thus, if rain is ever found to have as great a charge as .15 electrostatic unit per c.c. of water we shall be justified in asserting that the charge cannot possibly have been obtained in the manner which the theory supposes.

Turning now to the results of actual measurements, we find that Gerdien* records charges as large as 2.25 electrostatic units per gram of water: Weiss† 5.3 units of negative and 13.6 units of positive electricity per gram of rain; and I myself have measured during rain in Simla charges as high as 19 electrostatic units of negative electricity per gram of rainwater. Thus it appears that in this particular also the theory signally fails to account for the observed facts.

(iii.) We will now turn to a still more important consideration and discuss the probability of any considerable separation of electricity taking place at the condensation layer.

To do this it will be necessary to go somewhat closely into the processes taking place at the condensation layer. We will imagine a volume of supersaturated air just arriving at the layer. Condensation takes place, and on the 3×10^9 negative ions in the cubic metre 6.5 grams of water are

* Gerdien, *Jahrb. der Rad. und Elect.* vol. i. p. 15 (1904).

† Weiss, *Wien. Ber.* cxv. pp. 1285-1320 (1906).

condensed; now if each ion receives an equal share of this water the volume of each of the resulting water-drops will be 2.2×10^{-9} c.c., and each drop will have a radius of 8×10^{-3} mm. According to the formula given by Mr. Wilson, in the letter to 'Nature' which we are discussing, such drops will fall relatively to the air in which they are formed at the rate of 9 mm. a second. Hence, if the air in the supersaturated stratum is rising with a greater velocity than this the drops will be carried upwards, and a cloud will form above the condensation layer.

It will at once be granted that the air must have a greater upward velocity than 1 cm. a second, for if it had not it would take a given mass of air nearly 70 hours to pass from the lower cloud to the condensation layer above. Hence, over the condensation layer we shall have an ever-increasing cloud containing the greater part of the separated negative ions each bound to a water-drop. Now through this cloud all the positive ions will have to pass, and it is quite obvious that even if the cloud is only a few metres thick very few positive ions will pass out through its upper surface. Thus the cloud will not only contain the negative ions on which the water had been deposited, but it will also contain practically the whole of the corresponding positive charge; that is, little or no separation of electricity will take place.

The only escape from this reasoning is to assume that the number of ions in a cubic metre of air has been over-estimated, and that there are in reality so few ions present that drops large enough to fall immediately through the ascending current are formed. Such an assumption is perfectly valid, but it would only get over this difficulty by greatly increasing those treated of in (ii.) above and (v.) below.

It may be as well to point out here that the theory we are discussing is only partially based on experiment; for although experiment has shown that when air is supersaturated to a fourfold degree condensation takes place on negative ions, no experiments have ever been made in which measurable quantities of electricity have been separated by the condensation process; until this has been done the theory cannot be said to be entirely satisfactory.

We will now pass on to discuss the theory from the meteorological point of view.

(iv.) In all the reasoning up to this point we have assumed the main proposition of the theory, namely, that the formation of a supersaturated stratum in the atmosphere is a possibility. Now it may not be possible to prove that the formation of a supersaturated stratum in the atmosphere would

under all circumstances be an impossibility, but it is not difficult to show that we have here one of those cases in which the probability is so small that for practical purposes the word "improbable" may be replaced by "impossible."

The steps of the process leading up to the formation of dust-free air and then to a fourfold supersaturation have been so carefully described by Wilson and Gerdien that it is surprising to find that they both have left out of account a step which appears to be of vital importance to the theory ; that is, the step by which the first few metres of the super-saturated stratum come into existence.

Wilson and Gerdien both commence their reasoning with a cumulus cloud ready formed, and having a supersaturated layer above it, into which more dust-free air is continually being passed from the cloud itself. We will start still further back in the history of the process, and consider the formation of the cloud and then of the supersaturated stratum.

Thunderstorms generally occur on warm calm days, and usually follow hot cloudless mornings. This is not always the case, but as we can only go into typical instances, it will be as well to consider a storm forming under some such conditions. We have, then, to start with a cloudless sky, and the air near the ground rapidly increasing in temperature as the day advances. First weak ascending currents form ; then as the ascending currents become stronger and penetrate higher condensation takes place at the upper surface of one of the more extensive of them. Thus a cumulus cloud is formed in dusty air with dusty air above it as well as below it. Now the ascending current will go on rising, and so will be continually pushing its way into the dusty air above. We will grant that the top of the ascending current is composed of air which has passed through the cloud, and is in consequence dust free, still it must be remembered that this air cannot push back the air around it without mixing with it to some extent, and so will have little chance of becoming supersaturated. Thus the dust-free air which rises from the cloud becomes contaminated with the surrounding dusty air, condensation takes place on it, and the cloud grows as fast as the ascending current rises. It may be urged that the first dust-free air which gets through the cloud may mix with the dusty air around, but that the resulting mixture may not be saturated ; in this way a partition might be formed between the supersaturated air which subsequently passes out of the cloud and the dusty air above. This, however, neglects the fact that on account of the continuous rises there will always be ultimately, above the supersaturated air,

a surface in which the air is both dusty and saturated, and in this layer condensation will take place. As soon as this occurs the drops will lag behind the ascending current, and will soon be overtaken by the region of supersaturated air. Drops thus fed by supersaturated air will grow more rapidly than the drops in the upper layer of the cloud below, and in consequence will lag behind the ascending current at a greater rate than the latter. Thus the supersaturated stratum which had begun to form will rapidly decrease in size and finally disappear, the top of the cloud becoming once more the boundary between the ascending air and dusty air which it is displacing.

In other words, an extensive supersaturated stratum could never form above an ascending current.

(v.) We will, however, for the sake of the discussion, assume that a supersaturated stratum may, as proposed by the theory, come into existence, and then consider whether such a stratum could exist during a thunderstorm.

It has been pointed out several times that in order to have a supersaturated stratum we must have a total absence of nuclei on which the water could condense. Now every drop of water which leaves the condensation layer has to pass through the supersaturated stratum on its way to the earth, and if the electrical discharges are to be as violent and frequent as we often observe during thunderstorms there must be a very considerable rainfall from the condensation layer. It must be admitted that each rain-drop as it passes through the supersaturated stratum will act as a very efficient nucleus for the condensation of the water vapour, and that no mass of air through which heavy rain is falling could long remain supersaturated.

Thus we see that a supersaturated stratum would be an impossibility underneath a condensation layer, condensing water at the rate necessary to bring into play the violent electrical discharges observed during thunderstorms. Hence, we are forced to conclude that the probability of the formation of an extensive supersaturated stratum in the atmosphere is extremely small, and that if even once formed it could not long exist after precipitation commenced from the atmosphere above it.

(vi.) We will now turn to actual phenomena observed during thunderstorms, and see in how far they support the theory.

A peculiar formation for thunder-clouds is demanded by the theory : there must be a lower cloud to separate out the dust ; then a dust- and cloud-free supersaturated layer of

considerable extent, which, if the storm is to last long, must be little less than 2,500 metres in vertical thickness; and, finally, a second cloud-layer where water is condensed on the negative ions, and the separation of electricity takes place.

From the time when the theory under consideration first came to my notice I have watched thunderstorms with special attention to the conditions which the theory presupposes. In no single case have I seen what appeared to be a large cumulus cloud with another cloud above it, but on the contrary, I have seen on several occasions distant clouds, from which heavy rain associated with thunder and lightning was falling, with sharply defined cumulus heads and quite clear sky above. Both Wilson and Gerdien refer to the false cirrus seen above thunder-clouds as very likely to be the cloud connected with the condensation layer. But every time that I have seen a layer of cirrus-cloud accompanying a thunder-cloud it has appeared to me as if the top of the cumulus-cloud had reached a part of the atmosphere which had robbed it of its rounded form and spread it out into a cirro-stratus cloud. I have never seen anything which would lead me to believe that between the heavy cumulus-cloud of the lower atmosphere and the cirro-stratus of the upper atmosphere there was a large cloudless region.

With regard to the electrical discharge of a thunderstorm; as far as my own observations go I should certainly say that the most violent and rapid discharges do not take place in the upper atmosphere: on the contrary they have always appeared to me to have their chief source in the centre of the large cumulo-nimbus cloud from which the rain was falling. The following account of a small thunderstorm may be interesting from the point of view of this theory. During an afternoon in August there had been heavy rain in Simla, but the sky commenced to clear before six o'clock. Soon after the sky had cleared so far as to leave the hills around visible, with large masses of clouds still clinging to them in places, a cloud in the form of an isolated pillar of comparatively small cross-section was seen to be rapidly extending upwards. The cloud was evidently the result of a rapid rise of moist air in a local current; it extended upwards with a beautifully rounded cumulus boundary at the top, and the centre was seen to be illuminated at intervals of about a minute by vivid internal lightning discharges. In an hour the cloud had lost its shape and become an ill-defined mass of cloud. In this case it could be said with absolute certainty that there were not two clouds separated by a supersaturated

layer, and, what is still more important, the cloud itself from base to summit during the time that the lightning-discharges were taking place was never 2,500 metres in height: thus supersaturation to a fourfold degree could not possibly have taken place.

To sum up the discussion, we may state that the following six reasons have been given for considering the theory, first proposed by Wilson, and subsequently expanded by Gerdien, unsatisfactory as an explanation of the electrical effects connected with thunderstorms.

A. Electrical.

- (1) The theory does not account for the frequent lightning discharges observed during thunderstorms.
- (2) It cannot account for the large charges of electricity carried down by rain during thunderstorms.
- (3) It is extremely improbable that any electrical separation would take place as the result of water vapour being condensed from fourfold supersaturated air onto negative ions.

B. Meteorological.

- (4) An extensive stratum of supersaturated air could not form above an ascending current.
- (5) Even if such a stratum were once formed it could not exist after precipitation commenced from the condensation layer above it.
- (6) The meteorological phenomena observed during thunderstorms do not lend any support to the theory.

Simla, Nov. 11, 1908.

LVI. *On Thunderstorm Electricity.*

By C. T. R. WILSON, M.A., F.R.S.*

MR. SIMPSON has been kind enough to send me a copy of his paper "On the Wilson-Gerdien Theory of Thunderstorm Electricity."

I should like at the outset to remark that I have nowhere myself attempted to account for the strong electric fields of thunderstorms by the theory which Mr. Simpson attributes to me. I have merely regarded condensation on negative ions as one of the possible factors in the production of such fields.

* Communicated by the Author.

The letter to 'Nature' from which he quotes was in defence of the view (first suggested by J. J. Thomson) that the positive charge of the atmosphere may be due to a preponderance of negative ions being carried down in rain, in consequence of their efficiency as condensation nuclei being greater than that of the positive ions.

There can, I think, be no doubt that in fine weather regions there is a positive current downwards from the atmosphere into the ground in accordance with the direction of the electrical field. The current per sq. cm. of the ground can in fact be measured indirectly from the potential gradient and conducting power of the air, as in the experiments of Gerdien and others, or more directly as in my own experiments (*Proc. Roy. Soc.* vol. lxxx. p. 537, 1908); and the measurements by different methods give values which agree. Until there is evidence to the contrary, it is natural to assume that in wet weather regions a compensating process is going on, due to a preponderance of negative electricity being carried down by rain, the electrical current in the atmosphere from the regions of precipitation to the fine weather regions being one of conduction (*Proc. Camb. Phil. Soc.* vol. xiii. p. 363) in the upper layers. It may eventually turn out that there is not the required excess of negatively charged rain, and that we shall be compelled to look to some other, possibly cosmical, source for the supply of negative electricity.

Both Elster and Geitel, and also Gerdien, have, however, found such an excess of negatively charged rain. If further investigations should confirm the apparent preponderance of negative rain found in those earlier experiments, then the superior efficiency of the negative ion as a nucleus may, I think, be looked upon as one possible factor in bringing about this preponderance. It is unfortunately exceedingly difficult in measurements of the charge carried down by rain in heavy showers and thunderstorms to avoid spurious effects.

In the only paper where I have discussed the possible factors causing strong electrical fields accompanying heavy showers or thunderstorms ('Nature,' vol. lxxviii. p. 102, 1903), I pointed out that the difference between positive and negative ions will not of itself account for intense fields. I should like to quote the passage—

"The following are possible factors in the production of the intense electrical fields which accompany heavy showers—

"A less degree of supersaturation is required to make water condense on the negative than on the positive ions (C. T. R. Wilson, *Phil. Trans.* vol. xciii. p. 289). Thus if condensation takes place from the supersaturated condition, the drops

formed are likely to be negatively charged ; that the drops, formed in ionized air by expansions slightly exceeding that required to cause condensation on negative ions, are actually negatively charged has been proved by H. A. Wilson (*Phil. Mag.* April 1903). Since, however, each drop will only carry the very small ionic charge, the electrical effect will be small if only a few large drops are formed ; if a large number of negative ions serve as nuclei of condensation, the drops will be small, and will only fall slowly relatively to the air ; the resulting electric field cannot exceed that which drives positive ions downwards as fast as the negatively charged drops fall under the action of gravity. The field initially produced may, however, be strong enough to induce coalescence of drops which come in contact (Lord Rayleigh, *Roy. Soc. Proc.* vol. xxviii. p. 406), and we may thus get drops carrying many times the charge of one ion, and large enough to fall rapidly. Strong fields may then result.

Again, we should expect (*Nature*, vol. lxii. p. 149) drops falling through ionized air to become negatively charged as a result of the difference in the mobility of the positive and negative ions. This effect has in fact been experimentally demonstrated by Schmauss (*Ann. d. Physik*, vol. ix. p. 224).

If collisions resulting in splashing occur between raindrops (and they are likely to be frequent in the up-rush of air in thunderstorms) positively charged rain may be formed. For, as Lenard has shown, when splashing of pure water occurs, as for example in waterfalls, the air in the neighbourhood acquires a negative, the water a positive charge.

Apart from the Lenard effect, the splashing resulting from the collision of drops in an electric field may have large effects, either in intensifying or diminishing the electric fields already existing, the action being like that of an electrostatic influence-machine. The result would be to increase the intensity of the field if the splashes were thrown out from the lower portion of the combined drop. If, for example, the field were such as to produce positive electrification on the lower surface of a neutral drop, a droplet leaving the lower surface would be positively charged, and being carried upwards by the air relatively to the large drop, would add to the intensity of the primary field."

The difficulty mentioned in the first paragraph of the passage quoted is in itself sufficient to make untenable the theory of thunderstorms which Mr. Simpson criticises. Taking the example given by him and assuming a cloud layer of continually increasing thickness to be formed, we can readily see that unless larger drops are produced by

coalescence, the maximum strength of field could not exceed one volt per centimetre; for in stronger fields the positive ions would travel downwards relatively to the air faster than the drops.

Small uncharged drops do not generally coalesce on coming in contact, but, as Lord Rayleigh has shown, electrification of the drops causes coalescence, and indeed he has suggested that the large drops of thunderstorms may be due to this cause. In his experiments, it is true, the conditions determining coalescence were such that the coalescing drops carried a charge of electricity while the drops condensed on the ions have a charge which is probably too small to have any effect of this kind. But there is little doubt that coalescence would also take place between neutral drops in an electric field, *i. e.* between drops of which the impinging surfaces were oppositely charged by induction. Experiments could alone decide what strength of field would be required to cause coalescence.

We should not necessarily get an upper cloud of more than very small vertical thickness, even if the upward air-current were sufficient to support the drops when first condensed on the negative ions from the supersaturated vapour.

For the critical supersaturation (*i. e.* the supersaturation necessary to cause condensation on the negative ions) will be reached at a level which will be continually rising. The critical supersaturation will be attained by a given portion of the air when it has risen to a definite height, not above the present position of the upper surface of the lower cloud, but above the level which that upper surface had when the portion of air under consideration escaped from it. The level of critical supersaturation would then in the absence of condensation travel upwards with a velocity depending upon that of the upper surface of the lower cloud.

The size of the drops which separate out when the critical supersaturation is first reached depends of course on the number of negative ions present, and can be calculated for given conditions. If these initially formed drops are sufficiently small to be carried upwards faster than the upward velocity of the level of critical supersaturation (which we may call the critical velocity), then a layer of cloud will be formed continually increasing in thickness by the condensation of drops at this level of critical supersaturation. Ice-particles would be specially likely to be carried upwards with more than the critical velocity. If, however, the drops condensed on the negative ions are large enough to be carried up with less than the critical velocity, the critical supersaturation

will never be attained below the first-formed drops, and the rising air will remain supersaturated to nearly the critical extent till it reaches these drops. They will thus grow extremely rapidly until large enough to fall through the supersaturated layer and lower cloud.

The supersaturated layer above the top of a cumulus cloud which is rapidly developing into cumulo-nimbus would probably in most cases be of quite small vertical thickness. Under the conditions just considered we might expect a thin cloud-cap to be formed suddenly over the head of the cumulus and sink rapidly into it. (I have on several occasions observed something very like this taking place; but it is of course very easy to be deceived in such matters.) It is possible that a process of this kind would determine the transition from cumulus to cumulo-nimbus. The drops formed on the ions may themselves fall through the lower cloud and reach the ground as rain. At the same time the electric field between the free positive ions above and the negatively charged drops will by the fall of the latter be extended through the upper part of the lower cloud. This field may help coalescence both between droplets of the lower cloud and between them and the larger drops which have come from above it. When this stage has been reached the effect of splashing referred to in the last paragraph quoted above from 'Nature' may be expected to become effective. It would be interesting to know from direct experiment what is the nature of the splash occurring when a large falling drop overtakes a smaller one. It seems not unlikely, however, in the light of Worthington and Cole's photographs of the "splash of a drop" falling into a vessel of water (*Phil. Trans.* vol. clxxxix. A. p. 137, 1897), that droplets may be thrown out from the underside of the combined drop if coalescence takes place. If we have a vertical electrical field already in existence the splashing process will in that case continually add to the intensity of the field.

It is in such ways that I think condensation upon negative ions may have to be taken into account in relation to thunderstorms; the first onset of the rain may be due to this cause, and it may also determine the direction of the initial electric field. The intense electrical fields of thunderstorms and the large charges carried down by the drops to the ground are possibly due to electrostatic induction effects accompanying collisions of the drops; the prevailing sign of the field and of the charge carried by the drops may be determined by the initial condensation on negative ions.

Elster and Geitel long ago suggested (*Wied. Ann.* vol. xxv.

p. 121, 1885) that the intense electric fields of thunderstorms might be due to electrostatic influence effects accompanying the fall of larger drops through a cloud of smaller ones—a weak initial electrical field being assumed.

To what extent condensation upon the ions is likely to be effective after the initial stages of the showers is a difficult question into which I do not propose to enter. I have never regarded it as at all likely that the supersaturation necessary to cause condensation upon the positive ions could ever occur in the atmosphere, as Elster and Geitel and others have imagined.

The question of the occurrence of supersaturation in the atmosphere is of course fundamental in relation to the question under discussion. In the article in '*Nature*' quoted by Mr. Simpson nothing was said about the condition of the atmosphere immediately above the cloud. The argument was intended to show how, even if the lower moisture-charged layers were highly charged with dust-particles, a supersaturated layer might still result, and the upper atmosphere remain dust free.

To make the matter clearer, let us assume that above a lower dust- and moisture-charged layer we have a dry and dust-free layer. (It may be remarked that a large mass of air in the atmosphere can hardly have become dry otherwise than by the condensation and subsequent precipitation of the water from it—a process which is likely to remove from it the dust-particles as well. It is of course possible that there may be in the upper atmosphere a continual production of dust-particles—under which title we may include all persistent nuclei more efficient than the ordinary negative ion.) If we imagine a local ascensional current to begin, the surface of separation of the two layers will rise; and when it has reached the level corresponding to saturation for the lower layer a cumulus cloud will begin to form. The upper surface of this cumulus cloud marks approximately the surface of separation between the dry and moist layers. The transition from the dry to the moist layer cannot be absolutely sudden on account of diffusion; in consequence condensation will first begin somewhat below the upper limit of the dust-charged air. As the air continues to rise the cloud will thus at first grow partly by additions to its upper edge, the newest and smallest drops being at the top. If the transition layer between the dry and moist air continues to ascend, a stage will be reached when saturation extends to the highest level reached by the dust-particles. Beyond this stage the cloud will cease to grow by addition of new drops at its upper

surface, and a super-saturated layer will begin to form above it.

Although I have never observed the peculiar formation described by Mr. Simpson as being required for thunder-clouds according to the theory he is criticising, I see no reason on that account to conclude that condensation on negative ions may not take place in the atmosphere. If the supersaturated layer be thin, the condensation on the negative ions will not in itself be at all a conspicuous phenomenon although its effects may be by no means unimportant. It is possible, however, that more striking phenomena may also be the result of condensation on the negative ions. In a typical fully developed cumulo-nimbus cloud, while the lower part of the cloud has the appearance of ordinary cumulus, the upper part has an entirely different appearance, of which there are several different varieties. In some cases the appearance is suggestive of rain or snow falling from some height above the lower cumuliform portion of the cloud. It is possible that this upper portion of the cumulo-nimbus (the false cirrus) may be due to condensation on the negative ions. Until more has been done by experiment and observation to throw light on the processes going on in clouds, it hardly seems profitable to consider in detail the possible cloud forms that might be expected to result from condensation on negative ions.

There is one statement in Mr. Simpson's paper which, while not seriously affecting his argument, appears to call for remark. In discussing the effect of a lightning-flash he says:—"Now Barus has shown that whenever ions are formed in dust-free air there is produced at the same time a large number of nuclei which exist for a long time after the ions have disappeared. As water-vapour is very readily deposited on the nuclei, fourfold supersaturation is prevented in air in which they are present unless the supersaturation is caused by exceedingly rapid rarefaction." It is quite true that spark-discharges in air, as well as other ionizing processes which are accompanied by chemical effects, do produce large and persistent nuclei in addition to the ordinary ions; a lightning-flash will therefore almost certainly produce such nuclei. But the ordinary ionizing radiations when of moderate intensity do not produce, in uncontaminated air, nuclei more efficient than the negative ions.

I shall conclude with the following three statements expressing summarily my views on the questions raised by Mr. Simpson.

1. I have never regarded condensation on the ions as in

itself a sufficient cause for the strong electrical fields of thunderstorms.

2. If, however, we assume the occurrence of supersaturation in the atmosphere, condensation on the negative ions is likely to be of importance in connexion with the production of precipitation, and as tending to cause a preponderance of negatively charged rain to be carried down to the earth.

3. The question of the occurrence of supersaturation in the atmosphere must still, I think, be regarded as an open one.

LVII. *On the Faraday-Maxwell Mechanical Stress; and on Ætherial Stress and Momentum, in general.* By C. V. BURTON, D.Sc.*

1. **A**LTHOUGH the more recent development of electrical theory has shown the difficulty of accepting as anything more than an analogy the Faraday-Maxwell "tension along the lines of force and equal pressure across them," this specification of ætherial stress must always be regarded as one of the most profound and important contributions ever made to the progress of physical science. Maxwell's analysis from which this mechanical ætherial stress was deduced†, and which gave a precise form to Faraday's theory, was probably as cogent in directing attention to the possibilities of an "intervening medium" as was his great and enduring system of electromagnetic equations. As the merely analogical significance of Maxwell's "mechanical stress" does not seem to be quite generally appreciated—that specification of stress being still spoken of by physicists of distinction as if it necessarily represented the actual stress in the medium—the brief exposition which follows is perhaps not wholly uncalled for.

2. For an electrostatic field in free æther, the Faraday-Maxwell mechanical stress at any point is a tension $R^2/8\pi$ along the lines of force, with a pressure of like magnitude in all directions perpendicular to the lines of force; R being, in electrostatic measure, the electromotive intensity at the point in question. To the student of twenty odd years ago this stress was certainly a stumbling-block; for in a medium whose electromagnetic properties are expressible by a system of linear equations, and in which electromagnetic disturbances are propagated with a velocity independent of amplitude and of wave-length, it is not easy to realize why any

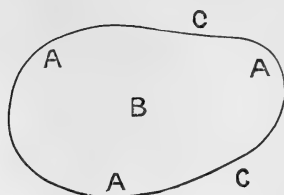
* Communicated by the Author.

† 'Treatise,' vol. i. chap. v.

mechanical stress which may exist should be proportional to the *square* of the electromotive intensity at each point, instead of following a like linear law.

3. To realize the nature of the assumption implicitly involved in Maxwell's theory, consider first a body B (fig. 1)

Fig. 1.



bounded by a closed surface A of any form, and surrounded by a medium C which may be treated as continuous. The nature of the substances composing the body B and the medium C need not be specified; they may be identical in nature or not. What must be understood here is that the material, of whatever kind, contained within the surface A is always the same identical portion; that surface, however its position and form may vary, being regarded as an impermeable barrier. In these circumstances, a given distribution of stress in the medium C will give rise to determinate forces acting upon the body B, the components of stress in the medium at every point of the surface A being all that we require to know in order to deduce those forces. In particular, if at each point of the surface A the normal is a principal axis of stress in the medium C, a specification of the normal stress in the medium over the whole of A is sufficient to determine the forces to which the body B is subject. This corresponds in Maxwell's theory to the case of a conducting body surrounded by a dielectric; and from the principle of virtual work it is evident that, in order to apply directly the ordinary theory of stresses as Maxwell does, we must assume the fulfilment of the conditions already stated in this paragraph: the surface A, which is the bounding surface of the conductor in question, must be regarded as a boundary dividing that body from the medium C; so that wherever the surface of the conductor advances in the direction of the outwardly drawn normal, the dielectric medium recedes to the same extent, and *vice versa*. If the surface of the conductor were in some degree permeable by the dielectric medium, the force exerted on the body would be no longer thus simply derivable from the stresses in the medium. If

the conductor is in a vacuous space, so that the dielectric is simply æther, the conclusion to which our assumption leads may be expressed by saying that where the conductor is, the æther is not, the surface of the conductor being that boundary where the æther ceases and the substance of the conductor begins. We are thus treating the conducting body as foreign to and wholly distinct from the æther.

4. On the other hand, suppose that we are dealing with a medium in which it is possible for a "strain-figure" to exist: this term being understood to mean a distribution of strain which requires no independent source of stress to maintain it, but is entirely self-sustaining. Such a strain-figure is at least a possible aspect of the modern electron, whether positive or negative. It is necessary to conceive of it as freely mobile through the æther, the displacement of the strain-figure from some given position to a neighbouring position being equivalent to imposing a differential strain upon the æther. When the æther in the neighbourhood of the strain-figure is stressed in a definite manner by some independent agency, it is at once evident, on applying the principle of virtual work, that the tendency of the strain-figure to move through the medium in one direction or another *depends essentially on the type of the strains* of which the strain-figure is made up. For example, when a material body is regarded as an assemblage of self-equilibrating ætherial strain-distributions (electrons), it must not be assumed that an excess of ætherial pressure* on one side will necessarily tend to displace the body through the æther towards regions where the pressure is less; this will only be the case if the constitution of the body is such that the region which it occupies contains less ætherial substance than a like volume of free æther†. If the region occupied by the body contains an excess of ætherial substance, the body will tend to move from places of lower towards places of higher ætherial pressure. In the intermediate neutral case, the existence of

* Without assuming any special constitution for the æther, we may take the hydrostatic pressure to be one of the constituents into which the stress at each point is resolved.

† Let the *defect* of æther in the region occupied by the body be measured by the mass vp , where ρ is the density of free æther, and v is of the nature of a volume. Then a pressure-gradient in the æther causes a force whose components are

$$-v\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right),$$

to act on the body, tending to displace it with respect to the æther. This result is easily proved, and is equivalent to equation (3) p. 76 of Phil. Mag. January 1909.

an ætherial pressure-gradient has no tendency to displace material bodies one way or the other with respect to the æther. Though (pressural) forces of the kind just referred to may conceivably be concerned in gravitation, it seems probable that the mechanism of the action of electrically charged bodies and of magnets on one another are of types entirely distinct.

5. Again, when dealing with exchanges of momentum between the æther and "ordinary" matter, we have no right to treat the momentum of a material particle, measured by the product of its mass and velocity, as equivalent to ætherial momentum similarly measured as a product of mass and velocity. Experience teaches us that, through a wide range of phenomena, the principle of conservation of momentum holds good with great exactitude for ordinary material systems, every particle or element of matter being regarded merely as an entity endowed with definite inertia and with perfect mobility, except in so far as its motion may suffer interference from the neighbourhood of other particles. On such a basis the dynamics of matter reached a high degree of development before the existence of an æther was realized; but when the interaction of matter and æther is in question, this primitive conception of material inertia and momentum is no longer adequate. The motion of a material particle with respect to the æther represents kinetic energy (T) which is a function of the velocity $(\dot{x}, \dot{y}, \dot{z})$ of the particle. This kinetic energy we suppose to be due to some type of ætherial motion, since the motion of the particle through the æther amounts to a progressive readjustment of those ætherial strains, the aggregate of which constitutes the particle. The ætherial motion in question may not (and according to §§ 9-21 below does not) involve any bodily drift in the direction of motion of the particle; though in any case the components of momentum of the particle corresponding to the coordinates (x, y, z) are, in the generalized dynamical sense $(\partial T / \partial \dot{x}, \partial T / \partial \dot{y}, \partial T / \partial \dot{z})$; which together also express the momentum of the particle as ordinarily understood.

6. The implicit assumption referred to in § 3 above leads further to the conclusion that the æther is *indefinitely compressible*. For consider a condenser made up of an inner conducting sphere and a concentric spherical cavity in an outer conducting body, the interspace being vacuous. Let the inner sphere be of radius r_1 and the concentric cavity of radius r_2 ; also let the inner sphere carry a charge e , the charge upon the hollow spherical surface of the outer conductor being therefore $-e$. As we pass across from the

inner to the outer sphere, the electromotive intensity has values ranging from e/r_1^2 to e/r_2^2 , measured in the direction of the outwardly drawn radius. Now keeping the inner sphere insulated, so that its charge e remains unaltered, let this sphere expand symmetrically and infinitesimally, its radius becoming $r_1 + dr_1$. The electrostatic energy of the system is thus changed by

$$\frac{d}{dr_1} \left\{ \frac{1}{2} e^2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right\} dr_1 = -\frac{1}{2} \frac{e^2}{r_1^2} dr_1; \quad \dots \quad (1)$$

while the Faraday-Maxwell radial tension, reckoned per unit area of the inner sphere, is

$$\frac{1}{8\pi} \left(\frac{e}{r_1^2} \right)^2, \quad \dots \quad (2)$$

and the work done by this tension per unit area is

$$\frac{1}{8\pi} \frac{e^2}{r_1^4} dr_1. \quad \dots \quad (3)$$

Thus $\frac{1}{2} \frac{e^2}{r_1^2} dr_1$ is the work done by the ætherial "mechanical stress" during the expansion dr_1 , and this work is equivalent to (1) with sign reversed, that is to the loss of electrostatic energy of the system. But, as already pointed out, in order that the radial tension (2) may contribute the work (3) in the displacement dr_1 , it is essential that the æther between the spheres should contract by the amount dr_1 along every radius, its volume being thus reduced precisely as much as the volume of the inner sphere is increased. Proceeding in the same way, the inner sphere could be expanded until $r_2 - r_1$, which originally might be as large as we pleased, was reduced to an insignificant amount; the æther between r_1 and r_2 retaining all the while a radial tension which is nowhere less than e/r_2^2 , and being ultimately made to occupy only a minute fraction of its original volume.

7. Let us now contrast two interpretations of the virtual work corresponding to a small change of configuration of an electrified system; one interpretation being that of the Faraday-Maxwell mechanical stress, the other that of the electronic theory. Let the system consist of any number of bodies carrying surface charges, the intervening spaces, for simplicity of illustration, being supposed vacuous; and consider the virtual work of the electrostatic forces when the system suffers a determinate change of configuration $\delta\theta$, specifiable in terms of the geometrical coordinates. If, without making any other change, we increase the charge

of every element until it is n times as great as before, the virtual work corresponding to $\delta\theta$ will be n^2 times as great as before. Now on the Faraday-Maxwell view, since we regard the æther as simply partaking of the normal displacement of each electrified surface, the displacement of the æther* involved in the change of configuration $\delta\theta$ will depend on $\delta\theta$ only, and not on the electrical charges. Hence when all the charges are raised to n -fold, involving an n^2 -fold value for the virtual work in $\delta\theta$, we must have n^2 -fold values for the ætherial stresses.

8. But on the electronic theory, any displacement $\delta\theta$ of the bodies in question is a displacement of the electrons which make up those bodies and their surface charges, and the corresponding displacement of the æther at any point † is the vector-increment of electric polarization at that point. On this view, then, the n^2 -fold virtual work corresponding to $\delta\theta$, due to n -fold charges throughout the system, is accounted for by n -fold virtual displacement everywhere in the æther surrounding the bodies, together with n -fold ætherial stress. On the electronic theory, therefore, the stress in the æther is simply proportional to the electromotive intensity, conformably to the linear character of the equations of electromagnetism.

9. Maxwell's investigation of the stress in a magnetic field ‡ follows essentially the same course as that relating to an electrostatic field, and for non-magnetizable media an identical type of stress is found. Here again the assumption referred to in § 3 above is implicitly made, and it is further implicitly assumed that, in the magnetic as in the electrostatic case, the medium which transmits the forces in question may be treated as being at rest. The conclusion that the forces mutually exerted by electrified bodies on the one hand and by magnets on the other hand, are due to ætherial stresses of identical type is indeed somewhat disconcerting; but for isotropic media Maxwell's "mechanical stress" must be regarded as a perfect mathematical analogy, affording in certain cases a ready means of reaching results which are not so immediately obvious when otherwise approached. The most conspicuous example is Maxwell's own discovery of "radiation pressure," the general phenomena of which

* The term (infinitesimal) *displacement* is here used in the generalized sense commonly understood in relation to dynamical systems; not in the special sense associated with Maxwell's "electric displacement."

† See preceding footnote.

‡ 'Treatise,' vol. ii. chap. xi.

are immediately deducible from the analogy of a medium whose state of mechanical stress causes foreign bodies embedded in it to be urged this way or that.

The magnetic vector in free æther is not constituted by translational ætherial velocity.

10. In a previous communication to the Philosophical Magazine* an attempt was made to prove that (in free æther) neither the magnetic nor the electric vector is to be identified with translational velocity of the æther. As the cogency of the arguments then put forward has been called in question†, and as my own conviction remains unshaken, a more precise proof will now be offered, which I trust will be found satisfactory.

11. Consider a train of plane-polarized plane electromagnetic waves, which is being propagated in free æther in the direction of z -increasing, the electric vector being parallel to the axis of x , and the magnetic vector parallel to the axis of y . In this case the vector-potential reduces to a component which is parallel to the axis of x , and is a function of $z-Vt$, where V is the velocity of radiation. Writing—with Maxwell's notation—(F, G, H) for the vector-potential, (a, b, c) for the magnetic induction, and (P, Q, R) for the electromotive intensity at any point, let the suffix unity be used to distinguish the wave-train now considered, and let us put

$$(F_1, G_1, H_1) = \{\phi(z-Vt), 0, 0\}. \quad (4)$$

Then $(a_1, b_1, c_1) = \{0, \phi'(z-Vt), 0\}, \quad (5)$

and $(P_1, Q_1, R_1) = \{V\phi'(z-Vt), 0, 0\}; \quad (6)$

so that $P_1 = Vb_1. \quad (7)$

For a second train of plane waves polarized parallel to the first, and propagated in the direction of z -decreasing, we shall similarly have

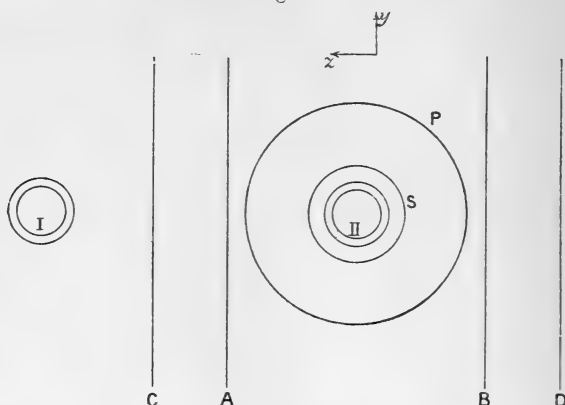
$$P_2 = -Vb_2. \quad (8)$$

* June 1907, pp. 694-696.

† Letters dealing with this subject have appeared in 'Nature' (1907): "The Structure of the Ether," O. Richardson (May 23), Lodge (June 16), C. V. Burton (June 13), E. Cunningham (July 4), C. V. Burton (July 18). "The Æther and Absolute Motion," Larmor (July 18).

12. In fig. 2 let the plane of the paper be parallel to yz , and let A, B, C, D be four planes parallel to xy . At a definite instant, say $t=0$, let the first wave-train be entirely confined to the region between C and D, and, moreover, between the limits A and B, and at the same instant let b_1 have the constant value b_0 , P_1 having therefore simultaneously,

Fig. 2.



between A and B, the constant value Vb_0 . Similarly at time $t=0$ let the second wave-train be confined to the region between C and D, while between A and B, b_2 has the constant value b_0 , and P_2 consequently the constant value $-Vb_0$. Thus the superposition of these two wave-trains gives rise, at time $t=0$, to uniform magnetic induction

$$\left. \begin{aligned} (a, b, c) &= (0, 2b_0, 0) \text{ between A and B} \\ \text{while } O &= P = Q = R \text{ between the same limits} \end{aligned} \right\} \dots (9)$$

At the same time there is neither magnetic induction nor electric force to the left of C or to the right of D.

13. Consider first the region to the left of C, where the æther is at rest, and is free from magnetic induction and electric force. Somewhere in this region let there be a material system, and to fix ideas let this take the form of a metal ring (I) which is rotating about an axis in its own plane. The rotation in this case can cause no selective circulation of positive and negative electrons around the ring, such as would constitute an electric current.

14. Next consider the region between A and B, where the æther (as we provisionally assume) is drifting in the direction of y -increasing with velocity $b_0/\sqrt{\pi\rho}$, but is otherwise in a condition identical with that of the æther to the left of C.

Within this region A ... B let there be a ring (II) in all respects similar to I, every electron in the one system having its counterpart in a like electron of the other. We shall suppose that, when $t=0$, not only is the assemblage of electrons II similar to the assemblage I and similarly oriented, but each electron of II has the same x -component of velocity as the corresponding electron of I and the same z -component, while the y -component of the II-electron's velocity exceeds that of the I-electron by $b_0/\sqrt{\pi\rho}$. Thus the compound system consisting of II with its surrounding æther is only distinguished from the system I with its surrounding æther by a general translational velocity $b_0/\sqrt{\pi\rho}$, which cannot modify the internal dynamics of the compound system II. This system accordingly, from the time $t=0$ onwards, imitates the compound system I in every detail of motion of every electron; the correlation being maintained so long as no electromagnetic disturbance has reached the region occupied by I, while the state of the region occupied by II continues to be defined by (9). In particular, as no electric current is flowing around the rotating ring I, neither does any current flow around the rotating ring II: and since the direction of the axes of rotation of the rings may be any we please, it must be concluded that the region between the planes A and B is free from magnetic induction; which is contrary to the supposition with which we started.

15. A further word may help to make the matter clearer. Let a sphere S be described completely enclosing the system II at time $t=0$ (fig. 1), and let P be a concentric sphere whose radius exceeds that of S by $V\tau$, where V is the velocity of radiation. Then, in accordance with well-known principles of wave-propagation, the state of things within S between $t=0$ and $t=\tau$ is completely determined by the state of things within P when $t=0$. If, then, the sphere P lies wholly between the planes A and B, the dynamical circumstances of the system II, from $t=0$ to $t=\tau$ at the least, will be precisely the same as if the range of equation (9)—instead of being limited by the planes A and B at time $t=0$ —extended to infinity in all directions. Thus no argument against the conclusions of §§ 11–14 above can be founded on the circumstance that the hypothetical æther-drift corresponding to (9) is limited by the planes A, B; or that the whole extent of quiescent æther beyond the limits C, D provides a standard relatively to which the velocity of the drift can be measured.

16. The foregoing is offered as a rigorous proof of the proposition that the magnetic vector in free æther is not to be identified with translational ætherial velocity—a

conclusion which may be admitted to have a *prima facie* reasonableness. For an observer who knows that his velocity relatively to the æther must at some time of the year amount to 28 kilometres per second at the least, might well hesitate to believe that between an intense magnetic field and a space free from magnetic force the only difference was a minute difference of ætherial velocity: a velocity of (perhaps) a fraction of a millimetre per second in the former case, as against zero velocity in the latter.

The electric vector in free æther is not constituted by translational ætherial velocity.

17. By suitably modifying the two trains of plane-polarized plane waves referred to in § 11, a uniform electrostatic field free from magnetic induction is equally readily obtained, and the proof of the proposition just enunciated follows on lines precisely similar to the proof given in the magnetic case. Such a demonstration, however, is hardly called for, in view of von Helmholtz's objection that if a continuous flow of æther were taking place along the lines of electric force, every charged body would behave as an unlimited source of æther or as a corresponding sink.

The Poynting vector in free æther is not identifiable with translational ætherial velocity.

18. In any purely progressive electromagnetic wave-train, the vector product of the electric and magnetic vectors (Prof. Poynting's vector) is persistently in the direction of propagation, and never in the opposite direction. For example, if we take the train represented by (5), (6), putting for $\phi'(z-Vt)$ the more definite expression $A \cos s(z-Vt)$, the Poynting vector is

$$\{0, 0, A^2 V \cos^2 s(z-Vt)\},$$

that is

$$\{0, 0, \frac{1}{2}A^2 V + \frac{1}{2}A^2 V \cos 2s(z-Vt)\} : \dots (10)$$

and if we suppose the æther to be flowing at each point in the direction of this vector, the motion corresponding to (10) will be a wave-train of compressional-rarefactional type (like sound-waves in air) superposed upon a general drift of æther in the direction of wave-propagation.

19. We have thus to face the conclusion that the æther is compressible; its compressibility being not merely of that minute residual order which has been assumed as a means of accounting for the phenomena of gravitation, but of such

fundamental import in the electromagnetic scheme that without it no propagation of electromagnetic waves could take place. Over and above this, we are met by essential difficulties.

20. One of these relates to the question of polarization. For if an essential feature of electromagnetic wave-motion is a motion of the æther parallel to the direction of propagation, it would hardly seem possible for a beam of radiation to be fully polarized.

21. Another difficulty arises from the conclusion that, in a progressive train, the motion of the æther is persistently in the direction of propagation. Such uni-directional flow involves something which cannot properly be described as wave-motion at all; while on the view now considered a body such as the sun, which continually emits more radiation than it receives, must be regarded as a *source of æther*.

22. We might here evade the conclusion that an actual creation of æther is involved in the process of radiation by supposing a hot body to be a centre of ætherial condensation, as compared with a colder body of like composition. But the difficulty of the position becomes more evident when we consider the conditions under which radiation is emitted by a glow-lamp filament, say of tungsten. Let AD (fig. 3) be a

Fig. 3.



portion of such a filament, and let BCEF be a closed surface made up of the narrow cylindrical surface FE, BC coaxial with the filament, and two plane ends BF, CE perpendicular to the filament. If we suppose that a bodily flow of æther takes place in the direction of the Poynting vector, then so long as the filament is maintained at a high temperature by an electric current passing through it, there will be a flow of æther outward through the cylindrical surface FE, BC. Through the plane surfaces BF, BC there will be on the whole no flow of æther, the circuital character of the electric current precluding the idea that anything directly associated with the current can flow into our closed surface (say) through BF, except what inevitably flows out again through CE*. Moreover, if the filament AD is in motion with respect to the æther, we may suppose it accompanied by the ideal

* We could even use an alternating current.

surface BCEF, and the motion in question will clearly have no tendency to cause either an accumulation or a depletion of æther within BCEF. Thus if we keep the filament heated by a current flowing along it, æther will on the whole be streaming out of the space BCEF, and provided the operation can be long enough continued, the quantity of æther emitted from the space in question can be made as great as we please. This would imply an actual creation of æther.

23. It might be argued that, in any realizable case, the flow of æther would be very minute, and if we proceed on the assumption that radiation pressure is due to the impact of bodily ætherial momentum this is probably true; it is certainly true that no operation of the kind just indicated has been carried out for more than a limited time; but it seems not unreasonable to suppose that the bulk-modulus of elasticity of the æther is enormously great, and is in some way responsible for the very high velocity with which gravitational influences are propagated. Thus even a minute emission of æther from within a given closed surface would appear likely to be promptly opposed, owing to the expansion which must then be experienced by some at least of the æther contained by that surface. Indeed, unless the systematic creation of æther can be accepted with equanimity, such an opposing reaction must set in from the first, and gradually increase in intensity with the duration of the outflow of æther, becoming in any case of importance if the outflow is long enough maintained.

24. Again, consider a train of electromagnetic waves in free æther, specified as follows

$$(F, G, H) = \{A/s \cdot \sin s(z - Vt), 0, 0\}, \quad \dots \quad (11)$$

so that

$$\left. \begin{aligned} (a, b, c) &= \{0, A \cos s(z - Vt), 0\} \\ (P, Q, R) &= \{VA \cos s(z - Vt), 0, 0\} \end{aligned} \right\} \quad \dots \quad (12)$$

If we substitute for A a different parameter B , the magnetic induction (a, b, c) and the electromotive intensity (P, Q, R) will be changed in magnitude, and in the same proportion, namely, $B : A$, the wave-motion remaining of precisely the same type as before, and being changed only in amplitude. But according to the doctrine now under consideration, the translational velocity of the æther at any point at any instant, being proportional to

$$(bR - cQ, cP - aR, aQ - bP),$$

will be changed, not in the ratio $B : A$ but in the ratio

$B^2 : A^2$. Such a conclusion seems hard to reconcile with the simple linear character of the electromagnetic equations of the æther.

25. Another aspect of the question is this : By the change of parameter from A to B the *total* energy of wave-motion within a given region at a given instant is changed in the ratio $B^2 : A^2$, while according to our assumption the *translational* energy of the æther within the same region is changed in the ratio $B^4 : A^4$. This implies that, notwithstanding the absence of any change of type in the wave-train considered, there has been a change in the partition of energy in the train ; the translational energy of the æther having become a greater or a less proportion of the whole energy according as B is greater than or less than A.

26. If the difficulties enumerated in the last eight paragraphs are to be regarded as insurmountable, the conclusion must be that the Poynting vector is not associated with translational ætherial velocity (in free æther)*, and this conclusion also seems, on other grounds, to be a reasonable one. For a steady electrostatic field and a steady magnetic field are, so far as we know, entirely without interaction, and where fields of these two kinds coexist, the resulting energy is simply the sum of the separate energies of the two fields in question, *without product terms* such as we should expect to appear if the æther were indeed in motion with velocity proportional to the vector product of the electric and magnetic vectors.

27. To sum up some of the objections which may be urged against the doctrine that a flow of æther takes place in the direction of the Poynting vector. This doctrine leads to the conclusion that the translational motion of the æther in a progressive wave-train is longitudinal, resembling the motion of air which is transmitting sound-waves, with a general bodily drift in the direction of propagation superposed. In the absence of special assumptions which might conceivably be devised to remove the difficulty, it leads also to the conclusion that radiation is accompanied by a continuous creation of æther. It assumes the existence of a translational energy-term of which the uncontested expressions for total electromagnetic energy (say in free æther) give no indication ; this assumed term constituting in fact a violation of the linear character which is an outstanding feature of the electromagnetic equations of the æther.

* The arguments above given are of course, *mutatis mutandis*, equally valid against the view that the Poynting vector with sign reversed corresponds to translational velocity of the æther.

How many distinct types of wave-motion are possible in free æther?

28. If, as is here suggested, neither the magnetic nor the electric vector, nor yet the vector product of these two, corresponds to a bodily flow of æther, it would seem a likely conclusion that no purely electromagnetic phenomenon involves the bodily displacement of any æther-element; that the æther remains, in the grosser sense, permanently at rest, experiencing only changes of "polarization." (The rapidly travelling compressional waves, which I have ventured to suggest as affording a possible basis for gravitation, involve only such slight deformations of the æther that they need not be supposed to interfere with electromagnetism.) But if ætherial waves of bodily transverse displacement have no part in electromagnetic or in gravitational phenomena, it is natural to inquire whether the æther is (as a frictionless fluid would be) incapable of transmitting waves of that type—or if such waves, having been generated by a suitable agency, would be propagated through the æther with a definite velocity, are we to infer that they are non-existent because no system of moving electrons is competent to produce them? If I may venture an opinion on so recondite a matter, it seems to me not improbable (in view especially of the considerations put forward in this paper) that by no manipulation of ordinary matter could ætherial waves of bodily transverse displacement be initiated; but in that case it might well be that waves of the type referred to, if existing in the æther, would fail to produce, through their interaction with atomic matter, any sort of observable phenomenon. At this point it would seem prudent to leave the question as to whether waves of transverse displacement are being propagated through the æther or not.

My thanks are due to Prof. G. M. Minchin, F.R.S., who most kindly read the MS. of this paper. His valuable criticism having led me to rewrite certain sections, it may be hoped that some obscurities have thus been removed.

LVIII. *Notices respecting New Books.*

General Physics. HENRY CREW, Ph.D. New York: The Mac-Millan Co., 1908. Price 12s. net.

THIS is intended as a first year university course in physics. It obviously consists of the amplified outline of a course of lectures; and therefore we miss the development of the measuremental (as distinct from the experimental) side of the subject which we like to see in a book intended for students' use. The first half of the book consists of mechanics. We are not sure that the experimental illustrations chosen are always the best. The case in which

the student is told to *watch* several balls of the same size but of different material rolling on the ground, will certainly not give a vivid conception of the notion of mass. 'Heavy' weights that roll across a theatre stage often consist of the lightest materials, but give the false impression intended. A student should himself *start* or *stop* the balls in order to realize the difference between them. Again, to prove that the elasticity of indiarubber is different from that of glass by letting two balls of these materials fall on an oiled glass plate and comparing the patches, is to make use of an experiment far too recondite for the purpose. Why not merely stretch two rods with equal weights? The part dealing with elasticity is scarcely satisfactory even allowing for the omissions necessary in an elementary book.

The rest of the volume (Heat, Electricity, Magnetism, Light) is more to be commended, especially the Electricity. In connexion with electric oscillations we may mention that it was Savary, not Joseph Henry, who first magnetized needles in alternating layers by the discharge of a jar; and that Kelvin's prediction should be dated 1853 not 1855. Bernoulli's name is spelled wrongly.

Though the book does not attain to our ideal of what an elementary one should be, there is so much about it to be praised that we have no doubt that many will find it just the one they want.

LIX. *Intelligence and Miscellaneous Articles.*

NEW GEOLOGICAL VIEWS ON TERRESTRIAL MAGNETISM AND VOLCANISM. BY J. J. TAUDIN CHABOT.

THE development of heat by the disintegration of atoms needed only to be discovered to raise at once the question whether the thermal phenomena of our planet are of the same kind, even as previously the ascertained magnetic properties of some substances raised the question whether the magnetic state of the globe arises from its being constructed of such materials.

With regard to the first question, observed data tend to support the possibility of its truth; concerning the second we have reached, so far, no satisfactory conclusion.

The results of recent investigations appear, however, to open up paths along which our knowledge may be extended, the most remarkable of which is one that gives rise to the unexpected idea of a possible connexion between the fundamental causes of the thermal and magnetic phenomena of our planet.

The spontaneous radioactivity (so-called) exhibits a definite dependence upon atomic weights inasmuch as it has been detected in those substances whose atomic weights have the highest known values. Thus it was discovered in a uranite, and immediately afterwards it was observed in thorium; whilst the more recently found radium has also a high (perhaps even the highest known) atomic weight. This last substance follows bismuth in a direct series leading up from radioactive lead; and of bismuth we

know, from a recent discovery, that it forms in combination with manganese a strongly magnetic substance the special property of which is that the magnetic quality is first brought into evidence (in a similar manner to that for several other existing combinations of non-magnetic substances) in the *heated* state at temperatures between 2000° and 2300° C.

This fact, so extraordinary and surprising when compared with all hitherto ascertained results, appears to me to be worthy of attention, especially by geophysicists.

The question arises: If radioactive processes constitute the source of the internal heat of our planet, to what extent may they at the same time determine its magnetic state?

Except this more omnipresent radioactive heat there are local sources of increased temperature, the principal one amongst them caused, as it seems to me, by friction and pressure of a few incessantly moving enormous parts of our planet's crust against each other; so that here and there in the contact-regions matter is liquefied and, sometimes, by the same reason, pushed through the earth's surface, thus showing us the what we call volcanic phenomena.

Rotterdam, February 13, 1909.

THE GRATING SPECTRUM OF THE RADIUM EMANATION.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

PROF. E. B. FROST has called my attention to his publication of a portion of the secondary hydrogen spectrum in the 'Astrophysical Journal,' vol. xvi. p. 104 (1902), with reference to my paper on "The Grating Spectrum of the Radium Emanation," published in the January Number of the Phil. Mag. From Prof. Frost's list it appears that, in addition to the hydrogen lines already marked, the four lines which were marked as not having been seen in the prism spectrum are due to the secondary spectrum of hydrogen. These four lines were not observed in the prism spectrum, although they were at more than resolvable distance from other lines, and it was therefore probable that they were due to hydrogen which it had not been possible to remove so completely in the experiments for the grating photographs. Not being aware of Prof. Frost's determination at low pressures, however, I had not been able to trace these lines to hydrogen from the determinations of Hasselberg and Ames.

I should like to take this opportunity of correcting a printers' omission of a footnote to the effect that the strong emanation line 4308.3 in the prism spectrum was masked in the grating photographs by the strong iron line of the comparison spectrum at 4307.96 (G in the solar spectrum), and its wave-length could not be measured more accurately.

Yours truly,
T. ROYDS.

Physikalisches Institut,
Tübingen, Germany.
March 3, 1909.

FIG. 4.

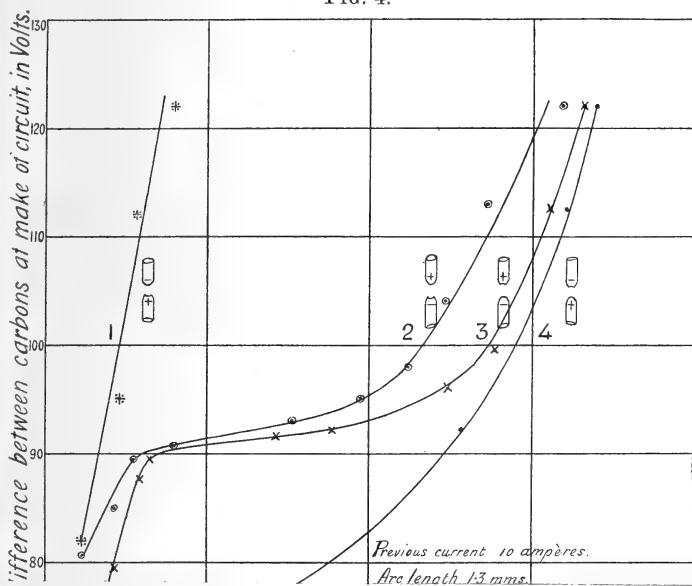
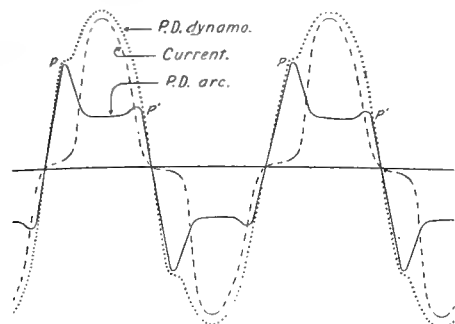
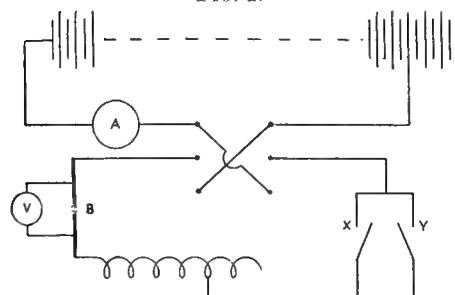


FIG. 1.



Waves of current and potential-difference in connexion with an alternating-current arc-lamp.—Duddell & Marchant.

FIG. 2.



Scheme of connexions.

FIG. 5.

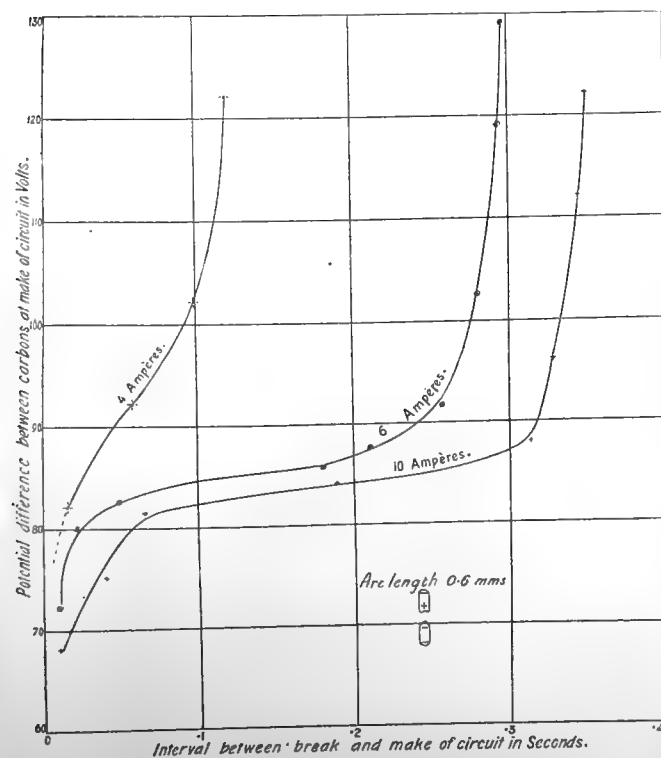
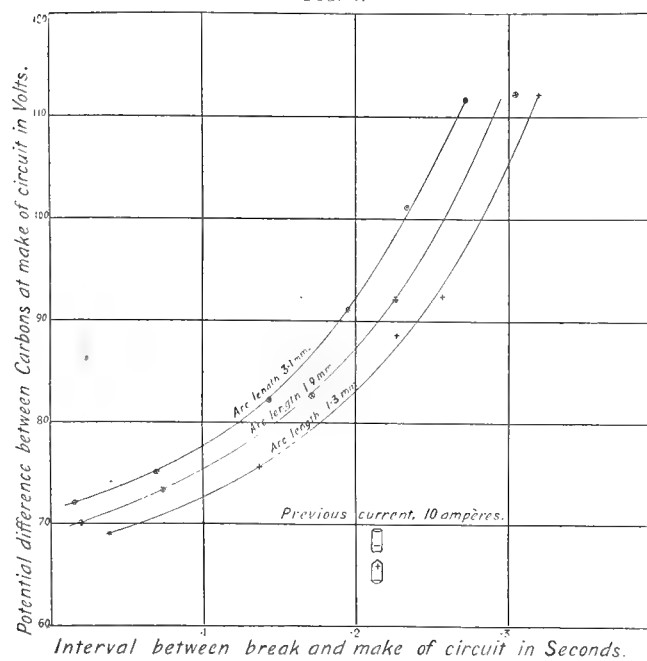


FIG. 3.



Relation for critical relighting; carbons in normal position, potentials reversed.

FIG. 6.

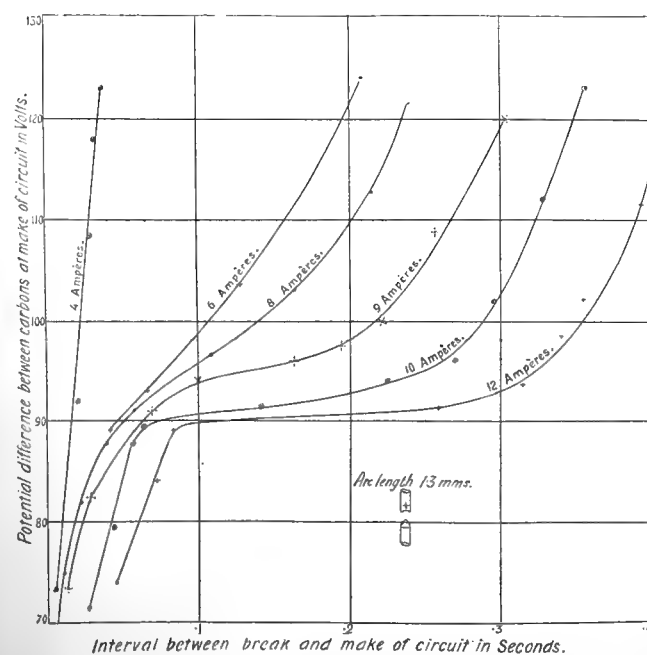
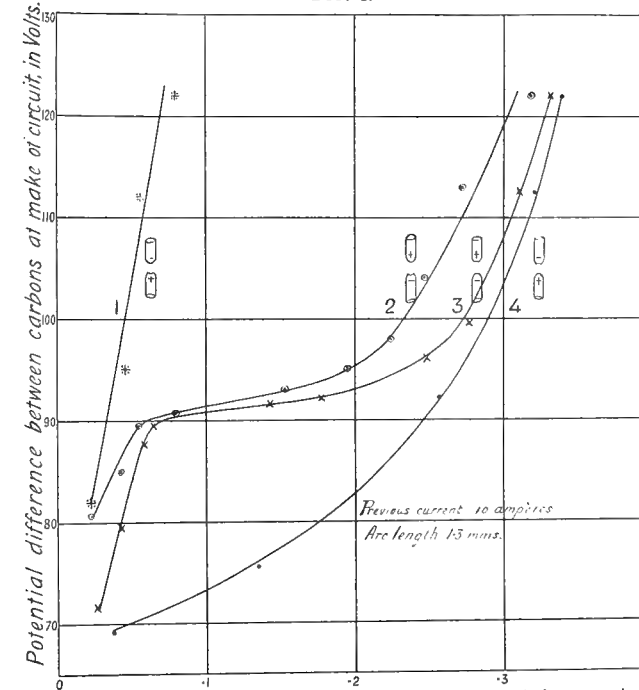
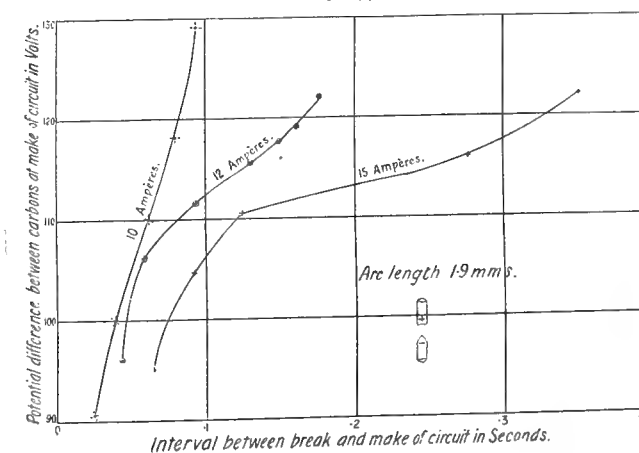


FIG. 4.



Relation for critical relighting as affected by changes in the relative positions of the carbons.

FIG. 7.



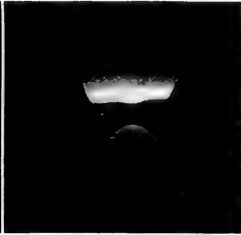




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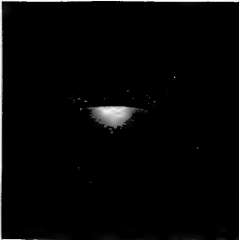
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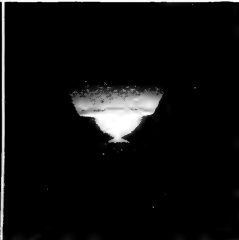
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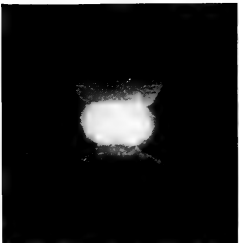
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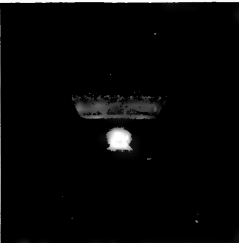
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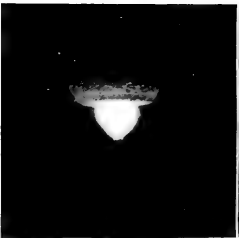
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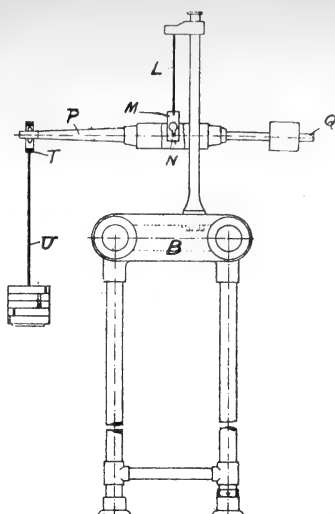
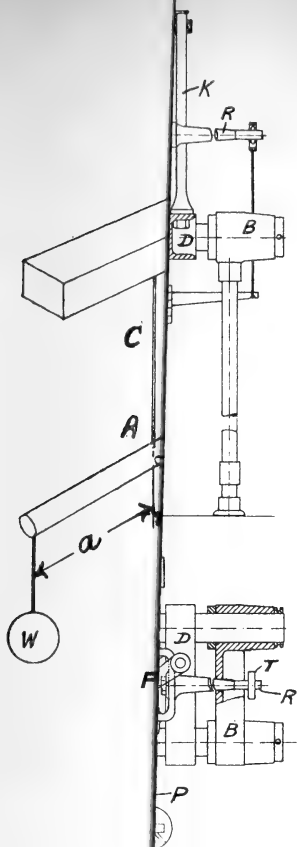


FIG 4

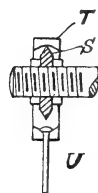


FIG 6

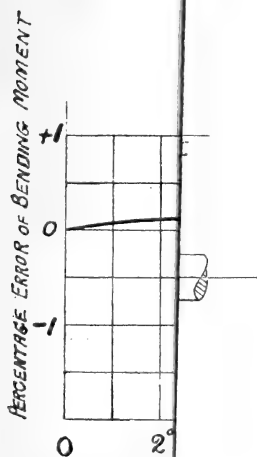
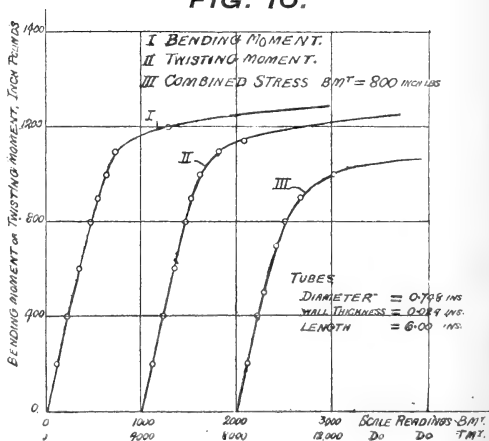


FIG. 10.



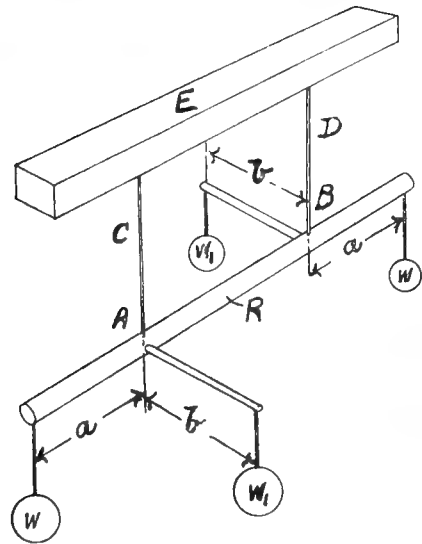


FIG. 1.

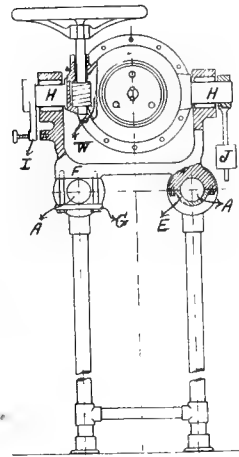


FIG 3

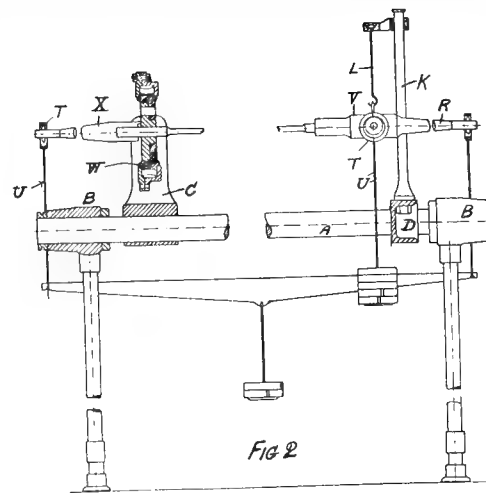


FIG 2

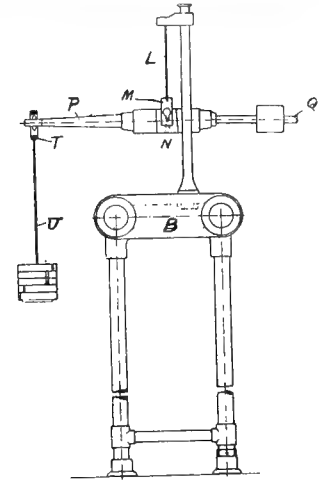


FIG 4

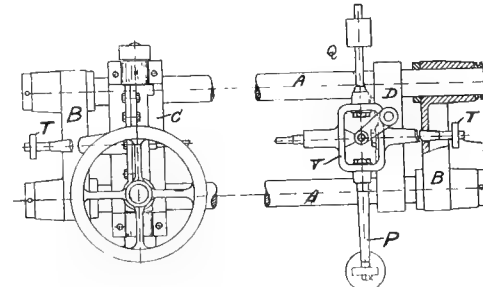


FIG 5



FIG 6

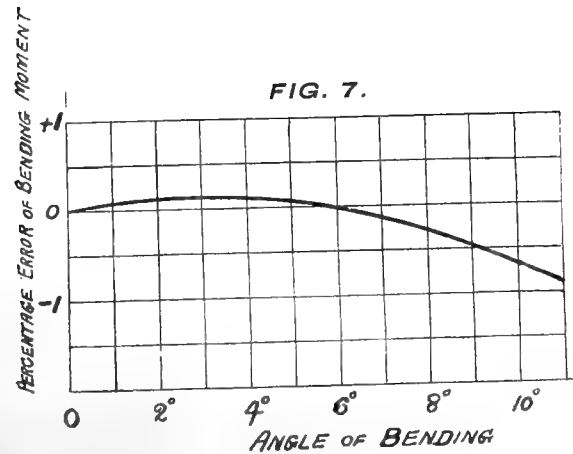


FIG. 7.

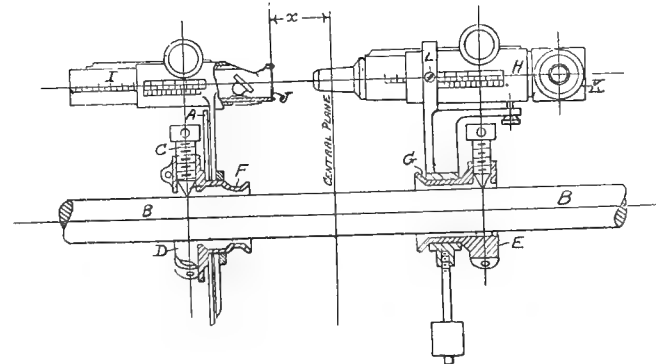


FIG. 9.

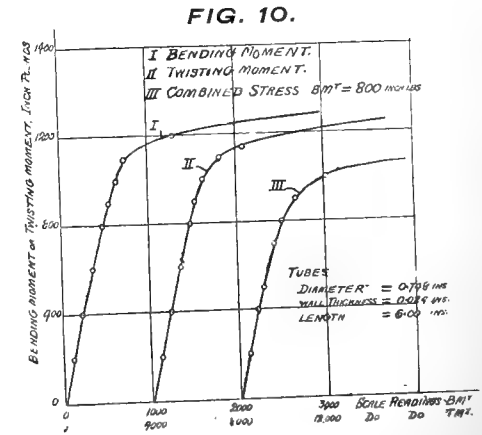


FIG. 10.

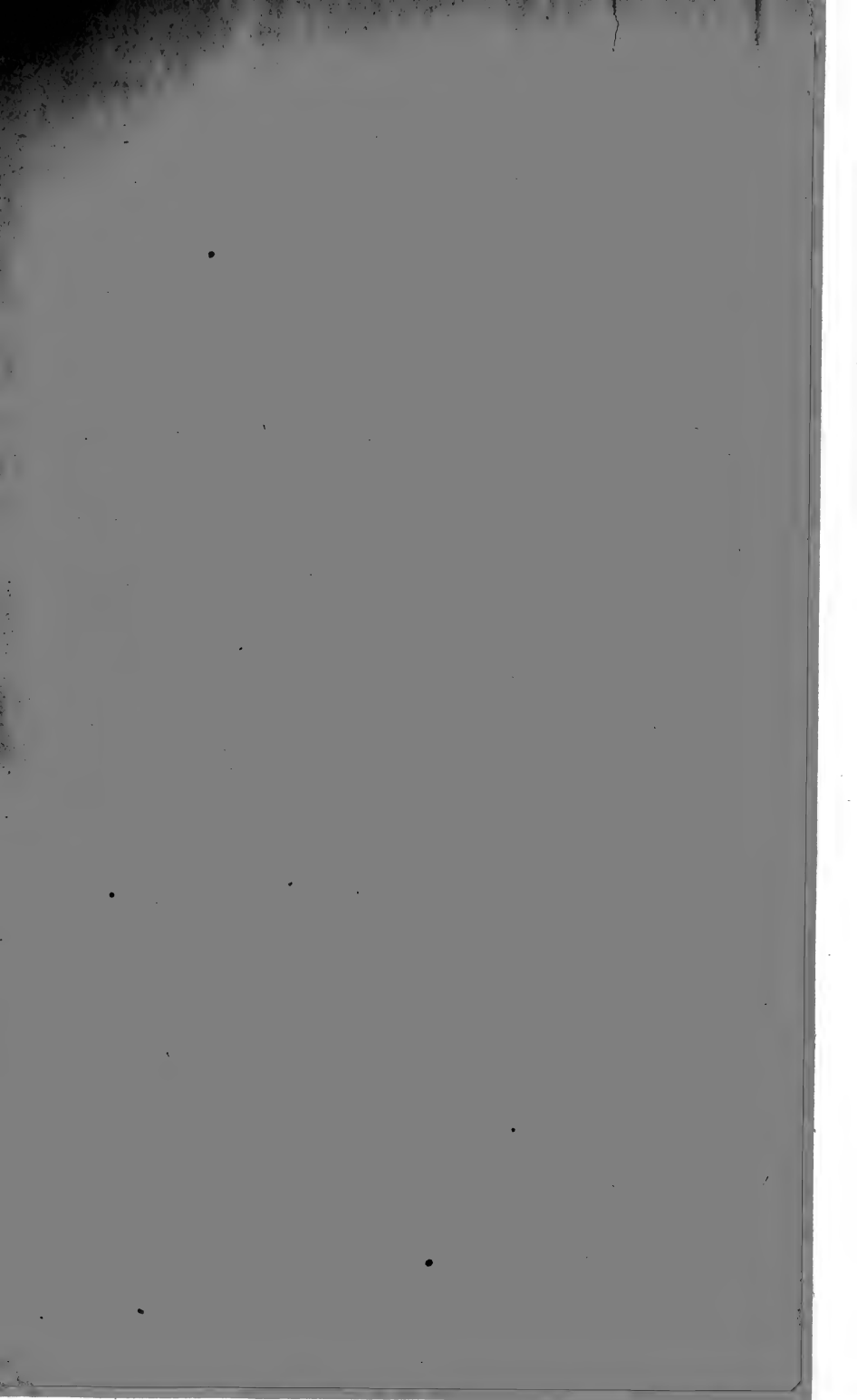
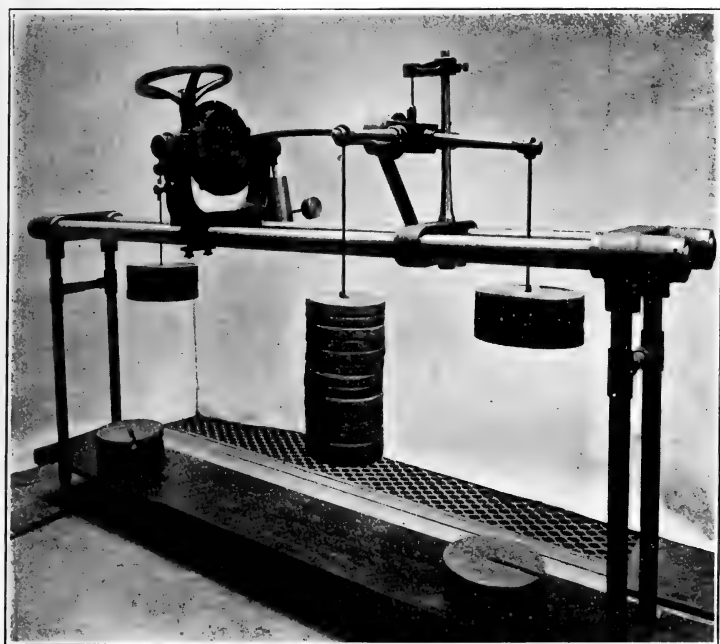


FIG. 8.



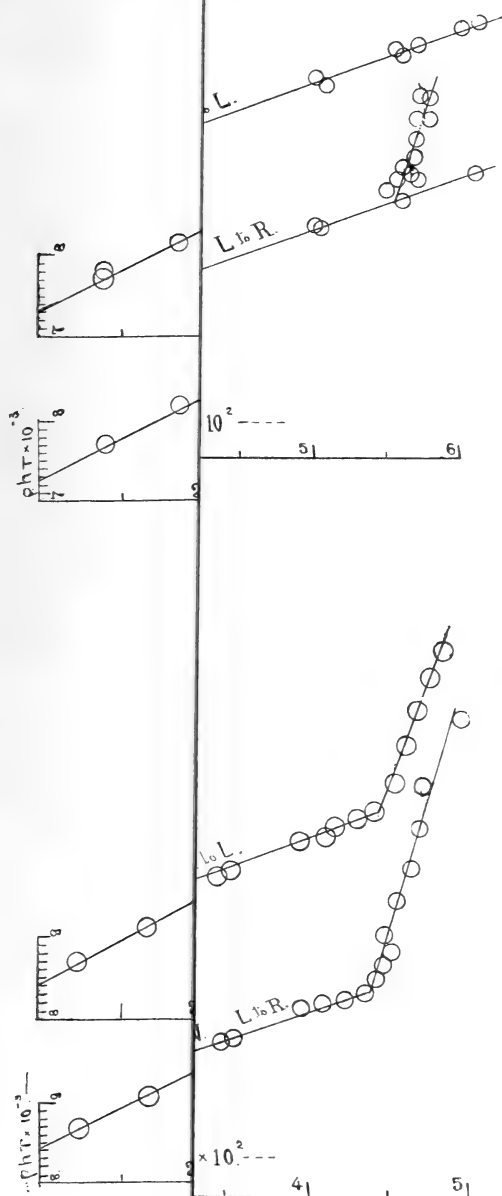


FIG. 2.

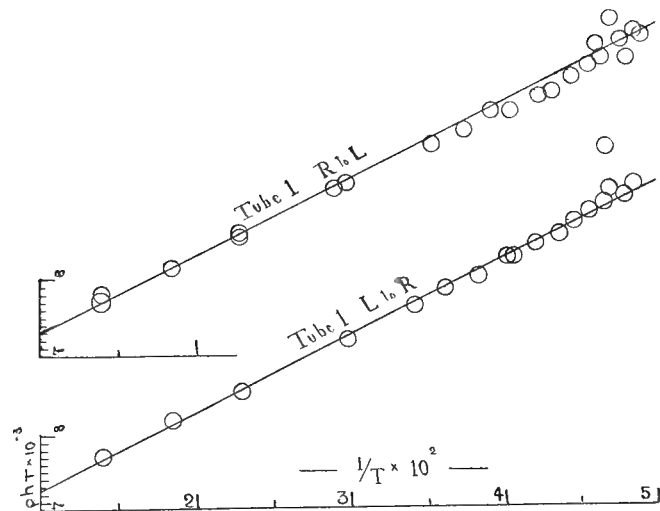


FIG. 3.

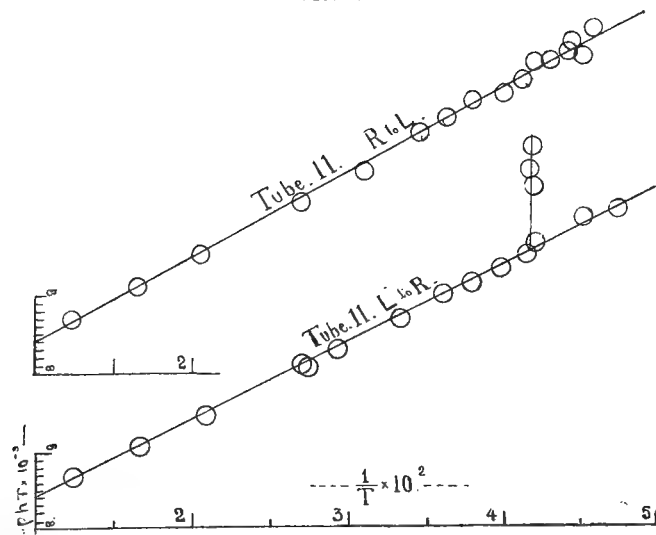


FIG. 4.

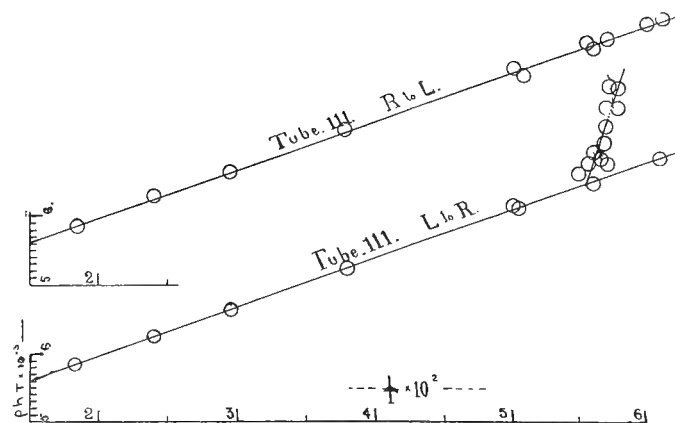


FIG. 5.

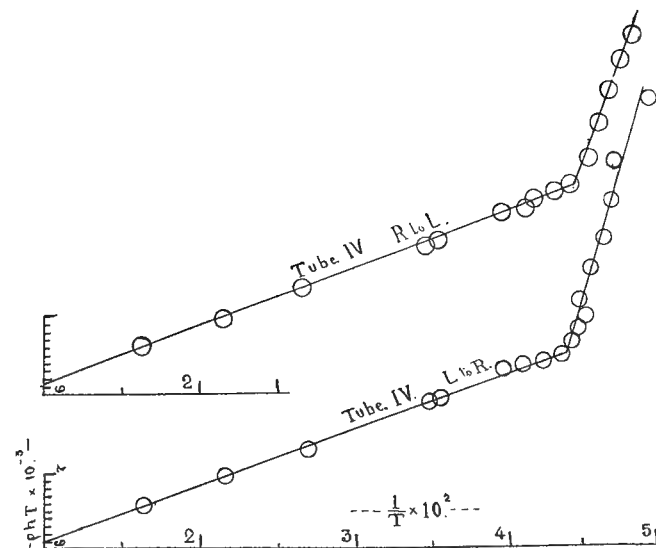


FIG. 2.

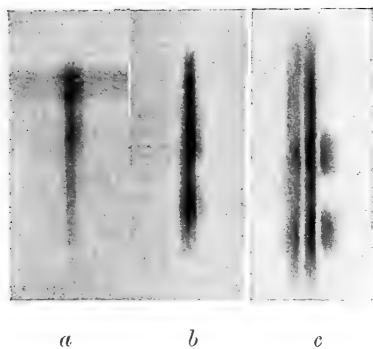
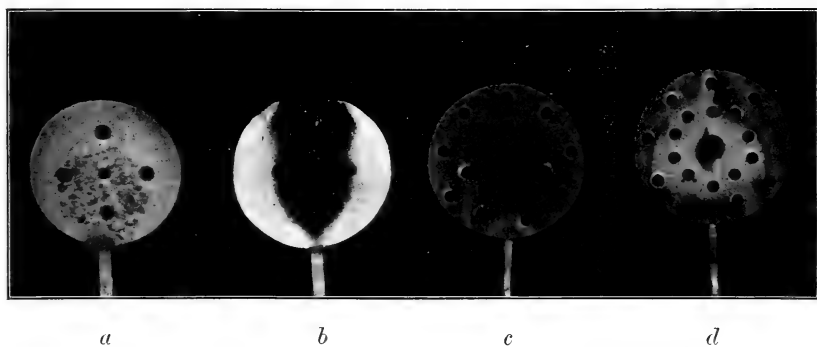


FIG. 4.



FIG. 3.



THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MAY 1909.

LX. *The Electric Origin of Molecular Attraction.*
By WILLIAM SUTHERLAND*.

IN connexion with this subject there is one chief outstanding difficulty to be discussed in this communication. Its nature will appear in the following few facts from the recent history of the subject. In a series of articles in this Magazine, beginning in August 1886 ([5] xxii. p. 81), I brought forward the evidence in support of the law of the inverse fourth power for molecular attraction, which may be written $3Am_1m_2/r^4$ in analogy with the Newtonian law of gravitation. It appeared subsequently that this analogy did not hold good in fact, that the attracting powers of molecules do not depend directly on their masses, and therefore that $3Am_1m_2$ should be replaced by $3a_1a_2$, in which a_1 and a_2 are parameters characteristic of the molecular masses m_1 and m_2 attracting one another. In connexion with this law of the inverse fourth power I was naturally led to give much consideration to the fact that the attraction between two small magnets similarly directed along a straight line varies inversely as the fourth power of the distance between their centres, that a similar result holds for atomic vortexes, and also for molecules containing separated electrons of opposite sign, for the significance of the electron had been emphasized in Helmholtz's Faraday lecture. But the great difficulty of connecting molecular attraction with these results for polar pairs was this, that polar pairs when oppositely

* Communicated by the Author.

directed repel one another with a force equal to their attraction when similarly directed. According to the prevailing conceptions of the kinetic theory of gases, it seemed at that time necessary to assume that any two molecules in a gas would have their polar axes, if they possessed such, oppositely directed for as long a time as similarly directed, that therefore they would repel one another as much as they could attract, and that polar forces would not account for cohesion such as had been proved to exist in gases. Ignoring this difficulty, Fessenden in America proposed to trace cohesion to the attraction of opposite electrical charges for one another (see *Phys. Review*, x. 1900, with references to *El. World*, 1891, and *Science*, 1892, 1893). He seems to have obtained most encouraging results, even at these early dates, from statical considerations alone, neglecting the kinetics of molecules. This ignoring of kinetics and of the repulsion between like charges of electricity was the probable cause of the failure of Fessenden's ideas to secure the attention of physicists. They remained almost unknown. Reinganum (*Phys. Zeitschr.* 1900; *Ann. der Phys.* [4] x. 1903) treated cohesion as of electrostatic origin, considering each molecule to have a pair of electrons at a certain distance apart. This of course gave him attractions and repulsions according to the law of the inverse fourth power. He eliminated repulsions by the consideration that by their mutual directive influence two neighbour molecules would cause one another's electric axes to be similarly directed along the line joining their centres. This is one step towards accounting for molecular attraction by means of electric bipoles, but it needs extension to show how the cohesive forces in gases are to be explained satisfactorily.

As regards gases, the main difficulty is still for the most part ignored. In solids another difficulty is possible, namely, that the distance between the centres of two neighbour molecules may be nearly equal to the distance between the two opposite electrons of a molecule. In this case the law of the inverse fourth power breaks down, and must be replaced by the more complex law for two large magnets almost in contact. This difficulty will be considered immediately. In "The Electric Origin of Molecular Attraction" (*Phil Mag.* [6] iv. 1902, p. 625) I sought to show that the cumulative effect of the evidence in my papers on molecular attraction between 1886 and 1902 pointed to electrical polarity as the cause of molecular attraction. To avoid the difficulty of the repulsion between molecules having their electric axes oppositely directed, especially in the case of gases, I introduced the consideration that a pair of attracting

molecules being brought nearer to one another by their attraction tend to increase its amount, while a pair repelling one another cause the repulsion to diminish by increase of the distance between them. There is thus a tendency for the attractive effect to preponderate. To show that the preponderance must be considerable it was argued that a pair of neighbours attracting one another would approach one another so comparatively closely as to introduce a comparatively large attraction. This principle taken with the tendency of the directive couples to promote similarity of direction in electric axes, was held in general to explain the preponderance of attraction, though no attempt was made to calculate definitely the amount of that preponderance. In a general way it was argued that, in reasoning about molecular attraction, we can replace the complex medley of Nature by a representative pair of neighbour molecules having similarly directed electric axes along the line joining their centres, and therefore attracting one another.

The attractive and repulsive effects of the more remote molecules in Nature are assumed to neutralize one another. In this way I sought to give a new interpretation and use to the idea of a finite range of molecular force. The actual range of the electric forces is infinite, but their effective range is the distance between our two schematic representative neighbours having their electric axes similarly directed in the same straight line, or nearly so. The object of the present communication is to establish this conception more rigorously, and to formulate it more systematically in detail. In "The Pressure of Gases and the Equation of Virial" (Phil. Mag. [6] ix. 1905, p. 494) Rayleigh has investigated some of the general conditions under which the preponderance of attraction over repulsion can come to pass, and obtains strong confirmation of its reality. He considered briefly the important point as to the time during which any particular value of the virial prevails. Van der Waals, Jr., has recently attacked the problem of the law of attraction for electrical double points in the molecules of a gas (*Kon. Akad. van Wet. te Amsterdam*, 8 Sept. 1908, p. 132). He takes any two molecules at random and writes down the expression for their potential energy in terms of their distance apart and the directions of their axes. He then uses the law of Boltzmann to express the number of pairs of molecules which are nearly in this same relative position within a range expressed in the usual way by differentials, and then evaluates the average attraction between two molecules at distance r apart, finding it to be a sum of powers of r^{-1} , the lowest of which is the seventh. He asserts, therefore, that the theory

of electron pairs is excluded by the law of the inverse fourth power for molecular attraction. I think that this result about the inverse seventh power is erroneous because the writer uses in Boltzmann's law the potential energy of a pair of molecules instead of the mean potential energy of all the molecules, which is the main object of investigation. Moreover, in this case Boltzmann's law is made to give a relation between the coordinates of the two molecules, which coordinates cease to be independent variables. For these reasons I believe Boltzmann's law is made to yield a result which is not correct. Moreover, in the treatment of the problem no account is taken of the principle that attractive forces preponderate because by their own action they increase themselves, while repulsive forces diminish themselves. As a large mass of experimental evidence now furnishes cumulative support to the theory of the electric origin of molecular attraction, I shall not attempt to discuss the analysis of van der Waals, Jr., in detail, as it seems to me that the question raised by him is rather that of the correct use of the Boltzmann-Gibbs theorem than that of the origin of molecular attraction.

The ideally simplest case in which molecular force can be investigated is at the absolute zero of temperature, at which the kinetics of molecules disappear, leaving the statics for unencumbered study. In "A Kinetic Theory of Solids" (Phil. Mag. [5] xxxii.) it was shown that the rigidity of metals at absolute zero could be found with considerable accuracy by a safe extrapolation from experimental data. Then it was further shown in "The Electric Origin of Rigidity and Consequences" (Phil. Mag. [6] vii.) that at absolute zero the rigidity of a collection of electrically polarized molecules is equal to their electrostatic energy per unit volume. If we imagine the electric polarization to consist in each molecule's having an electron pair $b\#$ whose charges are e at distance $(m/\rho_0)^{1/3}$ apart, m being the mass of the molecule and ρ_0 the density of the metal, with K for its dielectric capacity and N for the rigidity at absolute zero, then this result is given by the formula

$$N = \frac{2\pi}{3K} \frac{(m/\rho_0)^{2/3}}{(m/\rho_0)^2}, \quad (1)$$

which was found to express the experimentally derived facts satisfactorily. This is simply Maxwell's expression $2\pi D^2/K$ for the energy in a dielectric associated with an electric displacement D , for, when we consider the electric displacement per unit area corresponding with e per molecule, we get $e/(m/\rho_0)^{2/3}$ to be used instead of D , the result being divided by 3 for statistical reasons.

But we must look more closely into the differences between

a natural collection of molecules and the dielectric in an electric field. The main one is that in the electric field there is a resultant direction of electric force, whereas in a collection of molecules there is none, the electric force inside a molecule being reversed between the opposite poles of two molecules. The very definite conception which reduces the polarity of a molecule to two point electrons at the ends of a diameter of the molecule's supposedly spherical surface, becomes almost a hindrance in the case of molecules in contact at absolute zero because of the tendency of two oppositely charged point electrons to come nearly into coincidence and nearly to destroy one another's effect. In the facts of Nature there is no indication of any tendency to such destruction of effect. We shall do better then to regard the molecule as polarized throughout its mass, just in the same way as the Earth is magnetically polarized, and can be replaced by an infinitely short magnet at its centre, if only the magnetic moment of this is equal to that of the Earth. Just as we should encounter difficulties if we treated the Earth as a sphere having magnetic poles at the ends of a diameter, we are confronted with similar ones if we treat the molecule as an electric bipole.

With bipoles in which the axis between the poles is a diameter or nearly a diameter, the law of the inverse fourth power would break down completely, as it holds only for bipoles in which the axis is small compared with the distance between the attracting bipoles. But if the molecule is taken to be a uniformly electrically polarized sphere, which may be called a uniformly electrized sphere, then its external effect is the same as that of an infinitely short bipole at its centre, the bipole having the same electric moment as the sphere. With uniformly electrized spheres having their axes similarly directed along the line joining their centres, the law of the inverse fourth power becomes exactly correct at all distances. I hope to show in a separate communication that the conception of a uniformly electrized molecule is helpful in studying the vibrations causing spectra.

As I have shown (Phil. Mag. [6] iv. p. 636) that for many molecules the electric moment $e(m/\rho_0)^{1/3}$ is actually nearly equal to the product of electron charge and molecular diameter, the uniform electrization of the molecule is intimately related with the electron charge e of electrolysis. When a number of uniformly electrized spheres are brought into contact at absolute zero, we get a mass which is uniformly electrized, but with the axis of electrization changing in direction so that the average electric force throughout the mass is nil. To get a schematic representation of such a state of affairs,

in "The Electric Origin of Rigidity and Consequences" I used the artifice of dividing the space into cubes each containing a molecule, and then imagining the electrization of the mass caused by alternate sheets of positive and negative electricity in the planes dividing the space into cubes, so that each of the three plane distributions had a third part of the potential energy of the whole mass. Another method of representation is to place negative electrons \flat and positive \sharp alternately at the corners of the cubes so that along any line forming edges \flat and \sharp recur alternately. If with \flat of amount e as origin we draw three rectangular axes forming the edges of the eight cubes which meet at \flat and place \sharp of amount e at distances s and $-s$ on each axis, we see that our arrangement provides for a large preponderance of attraction over repulsion, since the six nearest neighbours of \flat are \sharp . It provides a distribution of polarity similar to that in our mass of uniformly electrized molecules. I have sought to show in "Ionization in Solutions and Two New Types of Viscosity" (Phil. Mag. [6] xiv. 1907, p. 1) that the ions of solutions give such a distribution of polarity as that just described. An expression of the same form as (1) for the rigidity of such a distribution of electrons leads to correct results concerning the electric resistance of electrolytic solutions. Since the form (1) applies to metals at absolute zero and to uniformly spaced ions, it follows that a cubical arrangement of electrons is for some purposes a convenient and proper simplified representation of the electric polarity in a metal at absolute zero. But this leads to a scheme which is better for most purposes because liker to the natural.

Imagine a mass of molecules at absolute zero to consist of cubes represented in plan in the figure where the arrow at the centre of each square denotes the direction of the uniform electrization of the cube. Here the cubes along AD and parallel to it form prisms which are alternately electrized in opposite directions. The next row of cubes above AD would form a prism oppositely electrized

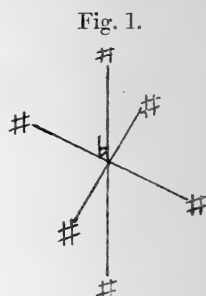
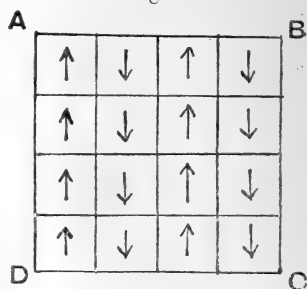


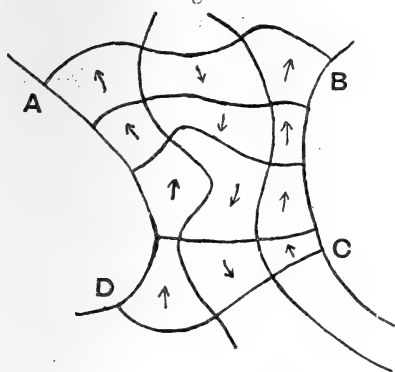
Fig. 2.



a prism oppositely electrized

to AD. If we call the direction of AD axial and directions at right angles lateral, we can say that a cube is electrized in the same direction as its two nearest axial neighbours and in a direction opposite to that of its four nearest lateral neighbours. Each cube attracts its six nearest neighbours. I believe that this constitutes the chief part of the solution of the main problem of cohesion. Consider two cubes in lateral relation and electrized in the same direction. They will repel one another till they reach the position of minimum potential energy when infinitely apart. But two cubes in lateral relation electrized in opposite directions will attract one another to the position of minimum potential energy when they are in contact. For a relative position between axial and lateral the force between two cubes vanishes. We conclude, then, that the principle of minimum potential energy in Nature acts so as to give such an arrangement as is set forth schematically in the last figure. In cubical crystals the scheme may be a still closer representation of fact. In the case of an amorphous mixture of substances whose molecules are of different size and shape, we can make a diagram of the state of affairs, such as that here given. Let AD and BC be two lines representing the prevailing

Fig. 3.



direction of electrization. They correspond with the straight lines AD and BC of fig. 2. They are like lines of electric force. Between them lie two similar curved lines. Then draw the system of lines that cut these orthogonally like equipotential surface. These two sets of lines divide up the space into regions corresponding with the squares of fig. 2, each region representing the domain of a molecule. The arrow indicates the direction of electrization of each molecule. Here we have the conditions for cohesion in an amorphous substance, the figure showing how in a body of finite size there may be no prevailing direction of electrization. Here we are reminded of the theoretically infinite range of the force of each molecule and its practical restriction almost to the nearest neighbours.

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But a most interesting problem arises when we extend these ideas from a solid at absolute zero to a gas at ordinary temperatures. The gas still shows cohesion, but how are we to imagine the axes of electrization of neighbour molecules in a gas to be related in a manner similar to that shown in figs. 2 and 3. Fessenden assumed that in a gas the molecules behave as if each molecule has an electric charge opposite to that of its immediate neighbours, and so he proposed to replace the equation of van der Waals $(p + a/v^2)(v - b) = RT$ by a form in which the cohesive term a/v^2 is replaced by $a/v^{4/3}$. I do not know of any experimental evidence that he ever submitted in support of this change. In "The Laws of Molecular Force" (Phil. Mag. [5] xxxv. 1893) I have shown that the equation of van der Waals applies to the element gases and methane, but not to compound gases in general, for which the cohesive virial tends to the form $l/(v + k)$ instead of l/v or the a/v of van der Waals. Now although it has just been proved that Fessenden's assumption is a convenient and proper simplification for a number of uniformly electrized molecules in contact, there is no warrant for it in the case of gases when we are studying the mutual potential energy of molecules. We shall see it apply to total potential energy. It does apply to the ions of a solution as the electrical evidence shows.

Since gases do not behave as completely ionized substances, the electrical evidence is also directly opposed to the literal truth of Fessenden's assumption for gases. As a fiction it has no schematic convenience in the study of characteristic equations, because it implies that the uniformly electrized spherical molecule of a gas converts the whole of its domain into a uniformly electrized sphere. We shall see that the whole potential energy, not merely the mutual, is that of a uniformly electrized spherical domain. Yet although there is no theoretical or experimental justification of the literal truth of Fessenden's assumption for gases, there has been discovered by J. E. Mills a remarkable relation which seems to verify it completely. This paradox contains matter of importance for the whole of molecular physics, and the rest of the present communication will be devoted to an attempt to elucidate it. This is the relation of Mills: Let L be the latent heat in calories for the evaporation of a liquid of density $\rho = d$ at temperature T to the state of saturated vapour of density $\rho = D$ at T , and let E be the thermal equivalent of the work done in changing the density from d to D against the saturation pressure, so that $L - E$ is the

internal latent heat, then at all temperatures up to the critical

$$\frac{L-E}{d^{1/3}-D^{1/3}} = \text{constant characteristic of liquid.} \quad (2)$$

This has been verified by a most exhaustive examination of the data for 31 liquids of diverse chemical types, compounds but not elements (Journal of Physical Chemistry, vi. 1902, p. 209, viii. 1904, pp. 383, 593, ix. 1905, p. 402, x. 1906, p. 1). Mills interprets his relation by treating cohesion as due to gravitation, taking $d^{1/3}-D^{1/3}$ as a measure of the change of the mutual gravitational potential energy of the molecules when they pass from the state of density d to that of density D . There are two well-known and perfectly definite reasons why gravitation cannot account for cohesion: first, the forces between neighbour molecules are enormously greater than their mutual gravitation; and, second, the latent heat required to evaporate molecules against their gravitation varies for unit mass with the size of the mass evaporated. This variation has never been discovered because the latent heat of evaporation against gravitation is so minute a fraction of the latent heat of evaporation against cohesive force, that its variations are beyond the reach of existing experimental refinements to detect. If Mills had traced his relation to the law of the inverse square as it operates in a uniform mixture of equal numbers of equal opposite charges of electricity, he would have been on the right track with Fessenden. In this case, the repulsions between the like charges introduce an element unlike gravitation. This mixture of electric charges has been shown above to be a convenient and proper schematic representation sometimes for a collection of uniformly electrized molecules in contact. Let us waive the stipulation about the molecules being in contact at absolute zero, and let us take (1) for the electrostatic energy of the molecules in unit volume of a liquid of density ρ or d . Then for a molecule of volume m/d the electrostatic energy will be $2\pi/3K(m/d)^{1/3}$. If with Fessenden we assume that exactly a similar condition of polarity prevails in the saturated vapour of density D , we can write $2\pi D^{1/3}/3Km^{1/3}$ for the electrostatic energy of the vapour, assuming for the moment that K does not change. The change is $2\pi(d^{1/3}-D^{1/3})/3Km^{1/3}$. If this is equivalent to the internal latent heat, then for a given substance

$$\frac{L-E}{d^{1/3}-D^{1/3}} = \text{constant at all temperatures.} \quad (3)$$

This is the result which appears to justify Fessenden's

treatment of a gas as a completely ionized fluid, and is one by which Mills might have given a dynamical interpretation of his discovery, instead of connecting it with gravitation. A notable fact in this relation is the disappearance of K , because in assuming K constant during the change of state from liquid to vapour, we have virtually made it disappear. Now in the laws of the parameter a in $3a_1a_2/r^4$ for molecular attraction in my various Phil. Mag. papers on this subject, it has never been necessary or advantageous to consider K as exercising any influence; it disappears from the scene. These two instances of the disappearance of K are most easily accounted for by the hypothesis that each molecule behaves as an electrized sphere. The external mutual effect of two such neighbour spheres depends only upon the electric moment assigned to each, just as the mutual effect of two neighbour magnetized spheres depends only upon the magnetic moment of each, and not upon the permeability of the matter of the spheres, if mutual induction is not operative. Thus in (1) K may be allowed to disappear by being merged in $e^2(m/\rho_0)^{2/3}$, the square of the electric moment of a molecule. Let us denote the electric moment by es and replace (1) by

$$N = \frac{2\pi}{3} \left(\frac{es}{m/\rho} \right)^2 \dots \dots \dots (4)$$

This discarding of K or this putting it equal to 1 expresses definitely the principle that cohesion is the attraction of immediate neighbours through the æther. In the theory of ions in solution K plays a prominent part, because neighbour ions are separated by the solvent of dielectric capacity K .

This equation (4) gives the electrostatic energy per unit volume: that per molecule is $2\pi e^2 s^2 / 3(m/\rho)$. If now s were equal to $(m/d)^{1/3}$ in the liquid and to $(m/D)^{1/3}$ in the vapour, we should obtain at once the formula of Mills. The most important point to understand then is this: How does the molecule of vapour behave as if its uniform electrization extends not only through its volume m/ρ or m/d , but through its domain m/D ? The answer is to be found by considering the electrostatic energy of a molecule in two parts, the intermolecular and the intramolecular, or the mutual and the self energy. For many purposes it is convenient with Maxwell to localize so much electric or magnetic energy in a given element of volume. Thus in our collection of molecules at absolute zero we assign electrostatic energy $-2\pi e^2 s^2 / 3(m/\rho)$ to each molecule, though we may drop the negative sign, as above, when it is not essential. But this

energy must be largely a mutual affair, and its localization in a molecule is only a convenience and not a necessity. When we expand the collection of molecules to density D in such a way that each always attracts its six nearest neighbours through the electrization, the mutual potential energy per molecule is proportional to $-e^2s^2/(m/D)$ and may be written $-2\pi e^2s^2/3(m/D)$. If the total potential energy is $-2\pi e^2/3(m/D)^{\frac{1}{3}}$, then the internal or self-energy is

$$-\frac{2\pi e^2}{3} \left\{ \left(\frac{D}{m} \right)^{\frac{1}{3}} - \frac{D}{m} s^2 \right\}. \quad . \quad . \quad . \quad (5)$$

Thus, then, for any density ρ we divide the potential energy per molecule up into two parts, the mutual $-2\pi e^2s^2/3(m/\rho)$, and the internal $-2\pi e^2\{(\rho/m)^{1/3} - s^2(\rho/m)\}/3$. At absolute zero this scheme makes the internal energy nil, the whole potential energy being mutual of amount $-2\pi e^2s^2/3(m/\rho_0)$. The total electrostatic energy $-2\pi e^2(\rho/m)^{1/3}/3$ does not appear in the equation of the virial or in the corresponding characteristic equation of gases, liquids, and solids, wherein the mutual relations of molecules are discussed. In these characteristic equations the mutual potential energy $-2\pi e^2s^2\rho/3m$ appears by itself or in association with other expressions for mutual potential energy during molecular collisions. The discovery of Mills may be stated as the following principle:—*The total potential energy of a number of like molecules is the same as if each caused its own domain to be uniformly electrized with an electric moment proportional to the linear dimension of the domain, the direction of electrization being such that in general any molecule attracts its six immediate neighbours.* Once again we are face to face with the difficulty in a gas of satisfying the condition that a molecule and its six immediate neighbours may have their electric axes related as in figs. 2 and 3. We seem to be led to the conclusion that in the molecules of a gas the electric axes change their direction with sufficient rapidity and in such a manner that as nearly as possible each molecule is attracting its six immediate neighbours. Thus we can apply fig. 3 to a gas, each of the four-sided regions corresponding with a six-faced region of space forming the domain of a gaseous molecule, the molecule being centrally situated. In this way then we trace cohesion in a gas to electric polarity, avoiding the difficulty of repulsions by the consideration that under a sort of mutual induction between neighbours and in accordance with the principle of minimum potential energy the polarities of neighbour molecules are related to one another according to the scheme of figs. 2 and 3. The same explanation applies

to the case of liquids and solids with only this difference, that the electric axis of a molecule may not change its position in the molecule so frequently through a large angle as in the case of a gas where the free motion causes numerous and rapid changes of neighbours.

In the further development of this conception interest centres round the laws of *es*, the electric moment of a molecule. These have been investigated in several of my Phil. Mag. papers on molecular attraction. In a broad way we can divide chemical compounds into two classes, those which are ionized in solution and those which are not. To the former belong the great majority of organic compounds and the compounds of the non-metals with one another. Typical of the latter are the salts of the metals. In both classes of compounds *es* for a molecule can be derived from values of *es* belonging to each atom. In those compounds which are not ionized in solution there is an approximation to the following simple rule, that in each atom *es* is proportional to the volume of the atom, and therefore that *es* for the molecule is proportional to the volume of the molecule. But as *es* divided by the volume of the molecule gives the intensity of electrization or the electric polarization per unit area, we have the simple approximate result that in this class of compounds the intensity of the electrization of molecules is an absolute constant. This result shows that for a detailed investigation of the structure of atoms, especially in connexion with the electron theory of matter, it will be necessary to carry out more extended studies of *es* or of $(M^{2l})^{1/2}$, as a proportional quantity is denoted in some of my papers.

Summary.

The electric polarity of molecule or atom may be represented as caused in two ways, either by the separation of two electrons *e* of opposite sign through a distance *s* giving an electron pair or bipole of moment *es*, or by considering the molecule or atom to be uniformly or variably electrized so as to have a total electric moment *es*. In the same way, the Earth as regards external magnetic effects may be treated either as a uniformly magnetized sphere or as a pair of opposite magnetic poles near the centre and having a magnetic moment equal to that of the Earth. At present we may regard the atom as electrized in both ways to account for the electron of electrolysis and for the electron structure of matter. Doubtless the two ways are but different aspects of one phenomenon, and for many purposes we can regard the resultant electric polarity of an atom as being due purely

to its electrization. As each atom is equivalent to an infinitely short bipole of equal moment at its centre, the law becomes rigorous that one atom acts on any other with a force inversely proportional to the fourth power of the distance between their centres. If the relative directions of the two electric axes of two atoms are random, repulsion will occur as frequently as attraction. This is the great difficulty in accounting for cohesion by means of electric polarity of atoms and molecules, especially in the case of gases in which the kinetic theory has made random distribution the ideal of generality. It was previously pointed out that as the attractive forces tend to increase themselves, while the repulsive tend by their own action to diminish themselves, there must be a tendency even in a random distribution of polarities for the attractive forces to preponderate. But it is now suggested that the electrization of neighbour molecules is not randomly directed, but is so directed that an atom is attracted by each of its six nearest neighbours. This satisfies the main condition for minimum electric potential energy in a set of moving molecules whose axes of electrization are free from constraint except that of being translated with the molecule. Beyond the range of the six nearest neighbours the attractions and repulsions of a molecule tend to become more nearly equal and opposite the greater the distance. Thus we have a range of force which is actually infinite but is effectively an attraction reaching only the six nearest neighbours of a molecule. The effective range of molecular attraction is the distance between a molecule and its immediate neighbours. This principle is verified by the fact that dielectric capacity does not appear in values of cohesion. Although in a small group of molecules, say a molecule and its six nearest neighbours, there is at any instant a prevailing direction of electrization, this varies with time, and at a given instant of time varies gradually from one group to the next, so that throughout a large number of molecules the directions of electrization are as many as if distributed at random. The electrostatic energy N per unit volume of a collection of such molecules of mass m and density ρ_0 in contact at absolute zero is $-2\pi e^2/3(m/\rho_0)^{4/3}$, and the energy per molecule is $-2\pi e^2/3(m/\rho_0)^{1/3}$, the dielectric capacity K not appearing. But Mills has discovered that the internal latent heat of evaporation of a liquid being $L - E$ at any temperature at which the densities of liquid and saturated vapour are $\rho = d$ and $\rho = D$, $(L - E)/(d^{1/3} - D^{1/3})$ is independent of temperature. This can be accounted for most readily by generalizing this last formula and making the

total electrostatic energy of a molecule at any density ρ equal to $-2\pi e^2/3(m/\rho)^{1/3}$. But according to the law of the inverse fourth power the mutual potential energy of the attracting molecules is $-2\pi e^2 s^2/3(m/\rho)$ per molecule. It appears then that $-2\pi e^2\{(m/\rho)^{2/3} - s^2\}/3(m/\rho)$ must be internal electrostatic energy per molecule. The total electrostatic energy of a number of molecules is of the same form as that of equal numbers of positive and negative electric charges uniformly mixed as in the ions of an electrolytic solution. Thus an hypothesis advanced years ago by Fessenden in America seems, when suitably interpreted, to be justified by the discovery of Mills made also in America, though both ignored fundamental physical difficulties in obtaining their results. To account for the facts it is proposed to look upon atoms as electrized, just as we speak of a magnet as being magnetized. But an electrized molecule causes the whole of its domain to be electrized as if for a given chemical substance each molecule possessed an invariable amount of positive electricity and an equal amount of negative, and as if these amounts were uniformly distributed through the domain of the molecule, whether its state is gaseous, liquid, or solid. This statement applies only to considerations respecting the total potential energy of molecules. In other words, as regards total energy each molecule behaves as if it had an electric moment proportional to the linear dimensions of its domain, whereas in the matter of mutual energy each molecule has an electric moment of amount *es* investigated in connexion with the laws of molecular attraction.

Melbourne, Feb. 1909.

LXI. *A Note on the Production of Steady Electric Oscillations in Closed Circuits and a Method of Testing Radiotelegraphic Receivers.* By J. A. FLEMING, M.A., D.Sc., F.R.S., and G. B. DYKE, B.Sc.*

IN testing radiotelegraphic detectors the difficulty is generally to obtain facilities for working in actual stations and at various distances. Thus, if an inventor desires to know whether an improvement which he has made in oscillation detectors is an advance on anything yet done, he must be able to test this receiver at a station in correspondence with others at various and at considerable distances, and even then quantitative measurements are difficult, or impossible, to obtain on account of the continually varying atmospheric conditions which, as is well known, introduce

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an element of difficulty in connexion with long-distance radiotelegraphy, or else on account of the limited or continually changing distance between the sending and receiving station.

The first-named author of this paper has therefore been seeking for some years past for a method of testing receivers within very moderate distances which can afford all the advantages to be obtained by working over long distances without any of the disadvantages.

This has now been achieved by the use of closed electric circuits or magnetic oscillators instead of electric oscillators. In a paper read before the Physical Society on October 25th, 1907 (see *Phil. Mag.* Dec. 1907 or *Proc. Phys. Soc. Lond.* vol. xxi. p. 47), "On Magnetic Oscillators as Radiators in Wireless Telegraphy," by J. A. Fleming, the author gave two formulæ: one for the radiation in watts from a linear oscillator of the Hertzian type of length l , and the other from a square closed circuit of area S , on the assumption that the oscillations were persistent oscillations having a root-mean square value a and a frequency N . These formulæ were as follows:—

$$W = 87 \times 10^{-20} l^2 a^2 N^2 \text{ (for the open or electric oscillator).}$$

$$W = 4 \times 10^{-38} S^2 a^2 N^4 \text{ (for the closed or magnetic oscillator).}$$

In the above formulæ W stands for the radiation in watts, l for the length of the linear oscillator, and S for the area of the closed or magnetic oscillator.

These formulæ show that in the case of the open or electric oscillator the power radiated varies as the square of the current-strength and as the square of the frequency, whereas in the case of the closed or magnetic oscillator it varies as the square of the current, but as the fourth power of the frequency. Hence, for any such frequencies as are used in radiotelegraphy and for such dimensions as are generally possible, an open or linear oscillator has much greater radiative power than a closed oscillator of about the same linear dimensions. Accordingly, if two closed-circuit oscillators are placed at a certain distance apart and oscillations set up in one of them, and the other one used as a receiving-circuit, the current in the receiving-circuit can be made extremely feeble when the oscillators are separated by not more than a few hundred yards, and we can avail ourselves of such a means to provide what is the equivalent to two radiotelegraphic stations with open or linear oscillators separated by many hundreds of miles. The convenience, therefore, of the closed or magnetic oscillator is very great because it is possible to set up in an ordinary building, such as a College or Technical Institution, two square circuits, both under cover and within a reasonable distance of each

other, which can be equivalent to two radiotelegraphic stations separated by several hundred miles.

In order to supply the necessary conditions for quantitative work, one of these closed circuits must be made the seat of perfectly constant oscillations, damped or undamped, by preference damped oscillations. The appliances which have now been in use in the radiotelegraphic research Laboratory at University College, London, for many years past for this purpose are as follows :—

The source of electromotive force may be an induction-coil or transformer. If a transformer, it must then be operated by a current from some supply circuit or from an alternator. If an induction-coil, it is preferably operated by large secondary cells, the primary current being interrupted by some good form of mercury coal-gas brake. We have used for this purpose with great advantage the Bécclère mercury coal-gas brake, in which a jet of mercury is raised by a rotary pump and squirted against the copper plate in an atmosphere of coal-gas. The mercury brakes that are generally sold for use with paraffin oil as the insulating medium for the mercury, are very messy in use and give great trouble by necessitating constant purification of the mercury and are by no means constant in action. Any mercury turbine brake, however, can be converted into a coal-gas brake by making the vessel gas-tight and employing, instead of paraffin oil, an atmosphere of coal-gas supplied from a small rubber bag under slight pressure. We have also used with even greater advantage for some time a mercury turbine brake by Schall, thus converted into a coal-gas brake which will work for hours at a time for many months without the slightest attention. The next element in the oscillatory circuit is the spark-gap. To obtain perfectly constant results, it is necessary to cause a jet of air to impinge upon the spark-gap to destroy the arcing which otherwise would take place, and, as shown in another paper, the result of this air-blast is to remove the causes of the irregularity in the discharge current (see the following paper on “The Effect of an Air-Blast upon the Spark-Discharge of a Condenser charged by an Induction-Coil or Transformer,” by J. A. Fleming and H. W. Richardson).

It is also desirable to enclose the spark-balls in a cast-iron chamber for silencing purposes.

The form of spark-gap therefore used by the authors consists of a cast-iron chamber S (see fig. 1) closed by a lid with a glazed peephole in it, the distance of the spark-balls being adjustable by a screw, and a glass jet J being arranged so as to cause a steady jet of air from a small Lennox blower under a pressure

of 16 to 20 inches of water, to impinge upon the spark-gap. An aperture is left in the cast-iron chamber, through which this air escapes. By using a small spark-gap not more than

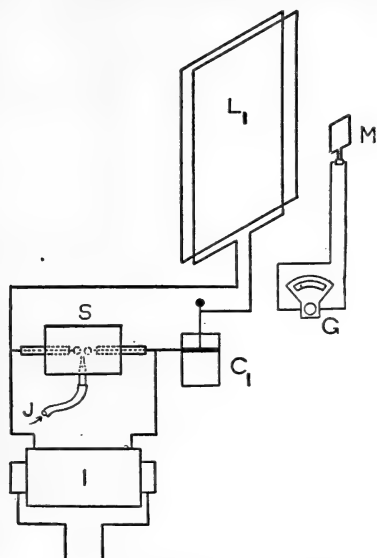


Fig. 1.—Transmitting Circuits.

3 mm. in length, and a suitable pressure of air, it is possible to maintain oscillations of great constancy in a circuit which includes the spark-gap S , a suitable condenser C_1 , and the closed circuit L_1 which constitutes the radiator. The radiator is preferably made by winding 8 or 10 turns of stranded insulated wire upon a square wooden frame, which may be anything from 2 feet to 8 or 10 feet in side. The condenser may be an ordinary leyden-jar, or preferably an oil-condenser consisting of metal plates placed in anhydrous paraffin oil. The high-frequency capacity of this condenser can then be measured accurately, and also the inductance of the radiative circuit and the frequency of the sparks can be ascertained by a spark-counter, as described previously by one of us. The mean-square value of the current in the oscillatory circuit can be determined by a hot-wire or thermoelectric ammeter inserted in it, or in a circuit M inductively coupled to it, and it will be found that if the above arrangements are adopted, the mean square value of the discharge current in the oscillatory circuit can be kept extremely constant for hours together. Signals can also be automatically sent by interrupting the primary circuit of the transformer or induction-coil by a key operated by a punched tape. In

this manner, a succession of Morse signals may be sent, or long and short signals of any kind for the purpose of testing the transmission of any particular words or letters. If the spark-chamber is made of thick cast-iron the apparatus will be nearly noiseless, and may therefore be set up in a Laboratory without disturbing other workers, which is not the case when an open oscillatory spark is employed. At a distance, say of 50 to 150 feet or more, another square circuit may be set up consisting of a similar square coil of insulated wire L_2 (see fig. 2) and a condenser C_2 , which is preferably a condenser of

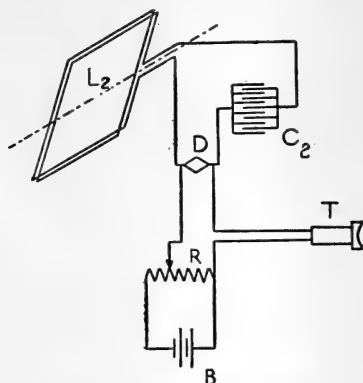


Fig. 2.—Receiving Circuits.

variable capacity for tuning purposes. A convenient form is one consisting of fixed semicircular plates and a number of movable semicircular plates fixed on a shaft, which can be rotated, so as to bring the second set of plates more or less in between the first set, the vessel being filled with a highly insulating oil. Two of such closed oscillatory circuits can be set up at a distance, say of 50, 100, or 200 feet within a large building, and even the interposition of brick walls makes no difference, provided they do not contain metal girders.

To test, then, a radiotelegraphic detector of any kind, it is necessary to be certain that the detector *per se*, when unconnected to the oscillatory receiving-circuit, is not directly affected by the spark at the distance at which the sending and receiving circuits are set up; but this can easily be done, and then any particular type of oscillation-detector, D (see fig. 2), whether of the current actuated type or the potential actuated type, can be tested as to sensibility by inserting it, either in series with the condenser of the receiving-circuit, or in parallel with the condenser of the receiving circuit. The detector D is associated with a telephone T and a battery B shunted by resistance R as usual. The use of the closed circuits has this great advantage, that being directive radiators and absorbers, it is possible, by a displacement of the planes of the magnetic

oscillators with reference to one another, to obtain a quantitative measure of the sensibility of any given oscillation-detector. Thus, for instance, it is generally possible, but not always, to find a position for the closed circuit of the receiver, such that at a certain distance no effect can be detected in that receiving circuit by any oscillation-detector, however sensitive. We may then call this the zero position. If the receiving circuit is moved out of the zero position by turning it through a certain angle round any axis, it will begin to be affected by the distant transmitting circuit, and a quantitative measure of any oscillation-detector can be obtained by noting the angle through which the receiving circuit must be turned, so that good audible signals may just be obtained. Whether the action of the sending on the receiving circuit is due to true electromagnetic radiation or to ordinary electromagnetic induction seems immaterial. The result in either case is that the receiving circuit is the seat of feeble electrical oscillations and the oscillation detector has to detect these if it can.

Another method is to maintain the receiving-circuit in its position of maximum effect, but to upset the tuning of the receiving-circuit by varying the capacity of the condenser or the inductance of the circuit. This varies the mean-square value of the received current and from the resonance-curve enables us to get the measure of the current or potential-difference which the particular oscillation-detector under test will just not detect. The authors have been employing arrangements of this kind very successfully for a long time past in investigations connected with improvements in the Fleming oscillation-valve.

It has been found convenient to denote the relative telegraphic value of detectors by stating the angle in degrees through which the receiving-coil has to be rotated from the zero position that good audible signals can be obtained on the telephone. If a note is made of the value in amperes or milliamperes of the current in the closed transmitting-circuit, this can always be recovered, and if the spark-length and spark-frequency are the same, we can always be sure that the sending circuit is in a constant and similar condition when comparative tests are made.

The closed receiving-coil is conveniently made by winding silk-covered copper wire, No. 16 S.W.G., on a square mahogany frame, which can be revolved on pivots carried on a baseboard which can itself be set at any required angle (see fig. 3, p. 676). A divided circle and pointer attached to the frame serve to show the angle through which the frame is rotated. In general appearance it resembles an instrument used in Physical Laboratories under the name of an earth-inductor, for obtaining small induced currents by means of the rotation of a coil in the terrestrial magnetic field.

If a closed transmitting-circuit is set up at a distance, then it is generally possible to find a position for the

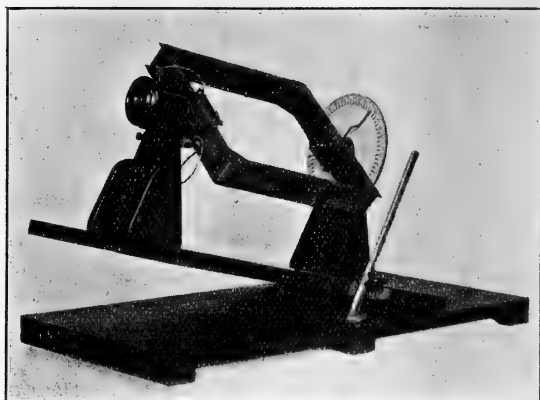


Fig. 3.—Radiotelegraphic Detector Tester.

receiving-coil, such that it will not detect any signals when coupled with a condenser and tuned and associated with a highly sensitive receiver. On turning the coil through a certain angle the signals will be heard. If a very sensitive oscillation-detector is employed, then there may be no position of absolutely null reception, but there will be a position of minimum reception. Thus, for instance, in a certain case, with a coil used at the Pender Electrical Laboratory, some Fleming oscillation-valves of a new type were found to be so sensitive to oscillations, that no position in which the receiving-circuit could be placed was so completely a position of zero mutual induction that these valves, when used with a telephone, gave no signals from a tuned transmitter. Such valves were called zero valves. Others, on the contrary, could not detect signals until the coil had been turned through 5° , 10° , or 20° from the zero or minimum position. A magnetic detector inserted in series with the coil could not detect the signals from the transmitter until the coil was turned through 15° . An electrolytic detector of a particular make required a rotation of 40° , and a carborundum detector required 45° rotation of the coil to give audible signals on the telephone. These measurements are not given as absolute and final measurements of the relative sensibility of all magnetic, electrolytic, or crystal detectors, but merely as examples of the ease with which the sensibility of these special samples of receivers could be tested for order of sensibility. The instrument has proved of great use in a research being conducted now in connexion with improvements in ionized gas radiotelegraphic detectors.

LXII. *The Effect of an Air-Blast upon the Spark Discharge of a Condenser charged by an Induction Coil or Transformer.*

*By J. A. FLEMING, M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London, and H. W. RICHARDSON, B.Sc.**

WHEN the oscillatory discharge of a condenser is caused to take place across a spark-gap in the usual manner by charging the condenser by means of an induction-coil or transformer, the intermittent spark which takes place between the spark-balls is a complex effect. It consists partly of the true oscillatory discharge of the condenser and partly of an electric arc, unidirectional or alternating, which is superimposed on the true condenser oscillatory spark. If a hot-wire ammeter or other means of measuring the effective or mean-square value of the discharge current is inserted in the condenser circuit, this current will generally be found to be irregular, and if a radiative circuit is coupled to the condenser circuit as in radiotelegraphy, the radiation from it will be found to consist of trains of waves whose initial amplitude is also variable. This irregularity is a source of difficulty in making radiotelegraphic or laboratory measurements of current, decrement, wave-length, &c., when originated by condenser discharges. The reason is that the moment the condenser begins to discharge, and the first so-called pilot spark takes place between the balls, the resistance of the spark-gap falls, and an arc discharge from the induction-coil or transformer commences across the gap. Until this arc is extinguished the condenser cannot again become charged to any high voltage, and the voltage to which it is charged will depend upon the state in which the ball surfaces are left as regards temperature and smoothness, since these are factors in determining the spark potential, and also on the condition of the air-space as regards conductivity. Accordingly, to produce a uniform oscillatory discharge this true arc-discharge must be either prevented or arrested at once, and the spark between the balls should arise wholly from energy which comes out of the condenser, and not from energy coming directly from the transformer or coil. When moderate power is being employed this arc-discharge can be best annulled by a blast of air thrown on the spark-gap. This has the effect of blowing away the arc, but does not stop the condenser oscillatory discharge. For a long time past the utility of this air-blast in connexion

* Communicated by the Physical Society: read March 26, 1909.

with practical radiotelegraphy has been noted by one of us (J. A. Fleming), but its use in purely scientific measurements is an advantage, as shown in the following paper.

If a large induction-coil has its secondary terminals connected to a pair of spark-balls, or to the outside pair of a series of balls, so arranged that each gap is not more than a millimetre in width, and if a condenser having a capacity say of 0.005 mfd. is connected across the outer balls, then when the coil is in action intermittent sparks pass at the gap. If these sparks are examined in a revolving mirror or photographed on a moving plate, they present themselves as bright images set at fairly equal distances. If a jet of air under a pressure of 16 or 18 inches of water is thrown on the gap by a glass nozzle, the images are then seen to have a ragged tail or aureole which is blown away from the spark, and this tail is generally reddish in colour and easily distinguished from the bright condenser spark. The tail is the image of the arc-discharge superimposed on the oscillatory spark. To determine the effect of this air-blast the following experiment was tried. A 10-inch induction-coil had its secondary circuit connected to brass spark-balls 3 cms. in diameter set with a gap of 1 mm., and the balls were also connected to a rectangular circuit of round copper wire, the diameter of the wire being 0.162 cm. and the sides of the rectangle respectively 14.21 cms. and 34.17 cms. The ordinary or steady resistance of this rectangle is 0.046 ohm and its high frequency resistance to currents of a frequency of the order of 1.25×10^6 is 0.31 ohm. The inductance of this circuit (calculated) is 5012 cms. In series with this circuit was placed a condenser consisting of metal plates immersed in paraffin oil, the capacity of which was 0.002645 of a microfarad.

In contiguity to the long side of the above rectangle was placed the bar of a Fleming Cymometer, two such instruments being used in the experiments, called respectively No. 2 and No. 3. The cymometer circuit can have two short fine wires of constantan each about 5 cms. long, inserted in it at pleasure, against one of which a bismuth-iron thermo-junction is attached. By passing measured small continuous currents through this fine wire and connecting the ends of the thermo-junction to a low resistance single-pivot Paul galvanometer, the arrangement can be calibrated as a hot-wire ammeter to indicate directly the mean-square value of the oscillations, which when passed through the fine wire cause a certain deflexion of the galvanometer attached to the ends of the thermo-junction. The other fine wire can be

inserted as an added resistance in the circuit of the cymometer. The cymometer consists of a circuit including a spiral wire having inductance L and a condenser of capacity C , which can be continuously varied in the same proportion, so that the oscillation constant (\sqrt{CL}) of the circuit can be given any value between certain limits. The cymometer was employed to take a resonance curve of the spark circuit by the usual Bjerknes-Drude method, (i.) when the spark-balls were not subjected to the air-blast, and (ii.) when the jet of air was thrown between them.

An extremely steady jet of air for this purpose can be obtained by the use of a small Lennox blower, which is a fan driven by an electric motor taking a current of about 1 ampere from any electric-lamp supply circuit. The resonance curve is obtained by plotting the values of the root mean-square current (a) expressed as a fraction of the maximum current (A) induced in the cymometer circuit by the damped oscillations in the spark-ball circuit corresponding to various values of the natural frequency n or oscillation constant for any setting of the variable inductance and capacity of the cymometer circuit. Then, if A is the maximum value of this secondary current in the cymometer corresponding to a certain natural frequency N or oscillation constant when resonance between the circuits exists, and if δ_1 and δ_2 are the logarithmic decrements per semiperiod of the oscillations in the spark and cymometer circuits respectively, we have by Bjerknes' formula

$$\delta_1 + \delta_2 = D = \pi \left(1 - \frac{n}{N}\right) \sqrt{\frac{a^2}{A^2 - a^2}},$$

provided that such values of n are selected that n does not differ from N by more than say 5 per cent. If then a known resistance R is added to the cymometer circuit, thus increasing its decrement by a known amount δ_2' , and a fresh set of observations taken, another resonance curve can be plotted which should lie wholly outside of the first when their maximum ordinates of both curves are taken as unity. From the formula given by Bjerknes we know then that

$$A^2 (\delta_1 + \delta_2) \delta_2 = A_1^2 (\delta_1 + \delta_2 + \delta_2') (\delta_2 + \delta_2'),$$

where A and A_1 are the mean-square values of the maximum or resonance currents in the two cases.

The actual observed quantity which gives us a is the deflexion of the needle of the galvanometer in connexion with the thermocouple pressed against the resistance wire inserted in the cymometer circuit. If there is no air-blast

on the balls this deflexion hardly ever remains steady, the needle wanders to and fro over the scale, and the observer can at best but take a mean reading. The result is that the points plotted for the resonance curve do not lie well on a smooth curve, and it is particularly difficult to plot the important part of the curve near the maximum value. This difficulty has been experienced by other observers. Thus, Mr. R. A. Houstoun has recently made a series of measurements of spark resistance and decrement with spark-balls of various metals (see Proc. Roy. Soc. Edin. 1908, vol. xxviii. p. 369, or 'The Electrician,' vol. 62, p. 636); and he remarks that even with great care taken, measurements of logarithmic decrement of oscillation made with the same spark-gap and circuit do not agree well together, the resistance of the spark-gap appears to vary irregularly. Again, Messrs. J. E. Taylor and W. Duddell in some radiotelegraphic measurements made in 1905 (see Journ. Inst. Elect. Eng. vol. 35, p. 321, 1905) also complained of the difficulty of obtaining good measurements with a spark-transmitter. These difficulties have their origin in the actions taking place in the spark-gap. If, however, a steady blast of air at a suitable pressure is thrown between the spark-balls, this irregularity is greatly reduced provided the spark-gap is not long. The deflexion of the galvanometer becomes greater and much more steady, and accurate observation of the values of the current a corresponding to known values of n becomes much facilitated.

With the above described arrangements a resonance curve can be easily taken as follows:—The jet of air from the Lennox blower conveyed by a rubber pipe ending on a glass nozzle is allowed to play between the spark-balls, and these are connected together by a condenser of known capacity in series with an inductance which has either been measured or predetermined. This inductance is preferably formed of round copper wire and may be rectangular in form, and its high frequency resistance and inductance can then be calculated by known formulæ.

The cymometer is then placed alongside this rectangle so as to have induced oscillations created in it, and by means of the hot-wire ammeter inserted in its circuit the mean-square value a of the cymometer current is taken for various settings of the cymometer circuit, which give it various assigned natural frequencies n . If A and N are the maximum values obtained we then calculate the value of the sum of the decrements of the spark and cymometer circuits from the Bjerknæs-Drude formula given above*. Since the resonance

* See 'The Principles of Electric Wave Telegraphy,' by J. A. Fleming. Longmans & Co., pp. 221–223.

curve is not symmetrical with respect to its maximum ordinate, the best mode of procedure is as follows:—Having taken a series of values of the cymometer current a without the extra damping resistance, and those of the current a_1 when the resistance is inserted in the cymometer circuit, and observed the maximum values A and A_1 of the quantities, for various values of n we plot two curves, each to abscissæ n/N , but one having ordinates a/A and the other ordinates a_1/A_1 . The latter curve should lie outside the former, but should have the same value for its maximum ordinate, viz., unity. We then draw a number of horizontal lines across the hump of the resonance curves, as shown in fig. 1, with

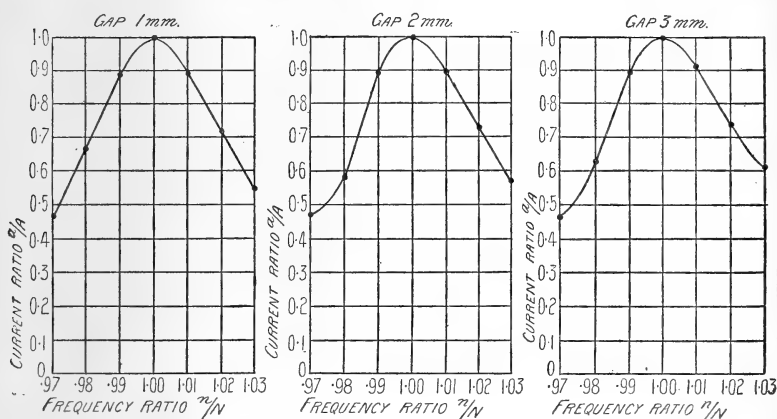


Fig. 1.—Resonance Curves taken with air-blast on spark-gap.

such ordinates that the greatest length intercepted by the curves is not more than 0.06 on the same scale that N is taken as unity. Let the half length of any such horizontal intercept be called x and the corresponding ordinate a/A be denoted by y , we then calculate the value of

$$D = \pi x \sqrt{\frac{y^2}{1-y^2}},$$

and this gives us the value of the sum of the decrements of the spark and cymometer circuits. It is well to calculate D from several, say four or five, measurements of x and y for intercepts of different lengths, and then take the mean value for D . In the same way from the outer resonance curve we can calculate the value of D_1 . The difference between D and D_1 is the increment in the decrement due to the added resistance wire, and if R is its high frequency resistance and

682 Prof. Fleming and Mr. Richardson on the *Effect of an*
if the frequency of resonance is N and the corresponding
inductance of the cymometer is L , we should have

$$D_1 - D = \frac{R}{4NL} = \delta_2'$$

as a check on the observations.

The following details of one set of experiments will show the advantage gained by the use of the blower. The experiments were made with cymometer No. 2. The primary circuit had an inductance of 5012 cms. and a capacity of 0.002645 mfd., and was charged by a 10 in. induction-coil worked with a coal-gas mercury turbine break. The spark-balls were brass balls 3 cms. in diameter set with spark-gaps of length of 1, 2, and 3 mm. respectively in various experiments. The resistance added in the cymometer circuit was a very fine constantan wire of which the resistance was 7.1 ohms, and a similar wire against which a thermojunction pressed has a resistance of 5.0 ohms.

When the cymometer circuit was adjusted to be in resonance with the primary circuit the resonance frequency N was found to be 1.25×10^6 and the corresponding inductance of the cymometer was 5500 cms. Hence for the added resistance, we have $R = 7.1 \times 10^9$ c.g.s., $L = 5500$ cms., and $n = 1.25 \times 10^6$. Therefore

$$\frac{R}{4nL} = \frac{7.1 \times 10^9}{4 \times 1.25 \times 10^6 \times 5500} = 0.0258,$$

and for the thermojunction

$$\frac{R}{4nL} = \frac{5 \times 10^9}{4 \times 1.25 \times 10^6 \times 5500} = 0.0181.$$

The thermojunction ammeter having been calibrated with direct currents so that from the deflexions of the Paul single pivot galvanometer the R.M.S. value of the oscillations passing through the wire which gave any observed deflexion could be obtained, a series of observations was taken by varying the setting of the cymometer slowly and continuously when it was placed in loose inductive coupling with the primary circuit so as to alter its oscillation constant $O = \sqrt{CL}$, or the product of its inductance L and capacity C , from which the natural frequency n of the circuit is at once obtained by the formula $n = 1/2\pi \sqrt{CL}$. At the same time the R.M.S. value of the current a in the cymometer was read off by the thermo-ammeter and the maximum value A also taken and the resonance frequency N . These observations then give the

means of calculating D or the sum of the decrements of the primary circuit δ_1 and that of the cymometer δ_2 , this last including that due to the resistance of the hot-wire ammeter. A resonance curve was then drawn and a series of horizontal intercepts measured off near the peak of the curve, and the ordinates of these intercepts also read off on the curve (see fig. 1). The half of the length of the intercept is taken as the mean-value

of $1 - \frac{n}{N}$, and the corresponding ordinate expressed as a fraction of the maximum ordinate of the curve is taken as a/A , and from this last the function $\pi \sqrt{\frac{a^2}{A^2 - a^2}}$ is calculated and the product of these two quantities, viz.,

$$\left(1 - \frac{n}{N}\right) \pi \sqrt{\frac{a^2}{A^2 - a^2}}$$

gives us $D = \delta_1 + \delta_2$, or the sum of the decrements of the primary and secondary circuits.

The resonance curves are plotted to such a scale that the maximum ordinate representing A is unity, and the abscissa of that ordinate representing N is also taken as unity.

The complete set of observations for the 1 mm. spark-gap when subjected to the air-blast was as follows :—

TABLE I.
1 mm. spark-gap with air-blast.
Resonance frequency = $N = 1.25 \times 10^6$.

Oscillation constant of Cymometer $O = \sqrt{CL}$.	Corresponding Frequency of Cymometer circuit = n .	Cymometer current in amps. = a .	Ratio n/N .	Corresponding actual frequency n .	Corresponding current from Resonance Curve a .	Ratio a/A .
4.15	1.205×10^6	0.046	0.96	1.2×10^6	0.041 amps.	.343
4.1	1.22 "	0.074	0.97	1.211 "	0.056 "	.469
4.05	1.235 "	0.10	0.98	1.224 "	0.0803 "	.672
4.025	1.242 "	0.113	0.99	1.238 "	0.106 "	.886
4.000	1.250 "	0.1195	1.00	1.250 "	0.1195 "	1.000
3.975	1.257 "	0.1146	1.01	1.262 "	0.107 "	.895
3.95	1.264 "	0.1025	1.02	1.275 "	0.0865 "	.724
3.925	1.272 "	0.086	1.03	1.281 "	0.0656 "	.549
3.9	1.282 "	0.077				
3.85	1.295 "	0.0558				

The maximum cymometer current = $A = 0.1195$ ampere without the added resistance in circuit, and the maximum current with the added resistance of 7.1 ohms = $A_1 = 0.0635$.

From the above observations and curve-measurements we calculate the following Table II. :—

TABLE II.
1 mm. spark-gap with air-blast on gap.

Cymometer Current as per cent. of max. current = $100 a/A$.	Value of $\pi \sqrt{\frac{a^2}{A^2 - a^2}}$.	Mean Value of $1 - \frac{n}{N}$.	Value of $D = \delta_1 + \delta_2$.
95	9.58	.0067	.0643
90	6.47	.0098	.0635
85	5.09	.0126	.0642
80	4.18	.0152	.0636
75	3.58	.0177	.0635
70	3.08	.0205	.0632
		Mean Value of $D = .0637$	

The values of D obtained from the various measurements of the resonance curves are seen to be in very fair agreement. We then calculate the value of δ_2 the cymometer decrement from the formula

$$A^2 D \delta_2 = A_1^2 (D + 0.0258)(\delta_2 + 0.0258),$$

where $A = .1195$, $A_1 = .0635$, $D = .0637$, and hence $\delta_2 = .017$ = decrement of cymometer including that due to the resistance of thermojunction wire.

$$\text{Hence } \delta_1 = D - \delta_2 = .0637 - .017 = .0467.$$

Accordingly the total high frequency resistance of the primary circuit = $R = \frac{4\pi N L_1 \delta_1}{10^9}$, where L_1 is the inductance = 5012 cms. and N is the natural frequency = 1.25×10^6 . Hence we have $R = 1.17$ ohms, and since the high frequency copper resistance of the rectangular circuit = 0.31 ohm, we find that the spark-gap resistance for this 1 mm. spark is 0.86 ohm.

Observations made with 2 mm. and 3 mm. spark-gaps with air-blast in each case gave results of which the final calculations and measurements are embodied in Tables III. and IV.

TABLE III.—2 mm. spark-gap with air-blast.

Cymometer Current as per cent. of max. current 100 a/A .	Value of $\pi \sqrt{\frac{a^2}{A^2 - a^2}}$.	Mean Value of $1 - \frac{n}{N}$ from Curve.	Calculated Value of $D = \delta_1 + \delta_2$.
95	9.58	.0069	.0662
90	6.47	.0010	.0647
85	5.09	.0127	.0647
80	4.18	.0144	.0609
75	3.58	.01695	.0605
70	3.08	.0192	.0595
Mean Value = .0627			

Maximum Cymometer Current = $A = .101$ ampere.

Maximum Cymometer Current

when resistance is added = $A_1 = 0.046$ ampere.

Calculated Value of $\delta_1 = .0443$.

„ „ $\delta_2 = .0184$.

„ „ of spark-gap resistance = 0.80 ohm.

TABLE IV.—3 mm. spark-gap with air-blast.

Cymometer Current as per cent. of max. current 100 a/A .	Value of $\pi \sqrt{\frac{a^2}{A^2 - a^2}}$.	Mean Value of $1 - n/N$ from Curve.	Calculated Value of $D = \delta_1 + \delta_2$.
95	9.58	.0063	.0604
90	6.47	.0095	.0615
85	5.09	.0117	.0595
80	4.18	.0138	.0579
75	3.58	.0161	.0577
70	3.08	.0187	.0577
Mean Value = .0591			

Maximum Cymometer Current = $A = 0.1195$ ampere.

Maximum Cymometer Current

with added resistance = $A_1 = 0.0652$ ampere.

Calculated Value of $\delta_1 = .0397$.

„ „ $\delta_2 = .0194$.

„ „ of spark-gap resistance = 0.68 ohm.

The same experiments were then repeated with the same spark-gap lengths but without air-blast. It was found to be more difficult to obtain good points for resonance curves, but the final results are as shown in the figures in Table V.

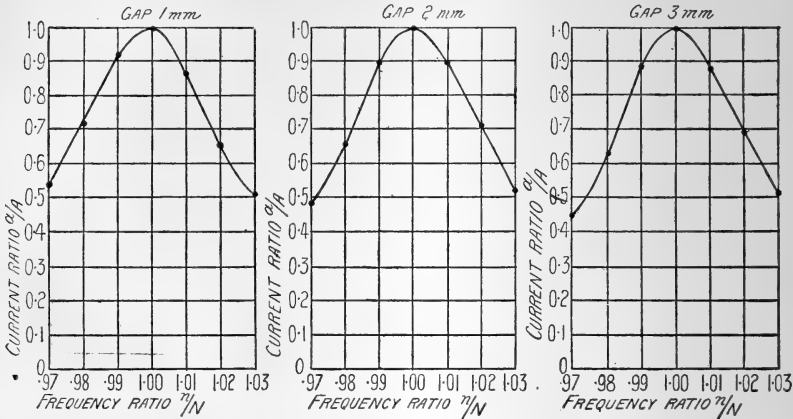


Fig. 2.—Resonance Curves taken without air-blast on spark-gap.

To economise printing we shall write X for the quantity $100 a/A$, and Y for $\pi\sqrt{\frac{a^2}{A^2 - a^2}}$, and Z for the mean value of $1 - n/N$.

Then since the values of X and Y are the same in all cases they apply to each spark-length, and need not be repeated if it be understood that the six rows of figures correspond respectively to $X=95, 90, 85, 80, 75$, and 70 per cent. We accordingly compress the final results for the 1, 2, and 3 mm. spark-gap unblown on in one Table V.

TABLE V.—1, 2, and 3 mm. spark-gaps without air-blast.

1 - n/N = Z.			D = $\delta_1 + \delta_2$.		
1 mm.	2 mm.	3 mm.	1 mm.	2 mm.	3 mm.
·0072	·008	·0067	·069	·0766	·0642
·00975	·01	·0092	·0631	·0647	·0595
·0017	·0124	·0116	·0595	·0631	·0591
·0141	·0146	·0139	·0592	·0612	·0582
·0164	·0172	·0162	·0587	·0617	·0581
·0187	·0193	·0185	·0576	·0595	·0570
Mean Values.....			·0613	·0645	·0593

The values of the maximum cymometer current in amperes without the resistance in its circuit ($=A$) and with added resistance ($=A_1$), and the calculated values of the decrements and spark-gap resistances are given in Table VI.

TABLE VI.

1, 2, and 3 mm. spark-gaps without air-blast.

Spark-length in mm.	Max. Cymo- meter Current in amperes $=A$.	With resis- tance added $=A_1$.	Primary decrement δ_1 .	Secondary crement δ_2 .	Spark re- sistance in ohms.
1.....	·0952	·052	·0423	·019	·75
2.....	·1202	·067	·0447	·0198	·81
3.....	·119	·066	·0383	·0216	·65

If we collect together the same quantities for the spark-gaps when blown on we have results as in Table VII.

TABLE VII.

1, 2, and 3 mm. spark-gaps with air-blast.

Spark-length in mm.	Max. Cymo- meter Current $=A$.	With resis- tance added $=A_1$.	Primary decrement δ_1 .	Secondary decrement δ_2 .	Spark-re- sistance in ohms.
1.....	·1195	·0635	·0467	·017	·86
2.....	·101	·046	·0443	·018	·80
3.....	·1195	·0652	·0397	·0194	·68

It will be seen, therefore, that when the air-blast is not applied the various values of $D=\delta_1+\delta_2$ calculated from the resonance curves are more irregular and deviate more from the mean, this being the result of the difficulty of obtaining correct galvanometer readings to delineate a good resonance curve.

The observations also appear to show that the air-blast has not much influence upon the primary decrement or upon the spark-resistance, or at most tends to slightly increase the

spark-resistance for very short sparks. The good effect of the air-blast is seen best when applied to short sparks, which tends to show that it is of assistance in destroying the arcing then occurring. This arc, however, is unable to persist with longer gaps; that is to say, it blows itself out, and accordingly for spark-lengths of 3 mm. or upwards and when using an ordinary 10 in. induction-coil with mercury break as interruptor, the blast is of no special advantage.

Another set of experiments was conducted, the object of which was to ascertain the effect of the air-blast upon multiple and upon very short spark-gaps. At one time some radiotelegraphists were of opinion that an advantage was gained by dividing up a spark-gap into smaller spark-gaps in series. To test this opinion more carefully, a series of six spark-balls were made, each consisting of a pair of brass balls 2 cm. in diameter adjustable as to distance by a screw of 0.5 mm. pitch, having a divided head. These spark-balls were arranged so that they could be put in series with one another, the series forming the spark-gap in an oscillatory circuit having an inductance of 125,000 cms. and a capacity of 0.0045 mfd. A number of glass jets were arranged so that an air-blast could be directed against each pair of spark-balls across the gap. In the oscillatory circuit a hot wire ammeter was placed, so arranged that a reading could be taken of the ammeter first with a single spark-gap of a definite length, and then with a series of 2, 3, 4, 5, and 6 gaps, each adjusted to be equal in length, the sum of them all being equal to that of the single gap employed. Thus, for instance, the effect of a single gap 0.5 mm. could be compared with that of five gaps in series each 0.1 mm., both with the air-blast and without the air-blast against the gaps. As in the case of very short gaps, the phenomenon of multiple sparks exists; that is to say, each interruption of the induction-coil in the primary circuit gives rise not merely to one discharge in the oscillatory circuit, but to a series of discharges, because a discharge takes place between the balls corresponding to the length of the spark-gap employed, and whilst the electromotive force in the secondary circuit of the coil increases and endures, during this time many sparks may take place.

Whether this occurs or not can be determined by means of a revolving mirror. If the image of an oscillatory discharge produced by an induction-coil is examined in a revolving mirror, then if there is only one spark corresponding to each interruption of the primary coil, a series of widely separated sharp images of the spark will be seen, but

if the phenomenon of multiple discharge is taking place, then the images are each seen to consist of 4, 5, 6, or more small sparks rapidly succeeding each other. Also, if there is any sensible arcing at the spark-gap it is at once seen in the image in the revolving mirror in the form of a trail of light of a different colour to that of the true oscillatory discharge-spark. On examining the spark-discharge with and without the air-blast for certain short spark distances, the effect of the air-blast is easily detected, because it is seen to blow away this trail which accompanies the image of the spark-discharge.

In order to avoid misinterpreting the results, it was necessary to be sure that when using say one single spark of 0.5 mm. in length and comparing it with the effect of 5 spark-gaps of 0.1 mm. in series, any increase in the current at the discharge circuit was not due to an alteration in the number of discharges.

As far as could be observed this was not the case when sufficiently short spark-gaps were employed.

The following Table VIII. (p. 690) embodies the results of the measurements. Employing single spark-gaps ranging in length from 0.1 mm. to 3 mm., observations were taken of the mean-square value of the current in the discharge circuit, both with and without the air-blast, and these values are recorded in the first two columns. It will be seen that up to a certain length of gap (about 2 mm. in the case of these experiments) the air-blast had a very decided effect in increasing the mean-square value of the discharge current, but beyond that point it seems to have the effect of diminishing it. On the other hand, if a single gap, say of 0.2 mm., is broken up into two gaps in series each of 0.1 mm. and so on for the other spark-lengths, it is found that dividing the spark-gap into two parts also increases the discharge current up to about 0.8 mm., and after that the current diminishes. If the double air-gap is blown upon, the current is increased as compared with the same two gaps not blown upon, but is not increased as compared with a single gap blown upon. The same is true when the spark-gap is broken up respectively into 3, 4, 5, and 6 gaps. The effect of dividing the gap up to a certain point is to increase the discharge current, but the effect of the air-blast on the multiple gaps becomes less and less marked in proportion as the number of gaps increases, so that the effect of dividing up a gap, say 0.6 mm., into five gaps of 0.12 mm. each, is to increase the discharge

TABLE VIII.
Comparisons of the Effect of breaking up a single Spark-gap into several Spark-gaps in series.

Total Gap.	In 1 Gap.		Separated into 2 Gaps.		Separated into 3 Gaps.		Separated into 4 Gaps.		Separated into 5 Gaps.		Separated into 6 Gaps.	
	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.	No Air Blast.	With Air Blast.
mm.	amp.	amp.										
1	.54	.66		.816	.71	.89688	.88		
2	.684	.816	.728	1.042	1.34	1.34	1.34		
3	.828	.912	.684	1.029	1.029	1.029		
4	.628	1.04	1.142	.937	1.042	...	1.029		
5	.71	1.095914	1.042896		
6	.755	1.253	.728	1.258	.88	1.017	1.017
8	.84	1.281	.84	1.228896	.896		
10	.896	1.35	.914	1.35		
12	.975	1.287	.975	1.35	1.042	1.142	.896	.896755	.755
15	1.142	1.142	1.029	1.029		
20	1.2	1.2	1.042	1.042755	.755		
30	1.258	1.042	1.017	1.017	.937	.937	9.5	9.5

current almost as much as by subjecting the single gap of 0.6 mm. to the air-blast. This increase appears to be due to the suppression of the arc-discharge, as shown by the appearance of the images of the spark in the revolving mirror, and hence the arcing is suppressed by the separation of the gap into multiple gaps, almost as effectively as by blowing upon the single gap. On the other hand, separating the spark-gap into more than five separate gaps seems to result in diminishing the total discharge current beyond a certain very short length of spark. The figures in the table, however, do not show what is observed in practice, namely, the greater steadiness of the discharge current under the operation of the air-blast; and the conclusion, therefore, is that in any experiments in which great constancy is required in the discharge current in the condenser circuit, a great advantage is obtained by using a short spark-gap, and by subjecting the discharge spark to an air-blast, as this both increases the charging voltage of the condenser and steadies the discharge current by abolishing the arc, which would otherwise take place even with an ordinary induction-coil.

A similar set of experiments was tried with a larger alternating current plant and a high tension transformer, constituting one of the wireless telegraph sets of the Radiotelegraphic Laboratory, University College. The alternating current was supplied from a 5-k.w. alternator and raised in pressure by a transformer discharging across a spark-gap between two iron balls, the spark-gap being shunted by a condenser in series with the primary circuit of an oscillation transformer, the secondary circuit of which was inserted between an antenna and an earth connexion, the antenna circuit being tuned to the condenser circuit. As no suitable hot-wire ammeter was available for measuring the large current in the condenser circuit, readings were taken by connecting the terminals of a hot-wire voltmeter to points on the earth-wire connexion of the antenna, consisting of a copper strip. These points being selected a few inches apart so as to give a convenient reading on the low reading hot-wire voltmeter. These readings, however, are approximately proportional to the currents flowing in the condenser circuit. Spark-gaps were then adjusted from 0.5 mm. to 3 mm. in length, and the reading of the voltmeter taken, both when the spark-gap was blown upon and when it was not blown upon by a jet of air.

The following table shows the readings :—

TABLE IX.

Spark Lengths.	Voltmeter Reading.	
	No Air Blast.	With Air Blast.
3.0 mm.	2.4	2.7
2.5 "	2.4	2.7
2 "	1.85	2.2
1.5 "	1.7	1.7
1.0 "	1.65	2.0
.5 "	.75	1.0
1.0 "	2.0	2.25
1.5 "	2.1	2.4
2.0 "	2.05	2.35
3.0 "	2.3	2.45

The Table shows in every case an increase in the current circulating in the condenser circuit when the spark-gap is subjected to the air-blast, but does not indicate the much greater steadiness of the voltmeter-needle which is seen with the air-blast. Without the air-blast the voltage was by no means constant, and the readings given in the first column are therefore a mean reading from which the extreme readings may differ by 10 per cent.

The final result of the experiments is to show that when using spark-gaps, 1, 2, or 3 mm. in width, in a condenser circuit, for the purpose of exciting oscillations, much greater uniformity in the discharge current can be obtained if the spark-gap is subjected to an air-blast as described.

LXIII. On the *Kinetic Criterion of Potential Energy.*

By C. V. BURTON, *D.Sc.**

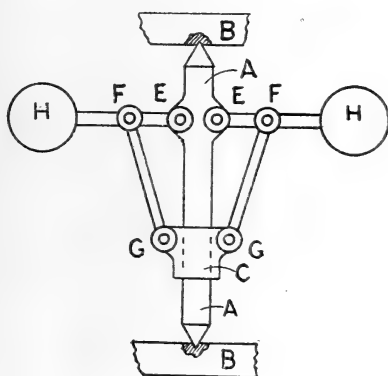
1. **I**N a large number of cases, if not in all, energy which we find it justifiable and convenient to treat as potential is found on a closer scrutiny to be essentially kinetic; and the object of this note is to supply an answer to the question: in what circumstances may kinetic energy be treated as potential?

* Communicated by the Author.

2. The potential energy of a conservative dynamical system is sometimes defined as that portion of the total energy which is a function of the coordinates of the system only, and is independent of the rates of change of those coordinates. Such a definition, however, is too inclusive, for in the energy-expressions of many dynamical systems there are terms which, although independent of the time-fluxes of the (working) coordinates *, do not fall into the category of potential energy.

3. Before treating the question more generally, it will be convenient to consider an example. Fig. 1 represents what

Fig. 1.



may be shortly called a governor, though we are not here concerned with its capacity for "governing." The spindle A A is frictionlessly journaled in a fixed frame B B, and the only other freedom of the system corresponds to a motion of the block C lengthwise of the spindle, with accompanying motion at the pivots E, F, G. For convenience of description, suppose the axis A A to be vertical, and as coordinates of the system take χ defining the azimuth of the governor, and z a length measured vertically downwards from a fixed horizontal plane to some point of the block C. Using dots over the coordinates to signify their time-fluxes, the kinetic energy may be expressed in the form

$$E = \frac{1}{2} M \dot{z}^2 + \frac{1}{2} I \dot{\chi}^2, \quad \dots \quad (1)$$

where I, the moment of inertia about A A, is a function of z , and M (a coefficient of the same kind as a mass) is also a function of z .

4. Let the only external force acting be a vertical force Z

* Cf. § 8 below.

7. It need hardly be pointed out that the potential energy (energy of rotation) with which we are concerned in §§ 3-5 may lose its potential character if the conditions of the motion are modified: for example, if the governor is acted on by forces having a moment about its axis of rotation, so that the angular momentum about that axis no longer remains constant; or if the axis AA is allowed to change its direction in space. In such cases the kinetic character of the rotational energy must be explicitly recognized in the dynamical equations.

8. In any conservative dynamical system let $\psi, \phi, \theta, \dots$ be the coordinates which we employ to define the configuration at each instant, and let $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$ stand for the time-fluxes of these *working coordinates* (as they may be called). Then the kinetic energy of the system as ordinarily understood may be expressed as a h.q.f.* of $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$, with coefficients in general functions of $\psi, \phi, \theta, \dots$, while the potential energy is a function of $\psi, \phi, \theta, \dots$ only. As soon as we admit the kinetic nature of the energy which we treat as potential, we realize that in addition to the working coordinates $\psi, \phi, \theta, \dots$ there must be others (say) $\chi, \chi', \chi'', \dots$ whose time-fluxes $\dot{\chi}, \dot{\chi}', \dot{\chi}'', \dots$ are involved in this so-called potential energy, and which may be distinguished as "ignored coordinates"†.

9. The Lagrangian function for any conservative dynamical system is the *difference* of the kinetic and potential energies, the energy (\mathcal{L}) being expressed in terms of the generalized velocities $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$; on the other hand, the Hamiltonian reciprocal function is the *sum* of the kinetic and potential energies, the kinetic energy being expressed in terms of the momenta $\partial\mathcal{L}/\partial\dot{\psi}, \partial\mathcal{L}/\partial\dot{\phi}, \dots$. In the first place, then, it is apparent that potential energy is energy expressed in the Hamiltonian form, in terms of momenta, not in the Lagrangian form, in terms of velocities.

10. Suppose at the outset that the entire energy T is recognized as kinetic, and is expressed as a h.q.f. of $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$ and the $\dot{\chi}$'s. T is then the Lagrangian function for the system, and the equations of motion corresponding to $\psi, \phi, \theta, \dots$ are of the type

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = \Psi, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$\Psi, \Phi, \Theta, \dots$ being the impressed forces of types corresponding to $\psi, \phi, \theta, \dots$ respectively.

* Here and below h.q.f. stands for "homogeneous quadratic function."

† Thomson (Kelvin) and Tait's 'Natural Philosophy, Part I. § 319.

11. Following Routh *, let us now *modify* the function T with respect to the coordinates χ, χ', \dots , the modified function will be

$$T' = T - \frac{\partial T}{\partial \dot{\chi}} \dot{\chi} - \frac{\partial T}{\partial \dot{\chi}'} \dot{\chi}' - \dots, \dots \dots (7)$$

the $\dot{\chi}$'s being supposed eliminated from the right-hand of (7) by means of the equations

$$\partial T / \partial \dot{\chi} = C, \quad \partial T / \partial \dot{\chi}' = C', \dots \dots \dots (8)$$

so that the C 's are the momenta corresponding to the χ 's respectively.

12. In a system such as we contemplate, the energy is made up of two parts: one a h.q.f. (\mathfrak{T}) of the velocities $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$, and the other (K) independent of those velocities; that is

$$T = \mathfrak{T} + K. \dots \dots \dots (9)$$

In this case \mathfrak{T} is the energy which we recognize as kinetic, and K is that which we call potential.

13. It is of course understood, when $\psi, \phi, \theta, \dots$ are the only "working coordinates," that the system is acted upon by no external forces except $\Psi, \Phi, \Theta, \dots$; in these circumstances it is shown in Kelvin and Tait that (9) will be fulfilled provided the χ 's do not appear in the coefficients of the energy expression T ; and it is easy to show that (9) will not be fulfilled otherwise. Hence in place of (9) we may write the conditions

$$\partial T / \partial \chi = 0, \quad \partial T / \partial \chi' = 0, \dots \dots \dots (9a)$$

14. Kelvin and Tait's analysis † relating to the ignorance of coordinates is therefore applicable, and we have

$$C, C', \dots \text{all constant}; \dots \dots \dots (10)$$

while K is a h.q.f. of the C 's.

15. From the modified function T' the equations of motion of the system may be obtained in the form

$$\frac{d}{dt} \frac{\partial T'}{\partial \dot{\psi}} - \frac{\partial T'}{\partial \psi} = \Psi, \dots \dots \dots (11)$$

with similar equations for Φ, Θ, \dots ; and these by a process

* 'Rigid Dynamics,' vol. i. chap. viii.

† *Loc. cit.*

similar to that followed in Kelvin and Tait, may be reduced to the form in which they are obtained in that treatise, namely,

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathfrak{T}}{\partial \dot{\psi}} \right) - \frac{\partial \mathfrak{T}}{\partial \psi} + C \left\{ \left(\frac{\partial M}{\partial \phi} - \frac{\partial N}{\partial \psi} \right) \dot{\phi} + \left(\frac{\partial M}{\partial \theta} - \frac{\partial O}{\partial \psi} \right) \dot{\theta} + \dots \right\} \\ + C' \left\{ \left(\frac{\partial M'}{\partial \phi} - \frac{\partial N'}{\partial \psi} \right) \dot{\phi} + \left(\frac{\partial M'}{\partial \theta} - \frac{\partial O'}{\partial \psi} \right) \dot{\theta} + \dots \right\} + \dots + \frac{\partial K}{\partial \psi} \end{aligned} \right\} = \Psi, \quad (12)$$

Here $M, N, O, \dots M', N', O', \dots$ are best defined for our purpose by the relations

$$\left. \begin{aligned} \dot{\chi} &= \frac{\partial K}{\partial C} - (M\dot{\psi} + N\dot{\phi} + O\dot{\theta} + \dots) = \frac{\partial K}{\partial C} + \dot{\chi}_1 \text{ (say),} \\ \dot{\chi}' &= \frac{\partial K}{\partial C'} - (M'\dot{\psi} + N'\dot{\phi} + O'\dot{\theta} + \dots) = \frac{\partial K}{\partial C'} + \dot{\chi}'_1. \end{aligned} \right\} \quad (13)$$

16. It is clear that (12) are precisely the equations of motion of a system in which \mathfrak{T} is the kinetic and K the potential energy, *provided that for all values of $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$*

$$\left. \begin{aligned} \Sigma C \left\{ \left(\frac{\partial M}{\partial \phi} - \frac{\partial N}{\partial \psi} \right) \dot{\phi} + \left(\frac{\partial M}{\partial \theta} - \frac{\partial O}{\partial \psi} \right) \dot{\theta} + \dots \right\} &= 0, \\ \Sigma C' \left\{ \left(\frac{\partial N}{\partial \psi} - \frac{\partial M}{\partial \phi} \right) \dot{\psi} + \left(\frac{\partial N}{\partial \theta} - \frac{\partial O}{\partial \phi} \right) \dot{\theta} + \dots \right\} &= 0, \\ \Sigma C \left\{ \left(\frac{\partial O}{\partial \psi} - \frac{\partial M}{\partial \theta} \right) \dot{\psi} + \left(\frac{\partial O}{\partial \phi} - \frac{\partial N}{\partial \theta} \right) \dot{\phi} + \dots \right\} &= 0. \end{aligned} \right\} \quad (14)$$

Let m be the number of the working coordinates $\psi, \phi, \theta, \dots$; then since in each of the m equations (14) the coefficients of $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$ will have to vanish separately, and since $\dot{\psi}$ is absent from the first equation, $\dot{\phi}$ from the second, and so on, we obtain $m(m-1)$ equations which must be satisfied if K is to be treated as the potential energy of the system. These conditional equations, however, are not all independent, but consist of $\frac{1}{2}m(m-1)$ independent equations, each occurring

twice; thus the conditions sought for, in addition to (9a), are

$$\left. \begin{aligned} \Sigma C \left(\frac{\partial M}{\partial \phi} - \frac{\partial N}{\partial \psi} \right) &= 0, & \Sigma C \left(\frac{\partial M}{\partial \theta} - \frac{\partial O}{\partial \psi} \right) &= 0, \dots \\ & & \Sigma C \left(\frac{\partial N}{\partial \theta} - \frac{\partial O}{\partial \phi} \right) &= 0, \dots \end{aligned} \right\} \quad (15)$$

17. *Special Class i.* An interesting case is that in which there is but a single working coordinate, say ψ ; so that the system, regarded from our standpoint, has only one degree of freedom. In this case the sole equation of motion is

$$\frac{d}{dt} \left(\frac{\partial \mathfrak{T}}{\partial \dot{\psi}} \right) - \frac{\partial \mathfrak{T}}{\partial \psi} + \frac{\partial K}{\partial \psi} = \Psi; \quad \dots \quad (16)$$

which indicates that the energy K , due to the "ignored" momenta, necessarily behaves as potential energy, without any condition having to be satisfied beyond those expressed by (9a).

18. Returning now to (7) and making use of (8) and (13), we see that

$$\begin{aligned} T' &= T - C \frac{\partial K}{\partial C} - C' \frac{\partial K}{\partial C'} - \dots \\ &\quad + C(M\dot{\psi} + N\dot{\phi} + \dots) + C'(M'\dot{\psi} + N'\dot{\phi} + \dots) + \dots \\ &= T - C \frac{\partial K}{\partial C} - C' \frac{\partial K}{\partial C'} - \dots - C\dot{\chi}_1 - C'\dot{\chi}_1' - \dots \quad (17) \end{aligned}$$

Remembering that K is a h.q.f. of C, C', \dots , we have accordingly

$$\begin{aligned} T' &= T - 2K - C\dot{\chi}_1 - C'\dot{\chi}_1' - \dots \\ &= \mathfrak{T} - K - C\dot{\chi}_1 - C'\dot{\chi}_1' - \dots, \quad \dots \quad (18) \end{aligned}$$

by (9). Thus the modified function T' , from which the equations of motion (11) are derived, becomes identical with $\mathfrak{T} - K$ provided

$$C\dot{\chi}_1 + C'\dot{\chi}_1' + \dots = 0; \quad \dots \quad (19)$$

that is, provided

$$\Sigma C(M\dot{\psi} + N\dot{\phi} + \dots) = 0, \quad \dots \quad (19a)$$

a condition which is the same as

$$\dot{\psi} \Sigma C M + \dot{\phi} \Sigma C N + \dots = 0. \quad \dots \quad (19b)$$

So long as this condition is satisfied, we may treat K as the potential energy of the system.

19. The condition (19), (19*a*) or (19*b*) necessarily implies (14), which must in any case be fulfilled if the energy K is to be treated as potential; in general, however, it includes more than is strictly demanded. But in most of the examples which readily present themselves, and in which K behaves as potential energy, the condition in question is in fact satisfied. If (19*b*) is to hold good *without any restriction being imposed on the velocities* $\dot{\psi}$, $\dot{\phi}$, ..., we shall have

$$\Sigma CM = 0, \quad \Sigma CN = 0. \quad . \quad . \quad . \quad (20)$$

20. The C 's being constants, it is evident that (14) will be satisfied provided

$$\Sigma CM, \quad \Sigma CN, \dots \text{are all independent of } \psi, \phi; \quad . \quad (21)$$

which relations, though in general expressing more than the requisite conditions (14), impose less restriction than do (20).

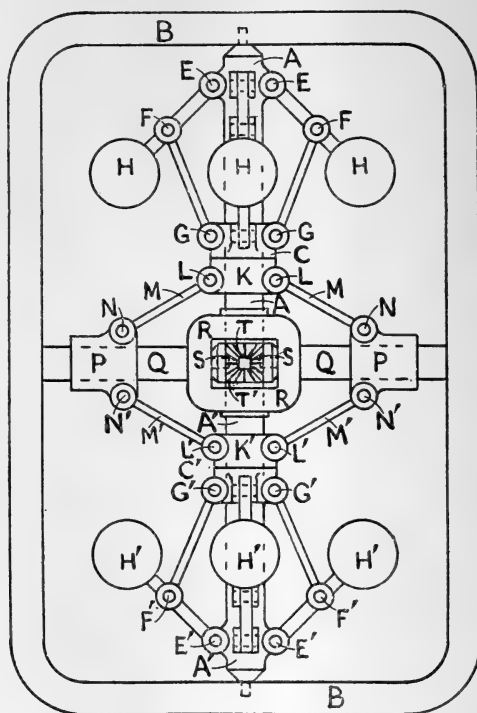
21. *Special Class ii.* A simple case, illustrating the condition (20), is when the M 's, N 's, ... all vanish, or in other words when, for given values of the working coordinates $\psi, \phi, \theta, \dots$ the velocities \dot{X}, \dot{X}', \dots of the ignored coordinates are determined solely by the momenta $C, C' \dots$, independently of the velocities $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$.

22. *Special Class iii.* Another simple case is when the state of the system, as defined by the working coordinates $\psi, \phi, \theta, \dots$, is one of *continued rest*. For when $\dot{\psi}, \dot{\phi}, \dot{\theta}, \dots$ are all constantly zero, (14) are satisfied. In any system, therefore, for which (9*a*) hold good, the energy K —due to the momenta of the ignored coordinates—may be treated as potential energy in computing what forces, corresponding to the working coordinates, must be applied to the system to maintain it "at rest." If, however, the system is moved from one configuration to another, even infinitely slowly, although at each instant the generalized forces required to maintain (infinitely nearly) equilibrium will only differ infinitesimally from $\partial K / \partial \psi, \partial K / \partial \phi, \dots$ yet the time-integrals of the forces in question will in general be finitely different from those of $\partial K / \partial \psi, \partial K / \partial \phi, \dots$ unless (20), or failing that (21), or in any case unless (14) are satisfied.

23. Though the rotational energy of the governor represented in fig. 1 can no longer be treated as potential when the frame BB is permitted to turn without restriction of direction, it is easy to devise a pair of governors carried by a single frame, and so connected or so set in motion that their

energy of rotation may properly be treated as potential energy. In fig. 2 let B be a rigid frame, and let there be

Fig. 2.



two equal and similar governors journalled in this frame, so as to be capable of rotation about a common (geometrical) axis. The construction of the governors $A C E F G H$, $A' C' E' F' G' H'$ is essentially similar to that of the governor shown in fig. 1, like letters indicating like parts. Each governor, however, has four arms (instead of only two), each carrying a ball, and all symmetrically arranged about the axis $A A$. The rotatable system which constitutes the governor proper has of course moments of inertia which vary according to the inclination of the arms $E H$; but the axis $A A$ is always a principal axis of inertia, and the moment of inertia is the same about all axes which intersect $A A$ perpendicularly at a given point. This condition has to be fulfilled in order that the angular coordinates χ , χ' (defining the integral angular rotations of the governors relatively to the frame) may not appear in the coefficients of the expression for the kinetic energy of the system.

24. The collar K can partake of the axial motion of the block C, as the latter slides lengthwise of the axis A A; K is incapable of rotating with respect to the frame B, but is so connected to C as not to interfere with C's rotation. The parts in the second governor are indicated by accented letters, and are in all respects equal and similar to those in the first governor. The guides Q Q are integral with the frame B on the one hand, and with the inner frame R on the other hand, and upon these guides the blocks P P can slide without rotation. The pieces P K P' K' are connected up as shown by the four links M M M' M', which are equivalent to a jointed parallelogram. For the present we leave out of consideration the mitre-gears S S T T', to which reference will be made later.

25. It is evident that, in addition to any freedom of motion which the frame B may have, and exclusive of the rotational motions of the governors, the system made up of the two governors and the parallelogram linkwork has one degree of freedom, corresponding to a motion of each of the blocks P P (say) towards the centre of the framework, with simultaneous motion of K K' away from the centre, and increase of the angle θ which the arm E H makes with the axis A A'. This angle θ may conveniently be taken as the coordinate corresponding to the freedom just defined. For the remaining coordinates of the system we may take x, y, z , the Cartesian coordinates of the centroid, and three angular coordinates defining the orientation of the frame B, together with the angles χ, χ' (measured from a standard configuration) through which the respective governors E H G, E' H' G' have turned with respect to the frame B.

26. Let the system, *with the exception of the rotatable governors*, have the moment of inertia I_1 about the axis A A', and at any instant (t) let the frame B be turning about the axis A A' with angular velocity ω_1 . Let I_2 be the moment of inertia of the whole system *including the governors* about the axis Q Q; and ω_2 the angular velocity of the frame B about the instantaneous axis Q Q; I_3, ω_3 being the corresponding quantities referred to an axis through the centroid of the system perpendicular to the plane of fig. 2. If M is the mass of the entire system, and \mathfrak{I} the moment of inertia of either governor about its mechanical axis, the kinetic energy may be written

$$T = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} I_0 \dot{\theta}^2 \\ + \frac{1}{2} \mathfrak{I} (\dot{\chi} + \omega_1)^2 + \frac{1}{2} \mathfrak{I} (\dot{\chi}' + \omega_1)^2. \dots \dots (22)$$

27. Here I_0 is a quantity of the same nature as a moment

of inertia, and it is to be understood that $\omega_1, \omega_2, \omega_3$ are to be replaced by their values in terms of the time-fluxes of the angular coordinates of the system. Of the coefficients, M and I_1 alone are constants, I_2, I_3, I_0 , and \mathfrak{F} being functions of θ . In the absence of any external forces acting on the system, T is the Lagrangian function, which we proceed to *modify* with respect to the coordinates χ, χ' . The angular momenta of the governors about their axis are

$$\text{and } \left. \begin{aligned} \partial T / \partial \dot{\chi} &= \mathfrak{F}(\dot{\chi} + \omega_1) = C \text{ (say),} \\ \partial T / \partial \dot{\chi}' &= \mathfrak{F}(\dot{\chi}' + \omega_1) = C' \end{aligned} \right\}; \quad . \quad . \quad (23)$$

Thus

$$\dot{\chi} = \frac{C}{\mathfrak{F}} - \omega_1, \quad \dot{\chi}' = \frac{C'}{\mathfrak{F}} - \omega_1. \quad . \quad . \quad . \quad (24)$$

The modified Lagrangian function is thus

$$\begin{aligned} T' &= T - C\dot{\chi} - C'\dot{\chi}' \\ &= \frac{1}{2} M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} I_0 \dot{\theta}^2 \\ &\quad + \frac{1}{2} \frac{C^2}{\mathfrak{F}} + \frac{1}{2} \frac{C'^2}{\mathfrak{F}} - C \left(\frac{C}{\mathfrak{F}} - \omega_1 \right) - C' \left(\frac{C'}{\mathfrak{F}} - \omega_1 \right); \end{aligned}$$

i. e.

$$\begin{aligned} T' &= \frac{1}{2} M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} I_0 \dot{\theta}^2 \\ &\quad - \frac{1}{2} \frac{C^2}{\mathfrak{F}} - \frac{1}{2} \frac{C'^2}{\mathfrak{F}} + (C + C')\omega_1. \quad . \quad . \quad . \quad (25) \end{aligned}$$

28. Since the rotatable governors are acted upon by no forces having a moment about the axes AA or $A'A'$, and since the coordinates χ, χ' do not appear in the coefficients of the expression T , the whole kinetic energy may be divided into two parts: one (to be called \mathfrak{T}) being a h.q.f. of the velocities $\dot{x}, \dot{y}, \dot{z}, \omega_1, \omega_2, \omega_3, \dot{\theta}$, and the other (to be called K) being a function of the coordinate θ , and involving besides only the constant momenta C, C' . (25) may in fact be written

$$T' = \mathfrak{T} - K + (C + C')\omega_1; \quad . \quad . \quad . \quad (26)$$

and we shall accordingly be able to treat the rotational energy K of the governors as potential, *provided the constant angular momenta C, C' are equal and opposite.*

29. Alternatively, instead of the two governors being rotatable independently of one another, we may suppose them to be positively connected in such a way that their rotations with respect to the frame B are always equal and opposite. The

mitre-wheels T, T', fixed upon the spindles A A, A'A' respectively and gearing with a mitre-wheel S, would effect this result; a fourth mitre-wheel S being introduced for the sake of symmetry and balance. Though the omission is not essential to the case now considered, it will be assumed for simplicity that the rotational energy of the mitre-wheels S, S is relatively small enough to be left out of account.

30. The expression for the whole kinetic energy is obtained from (22) by substituting $-\chi$ for χ' ; while the momentum corresponding to the χ -coordinate is now

$$\partial T / \partial \dot{\chi} = 2\mathfrak{K}\dot{\chi} = u \text{ (say); } \dots \dots (27)$$

and the modified Lagrangian function is accordingly

$$\begin{aligned} T' &= T - u\dot{\chi} \\ &= \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} (I_1 + 2\mathfrak{K}) \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 \\ &\quad + \frac{1}{2} I_0 \dot{\theta}^2 - \frac{1}{2} \frac{u^2}{2\mathfrak{K}}. \dots \dots \dots (28) \end{aligned}$$

If the momentum u remains constant, the last term on the right-hand of (28) is the energy of rotation of the governors about their axes, with sign reversed. When u is given, this term is a function of θ only, and T' , given by (28), may accordingly be taken as made up of the difference of the kinetic and potential energies of the system. As regards the equations of motion corresponding to the working coordinates, the present example differs from that of § 28 in that the moment of inertia I_1 of the frame B about the axis A A' is now effectively increased by $2\mathfrak{K}$, the sum of the moments of inertia of the two governors about their axes A A, A'A'.

31. An interesting example of potential energy is furnished by a system of perforated solids, immersed in a frictionless incompressible fluid which is circulating irrotationally through their various apertures. Let each of the solids be in the form of a thin rigid wire or wires, forming a closed loop or a framework. Then, provided no two solids approach one another very closely, the component of fluid motion contributed by any one of the solids is due almost exclusively to the cyclic constants of circulation associated with that solid, and is appreciably the same as if the remaining solids were non-existent. In the present case the working coordinates are any such as serve to define at each instant the position and orientation of every one of the solids, and each of the coordinates (χ, χ', \dots) to be subsequently ignored is the volume of liquid which, starting from a definite configuration, has flowed across one of the ideal geometrical surfaces required to close the various

apertures in the solids. We shall be able to treat the energy of the circulation-momenta as potential if certain conditions are satisfied which are equivalent to (9a) and (19). The condition (9a) (that the coordinates χ shall not appear in the coefficients of the expression for the total kinetic energy of the system) is obviously realized. But (19) will only be consistently fulfilled when the sum of the $C\dot{\chi}_1$'s for each solid is always zero; the momentum C corresponding to the coordinate χ being $\kappa\rho$ where κ is the cyclic constant of circulation for the aperture in question, and ρ is the density of the liquid.

32. For each solid therefore there is a condition to be satisfied of the form

$$\Sigma \kappa \rho \dot{\chi}_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

the $\dot{\chi}_1$'s being homogeneous linear functions of $\dot{x}, \dot{y}, \dot{z}, \omega_1, \omega_2, \omega_3$, where $\omega_1, \omega_2, \omega_3$ are the angular velocities of the solid about axes instantaneously coincident with a set of rectangular axes moving with the solid, and x, y, z are the Cartesian coordinates of the origin of those moving axes. For the particular solid under consideration, let

$$\dot{\chi}_1 = \left. \begin{aligned} & [\dot{x}] \dot{x} + [\dot{y}] \dot{y} + [\dot{z}] \dot{z} + [\omega_1] \omega_1 + [\omega_2] \omega_2 + [\omega_3] \omega_3 \\ & . \quad . \quad . \quad . \quad . \quad . \end{aligned} \right\}. \quad (30)$$

Then (19) or (19b) for that solid is equivalent to the six conditions

$$\left. \begin{aligned} 0 &= \Sigma \kappa \rho [\dot{x}] = \Sigma \kappa \rho [\dot{y}] = \Sigma \kappa \rho [\dot{z}], \\ 0 &= \Sigma \kappa \rho [\omega_1] = \Sigma \kappa \rho [\omega_2] = \Sigma \kappa \rho [\omega_3]. \end{aligned} \right\} \quad . \quad . \quad (31)$$

33. Let S, S', \dots be geometrical surfaces invariably related to the solid with which we are dealing, and sufficing to close all its apertures. Then since $[\dot{x}]\dot{x}, [\dot{x}']\dot{x}, \dots$ are the volumes of liquid flowing per unit time past the surfaces S, S', \dots , owing to the velocity-component \dot{x} of the body, we easily see that

$$[\dot{x}] = - \int \cos \nu \, dS, \quad [\dot{x}'] = - \int \cos \nu' \, dS', \quad . \quad . \quad (32)$$

where ν is the angle which the positively drawn normal at any point of the surface S makes with the axis of x , and so on. Thus the first of the conditions (31) may be written

$$\Sigma . \kappa \rho \int \cos \nu \, dS = 0. \quad . \quad . \quad . \quad . \quad (33)$$

34. Remembering that the $\kappa\rho$'s measure the impulsive pressures which must be applied over the surfaces S, S', \dots to

produce the circulations κ , we see that (33) amounts to this : that when all the impulsive pressures $\kappa\rho$ are so applied, there must, on the whole, be no component of impulse parallel to the axis of x (or of course to the axis of y or of z).

35. Similarly $\sum \kappa\rho[\omega_1]=0$, which is one of the conditions (31), may be put in the form

$$\sum . \kappa\rho \int r \cos \vartheta dS = 0, (34)$$

where r is the perpendicular distance of any point on the surface S from the axis about which ω_1 is measured, and ϑ is the angle which the normal to S at the point in question makes with a line perpendicular both to r and to the axis of ω_1 , the positive direction of this latter line corresponding (let us suppose for definiteness) to the positive sense of the angular velocity ω_1 . Now (34) expresses the condition that, when all the impulsive pressures $\kappa\rho$ act over the surfaces S , there shall be no resultant impulsive moment about the axis of ω_1 (or of course about the axis of ω_2 or of ω_3).

36. The results obtained in §§ 31–35 may be summarized as follows : If in a frictionless liquid free from vortex motion a number of solids are immersed, each consisting of a rigid framework of thin wires, then the energy of the circulation-momenta may be treated as potential energy, provided that for each single solid the impulses required to initiate all the circulations of that solid are such as, being applied to a rigid body, would be in equilibrium.

37. A dynamical system of the kind just considered may also be made to furnish an example of the fulfilment of the conditions (21). With our previous stipulation as to the thinness of the wires of which the solids are built up, it is evident that (21) will be satisfied, provided only that each body of the system is limited to translational freedom, without the possibility of rotation. In this case, no matter what may be the values of the circulation-momenta, the energy due to those momenta may be treated as potential energy, although in general the translational movement of any solid will involve reactions against the constraints arising from what may be called “want of balance” of the circulation-momenta.

38. The case of a mass of gas whose pressure is varied adiabatically may serve as a final example ; isothermal conditions are excluded from consideration, as in such case the system is not properly speaking conservative, although simulating a conservative system in its general dynamical behaviour. For simplicity let the gas be monatomic and be contained in a fixed cylinder, in which works a gas-tight

frictionless piston, both cylinder and piston being impermeable to heat, and the axis of the cylinder being vertical with the piston uppermost. If, as a first approximation, we neglect the inertia of the gas in comparison with that of the piston, the single working coordinate is the height (z) of the piston above the bottom of the cylinder, the kinetic energy being made up of $\frac{1}{2}M\dot{z}^2$ (where M is the mass of the piston) and the $\dot{\chi}$'s, the χ 's being the coordinates necessary for the complete specification of the distribution of gas-molecules when z is given. On differentiating with respect to the various $\dot{\chi}$'s it is evident that the momenta thus obtained are homogeneous linear functions of the $\dot{\chi}$'s, so that the $\dot{\chi}$'s are likewise homogeneous linear functions of the momenta, and the whole kinetic energy is equal to $\frac{1}{2}M\dot{z}^2$ together with a h.q.f. of the χ -momenta. The further condition which is sufficient to ensure that the energy of the χ -momenta shall have the potential character is that, when these momenta are given, none of the χ 's shall involve \dot{z} (though they may and do involve z). This is consonant with the assumptions which we make when we propose to treat as potential energy the translational energy of the gas-molecules.

39. Similar considerations are readily applied to a differential volume-element of a gas through which sound-waves are travelling. That energy of the element which we commonly treat as kinetic is its energy of translational motion, corresponding to velocity-components \dot{x} , \dot{y} , \dot{z} of its mass-centre: while the energy of the momenta corresponding to the remaining (ignored) coordinates of the gas-molecules which make up the element is independent of x , y , z , as are also the velocities of the ignored coordinates when the corresponding momenta have assigned values.

LXIV. *Direct Application of the Electron Theory to Induction Currents.* By R. J. A. BARNARD, M.A., Melbourne*.

A MOVING charge of electricity in a magnetic field is acted on by the electromagnetic force $e(\mathbf{v} \times \mathbf{H})$ in Gibbs's Vector notation, where \mathbf{v} is the velocity. Consequently, if electrons are moving about in a conductor, even when no current is flowing an electromagnetic force is acting on each electron when there is an external magnetic field. But since the electrons are moving in such a case impartially in all directions, there can be no resultant effect produced by these forces. Even if a current is flowing, the resultant effect of the electromagnetic force will be perpendicular to

* Communicated by the Author.

the conductor ; and therefore, if the electrons cannot escape from the conductor, no effect on the current, except the Hall effect, is produced in this way. But if the conductor is moved bodily, the motion of the positive and negative electrons so produced will cause a component electromagnetic force on the positive electrons to act along the conductor in one direction and on the negative in the opposite. These forces drive the positive electrons in one direction and the negative in the other, so that their effects add together and produce the induction current.

Consider then the motion of the secondary circuit in a fixed field, and attend at first to the positive electrons only.

Let

\mathbf{ds} be the vector representing the element ds of the conductor,

\mathbf{v} be the vector representing the velocity of this element,

$d\sigma = \mathbf{v}dt$ = vector distance travelled in an infinitesimal time dt ,

$n'ds$ = number of positive electrons in the element,

e' = charge of each positive electron,

m' = mass of each positive electron.

The electromagnetic force on each electron is

$$e'(\mathbf{v} \times \mathbf{H}).$$

The component of this along \mathbf{ds} is

$$e'(\mathbf{v} \times \mathbf{H}) \cdot \mathbf{ds}/ds.$$

Neglecting collisions for the present, we have the acceleration of an electron along the wire

$$\frac{e'}{m' ds} (\mathbf{v} \times \mathbf{H}) \cdot \mathbf{ds} ;$$

and the velocity generated in time dt

$$= \frac{e' dt}{m' ds} (\mathbf{v} \times \mathbf{H}) \cdot \mathbf{ds},$$

and the current generated in the time between two collisions is given by

$$\begin{aligned} C'ds &= e' n' ds \frac{e'}{m' ds} (d\sigma \times \mathbf{H}) \cdot \mathbf{ds} \\ &= \frac{n' e'^2}{m'} (d\sigma \times \mathbf{H}) \cdot \mathbf{ds}. \end{aligned}$$

If r = resistance per unit length, the E.M.F. generated in the element

$$\begin{aligned} &= r ds \cdot C' \\ &= \frac{r n' e'^2}{m'} dN', \end{aligned}$$

where $dN' = (\mathbf{d}\sigma \times \mathbf{H}) \cdot \mathbf{ds}$ = number of lines of force cut by the element in its motion.

Similarly, on account of the negative electrons the E.M.F. generated is

$$\frac{r n e^2}{m} dN,$$

undashed letters referring to negative electrons.

Now $\frac{m}{m'}$ is very small, while n and n' and also e and e' are of the same order of magnitude; hence the E.M.F. due to the effect on the negative electrons is very much greater than that due to the positive. (e' may be numerically a multiple of e and may not be the same multiple for all positive electrons, but the argument will not be affected.) If, then, we neglect the positive electrons we have the total E.M.F. generated in the whole circuit

$$= \frac{r n e^2}{m} dN,$$

where dN = total number of lines of force cut by the whole circuit in time dt .

To take the collisions into account we proceed as in J. J. Thomson, 'Corpuscular Theory,' p. 53. The average time between two collisions is $\frac{\lambda}{V}$, where λ is the mean free path and V the mean velocity of an electron treated as a gaseous molecule, and we have to replace dN by

$$\frac{1}{2} \frac{dN}{dt} \frac{\lambda}{V}.$$

We thus get, finally,

$$E = \frac{1}{2} \frac{r n e^2}{m} \frac{\lambda}{V} \frac{dN}{dt}.$$

This is the ordinary law of induction currents provided

$$r n e^2 \lambda = 2m V.$$

If

S = cross-section of conductor,

$\rho = rS$ = the specific resistance,

$\nu = \frac{n}{S}$ = number of negative electrons per cub. cm.,

we get

$$\nu = 2m V/e^2 \lambda \rho,$$

the numerical result given by Thomson, 'Corpuscular Theory,' p. 54.

The idea of relative motion naturally gives the same expression for the E.M.F. when the circuit is fixed and the field varying. But the force acting is no longer the electromagnetic force in the ordinary understanding of the term, but an electric force. I will consequently show, next, that the equation

$$\text{curl } \mathbf{E} = - \frac{d\mathbf{B}}{dt}$$

leads to the same numerical result as before.

If \mathbf{E} is the electric force at any point of the conductor, the acceleration of an electron along the conductor is

$$\frac{e}{m ds} \mathbf{E} \cdot d\mathbf{s},$$

and the E.M.F. generated in the element between collisions

$$= \frac{r n e^2}{m} dt \mathbf{E} \cdot d\mathbf{s},$$

and the total E.M.F.

$$\begin{aligned} &= \frac{r n e^2}{m} dt \int \mathbf{E} \cdot d\mathbf{s} \\ &= \frac{r n e^2}{m} dt \iint \text{curl } \mathbf{E} \cdot d\mathbf{A} \\ &= -\frac{r n e^2}{m} dt \iint \frac{d\mathbf{B}}{dt} \cdot d\mathbf{A} \\ &= -\frac{r n e^2}{m} dt \frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{A} \\ &= -\frac{r n e^2}{m} dt \frac{dN}{dt}. \end{aligned}$$

The same result as for the case of the moving circuit.

We further may note that the steps can be reversed; so that, if we assume the experimental fact of the dependence of induction currents on relative motion only, we may deduce the equation

$$\text{curl } \mathbf{E} = - \frac{d\mathbf{B}}{dt}$$

from the expression for the electromagnetic force, or, in other words, from the law $\frac{C ds \sin \theta}{r^2}$ for the magnetic force due to an element of a current together with the law of action and reaction.

In other words, again, the two circuital relations

$$\text{curl } \mathbf{H} = 4\pi \mathbf{C},$$

$$\text{curl } \mathbf{E} = - \frac{d\mathbf{B}}{dt}$$

are deducible from one another on the electron theory solely by the help of the experimental facts of relative motion with respect to induction currents.

LXV. *Note on the Absorption of Ultra-Violet Light by Dilute Solutions.* By F. B. PIDDUCK, B.A., Fellow of Queen's College, Oxford *.

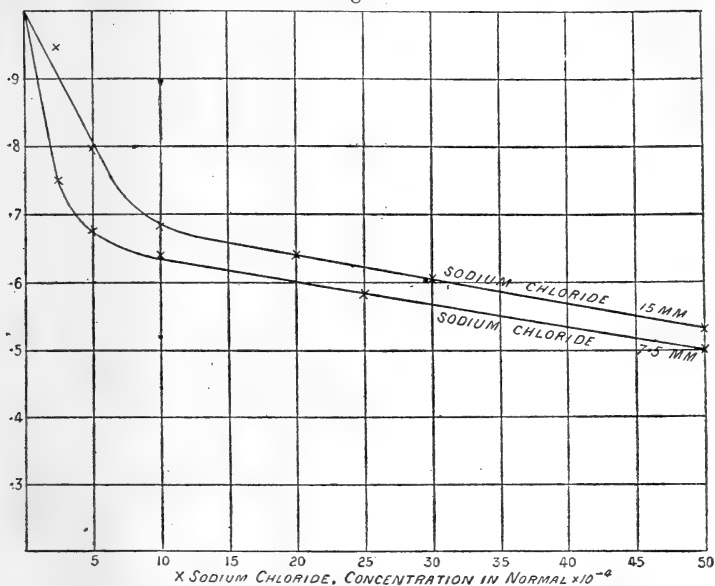
1. **T**HE experiments here described originated in the observation of a great decrease of the electrical action of ultra-violet light caused by transmission through ordinary clear tap-water as compared with the effect after passage through the same thickness of distilled water. In the attempt to elucidate the effect, a few very dilute solutions of known strength were made up and the relation of absorption to concentration and thickness traversed, as well as the result of mixture, was investigated.

The electrical connexions were made in the usual way. The ultra-violet light was produced by a spark between zinc electrodes in a Leyden-jar discharge circuit. The light passed through a wire grating which formed the positive plate of a parallel-plate condenser, the negative plate being of zinc and being attached to the insulated pair of quadrants of a Dolezalek electrometer. A force large enough to produce saturation was applied from a battery of lead cells, the negative pole of which, as well as the other pair of quadrants, was earthed. It was found that the electrometer readings were steadier when the zinc plate was allowed to remain some little time after cleaning, although the deflexions were thereby reduced.

* Communicated by Prof. J. S. Townsend, F.R.S.

2. The accompanying figure shows the results of experiments on solutions of sodium chloride for varying concentrations, using two thicknesses of absorbing solution. The

Fig. 1.



ordinate represents the ratio of the electrometer deflexion to that produced in the same time when the light passed through the same thickness of distilled water. The actual figures are given below.

Sodium Chloride.

Concentration in 10^{-4} normal.	Thickness 15 mm.	Thickness 7.5 mm.
2.5	.946	.750
5	.795	.677
10	.680	.639
20
25580
30	.604
50	.531	.500
100	.372
200	.350

The case of potassium bromide is particularly interesting

as a much weaker solution produces measurable increase of absorption.

Potassium Bromide.

Concentration in 10^{-6} normal.	Thickness 15 mm.	Thickness 7.5 mm.
5	.910
10884
12.5	.750
20785
25	.573	.696
33.3590
50	.407	.505
100	.334	.342
250	.245
1000199
50,000	.191

It was not found possible to obtain consistent results with tap-water, different specimens giving, with 15 mm. thickness, absorption ratios ranging between .114 and .173. The approximate analysis of Oxford tap-water, as regards the four principal constituents, is as follows :—

NaCl	2.8 parts by weight in 100,000
MgSO ₄	2.1 " " "
CaSO ₄	2.8 " " "
CaCO ₃	22 " " "

On making up an artificial tap-water of the above composition, it was found that the ratio of electrometer deflexion as compared with distilled water was .674, very much higher than in the natural product. Before entering into the question of the origin of the discrepancy, it is desirable to indicate the relative efficacy of the constituents. The ratio giving the absorption for the *separate* constituents at the above strengths were

NaCl804
MgSO ₄966
CaSO ₄853
CaCO ₃785

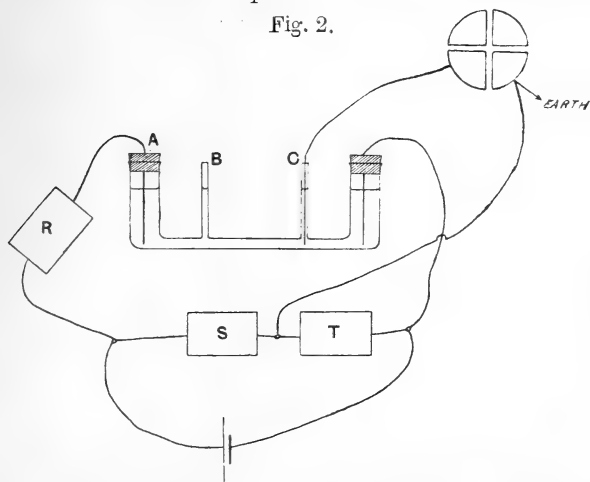
A mixture of the first three, an artificial tap-water with the calcium carbonate absent, gave a ratio .747. The effect of this last constituent is therefore less in combination than might be inferred from its separate effect. A moment's reflection will, however, show that this should be so, for the

source of light is composite, and it is possible and even probable that the wave-lengths chiefly absorbed by calcium carbonate are acted on by one or other of the other salts, and thus the addition of single extra salts is relatively ineffective. This phenomenon is also manifested when experiments on boiled and filtered tap-water are performed. The two fluids exhibit practically the same ratio, the original tap-water being in fact very slightly more transparent.

The large difference of absorption between natural and made-up tap-water may be conceived as due to three causes. Firstly, we have the possible efficacy of small suspended particles or even of micro-organisms in scattering the light. Then there may be a difference of dissociation of the salts in the two cases, or the preponderating cause of the greater absorption of natural tap-water may be the presence of a large number of salts in small quantities. We have seen that ordinary filtration has little effect. If a specimen of tap-water is sterilized and then passed through a porcelain filter, it is found to be rendered more instead of less opaque, although the process of sterilization removes the carbonate of calcium. It is quite evident, therefore, that the first of the three causes of the discrepancy is not of great power.

In order to find out the difference of dissociation, if any, the electrical resistances of the two fluids were found. I take this opportunity of describing the method employed, which is one that has been in use in the laboratory for some years, and seems free from the objections that can be urged against the use of the telephone. The connexions are shown

Fig. 2.



in the accompanying figure. The electrolytic cell is provided with two side tubes B, C into which a platinum rod can be

dipped. R, S, and T are resistance-boxes, R being large and the sum of S and T being always maintained constant. The wire which is shown connected to C can also be joined to A and B. Let S_1, S_2, S_3 be the values of S when contact is made with A, B, and C respectively and the electrometer is brought to zero. Let i be the current through the electrolytic cell, and j that through the boxes S and T. Then we have by Ohm's law

$$Ri = S_1j.$$

$$[\text{resistance of BC}] \times i = (S_3 - S_2)j.$$

$$\text{Hence, resistance of BC} = \frac{R(S_3 - S_2)}{S_1}.$$

The method can be made very accurate by using about 16 volts in the battery, as great sensitiveness is thereby obtained. The effect of polarization is eliminated altogether, and not merely reduced to a minimum, as when intermittent or alternate currents are employed.

The following results for the resistances of the same column of the various liquids were obtained. To get the specific resistance the figures should be divided by about 160. The experiments were performed at temperature 13°C .

Solution.	Resistance in ohms.
Tap-water	2.73×10^5
Do. boiled	2.90×10^5
Artificial tap-water	2.66×10^5
Do. without CaCO_3 ...	1.38×10^6
Distilled water used.....	2.63×10^7

Now it is doubtless a coincidence that the first and third figures should agree so nearly, but it is evident that no marked difference of dissociation is indicated. As regards the disagreement of the values for boiled tap-water and artificial tap-water without CaCO_3 , it should be mentioned that the boiling was not prolonged, as it was not desired to concentrate the remaining constituents more than necessary. Much of the carbonate may thus have remained in solution.

We now see that the most probable cause of the superior absorption of real tap-water is the existence of large numbers of different salts in very small quantities. The more varied the constituents the more chance there is of covering the whole range of the ultra-violet spectrum, and a glance at the curve in fig. 1 shows that small concentrations are relatively very effective. Absorption is to be particularly anticipated if coloured salts are present. Now the pipes of the laboratory water-supply are partly iron ones; it was consequently

found desirable to make an experiment on an iron salt at great dilution. Solutions of iron alum, $\text{Fe}_2.3\text{SO}_4.\text{K}_2\text{SO}_4.24\text{H}_2\text{O}$, having strengths in the salt of 1 in 100,000 and 1 in 1,000,000, were made up, and the absorption ratios were respectively $\cdot 679$ and $\cdot 924$. The latter solution is 1-20th the strength in salt of the weakest of the four principal constituents, and thus it is evident that a number of such impurities are quite likely to be able to bridge over the gap between the natural and made-up tap-water.

5. From the above it might be concluded that the absorption of ultra-violet light might be a delicate enough test to distinguish between different kinds of distilled water. But I have been unable to detect any difference. The specimens of distilled water available in the laboratory gave identical results, within the limits of experimental error, with two specimens of greater purity. One of these was of conductivity $\cdot 8 \times 10^{-6}$, and was also sterilized.

It need hardly be said that no mathematical formula can be given for the form of the curves in fig. 1. Apart from the fact that the non-homogeneity of the source of light prevents the simple exponential law of intensity from holding, it has been shown by Griffith * that in any case the intensity of the transmitted light is not proportional to the electrometer deflexion.

During the experiments I have been much helped by the advice of Professor Dreyer, and he has been kind enough to sterilize some of the solutions. Mr. D. H. Nagel has very kindly furnished me with some very pure distilled water. Lastly, I have to offer my thanks to Professor Townsend, in whose laboratory the work was performed, for his encouragement and suggestions during its progress.

LXVI. *The Dissociation of Water Vapour.*

By ALFRED HOLT, Junr.†

TWO years ago the author published in this Journal a short account of some experiments on the dissociation of water vapour by an incandescent platinum wire (Phil. Mag. 1907, p. 630). The percentage decomposition at a temperature of about 1000° (abs.)‡ was in close agreement with the values previously published by Nernst and Wartenburg (*Göttingen Nachrichten*, 1905), and Langmuir (Journ.

* I. O. Griffith, Phil. Mag., Aug. 1907.

† Communicated by the Author.

‡ All temperatures mentioned in this paper are absolute, unless stated to the contrary.

Amer. Chem. Soc. 1906, xxviii. p. 1357), but at 2000° the percentage was 1/7 the amount obtained by these authors.

The present communication contains an account of a repetition of the author's previous experiments.

It has not been found possible to determine the dissociation of water vapour with any accuracy for temperatures above 1650°, as it then becomes very difficult to calculate the temperature of a platinum wire from its resistance. The author's already published numbers for temperatures above this point are therefore of little value.

The errors in the determination below 1650° arose almost entirely from the fact that at the time the work was being carried out the author did not realize that the temperature-resistance curve for platinum approximated to a straight line above 1300°, and could not be calculated from the Heycock and Neville formula.

It will be seen from the following table that if the old temperatures are recalculated the values for the percentage decomposition do not greatly differ from those of other experimenters. The variation that remains results from a number of minor causes which have been corrected in the present research.

Old Temp.	Recalculated Temp.	Per cent. decomp. at 760 mm. determined experimentally.	Per cent. decomp. at 760 mm. calculated for new temp.
1220°	1220°	0·00108	0·001
1350	1318	0·00175	0·003
1500	1435	0·00887	0·0102
1540	1465	0·015	0·0158
1560	1480	0·0158	0·018
1630	1530	0·01842	0·027
1640	1540	0·0246	0·031

Although the method adopted by the author in the present and the previous research resembles that of Langmuir in that the decomposition of the water vapour in both is brought about by an electrically heated platinum wire, yet it differs in one important feature, namely, the wire is heated in a known volume of vapour at low pressure until equilibrium is established, instead of in a current of steam at atmospheric pressure.

The advantage of using water vapour at low pressure is that the amount of decomposition increases rapidly as the

pressure of the vapour diminishes, and as one of the experimental difficulties is the estimation of the electrolytic gas produced, it follows that the larger the amount the smaller the error in its estimation. The percentage decomposition was calculated for atmospheric pressure by the following formula :—

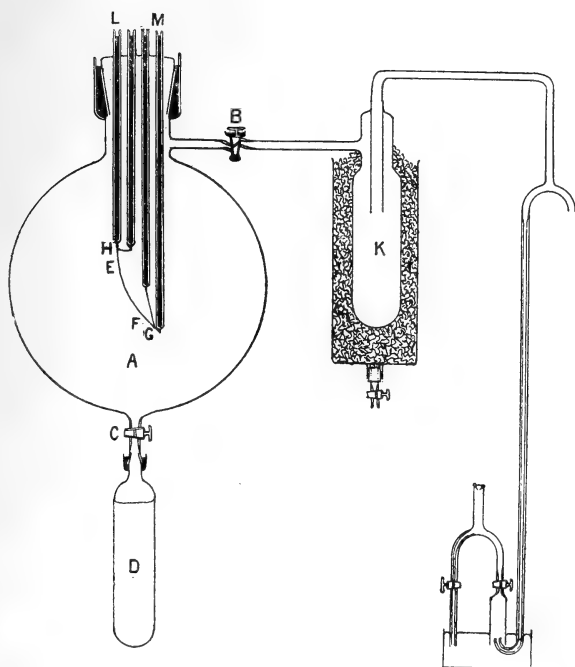
$$D_1 = D \times \sqrt[3]{\frac{P}{760}},$$

where D is the percentage decomposition at a water-vapour pressure P , and D_1 the decomposition at atmospheric pressure (760 mm.).

The results obtained between 1220° and 1640° are in good agreement with those of other experimenters, and, once the apparatus is set up and calibrated, the experiments can be performed easily and with expedition.

The apparatus is shown in fig. 1. It is essentially the

Fig. 1.



same as that previously described, but has various minor improvements in order to minimise possible errors.

The large glass bulb A had a capacity of 3948 c.c. at 15° C., and was connected by a ground-glass joint to the bulb D, which contained distilled water and a thermometer. The platinum wire EF was heated by the current led in through the mercury leads LM, but the resistance was only determined for the distance GH in order to avoid the error due to the cooling of the ends of the wire by the thick leads.

A thermometer was hung inside the globe A to determine the temperature of the vapour.

Each experiment was carried out as follows:—The whole apparatus being evacuated, the tap B was closed and the bulb A allowed to fill with water vapour. The tap C was left open until the water in D and the vapour in A had assumed the same temperature, and this had been noted. Then the tap C was closed and the wire heated to the required temperature until equilibrium was reached.

The wire was then allowed to cool, and the taps B and C being opened, the gases in the bulb A were pumped out through the condenser K surrounded by a freezing-mixture in order to condense water vapour.

The electrolytic gas was collected by a pump and analysed by explosion.

The platinum wire EF was about 7 cms. long and 0.5 mm. diameter, and gave a value for $\delta = 1.54$. At the ends it was welded to short pieces of thick platinum which communicated with the mercury leads LM. Extremely fine platinum wires, which would produce hardly any cooling effect, were welded on to it about 2 mm. from each end at G and H, and the resistance was determined between these two points.

The temperature of the wire up to 1300° was calculated from the resistance by the Heycock and Neville formula

$$\theta = \frac{R_t - R_0}{R_{100} - R_0} 100$$

$$t - \theta = \delta \left\{ \left(\frac{t}{100} \right)^2 - \frac{t}{100} \right\},$$

where θ is the temperature on the platinum thermometer, and t the temperature on the air thermometer.

But for higher temperatures, since the temperature-resistance curve becomes asymptotic to a straight line, the formula proposed by Langmuir has been employed:—

$$T(\text{abs.}) = 344.4 \frac{R_t}{R} - 247.$$

The resistance of the wire at 0° C. and 100° C. was determined from time to time during the course of the research, but it remained practically constant unless the wire was heated above 1670°. The irregularities in the resistance determination which begin about this temperature are due either to the volatilization of platinum, or to changes in the wire itself. On one occasion a wire was heated to 1760° for a short time and then examined, when it was found to have become brittle, and apparently crystalline.

It was noticed by Langmuir that when a wire was heated to a very high temperature in water vapour, the amount of apparent decomposition (*i. e.* volume of electrolytic gas collected) rapidly diminished, but that if the glass surfaces surrounding the wire were cleaned, the decomposition at once increased to the normal amount. He explained this on the supposition that platinum was sprayed on to the glass from the wire, and that it caused recombustion of the electrolytic gas. Platinum was undoubtedly deposited on the walls of the large bulb used by the author, but a number of experiments on this point showed that the amount of electrolytic gas which was caused to recombine was negligible, as, owing to the size of the bulb, the platinum deposit did not get heated sufficiently to exert a strong catalytic effect. Practically the same values were obtained before and after cleaning the globe.

The electrolytic gas resulting from the decomposition of the water vapour was collected and analysed in a vessel of the shape shown in fig. 1.

This gas-collector is a modified form of that previously used by Chapman and Lidbury (*J. C. S. Trans.* 1902, p. 1301), but it is specially suitable for the estimation of small quantities of gas, since the volume is determined by the weight of mercury displaced.

The collected gas always contained an excess of hydrogen. This varied considerably in amount, and did not seem to bear any direct relationship to the volume of electrolytic gas; sometimes it would be almost 30 per cent. of the total amount of gas, and at other times not more than 1 per cent. or 2 per cent.

A number of experiments undertaken to find the cause of the variations in the amount of this excess gave the following results:—

- (1) Moist oxygen is absorbed to a far greater extent than was imagined by the mercury in the fall tube of the pump. When the pump was working slowly,

or the volume of gas to be collected was very small, the percentage of oxygen absorbed was large, but if a considerable volume of gas was rapidly collected the percentage absorption diminished.

- (2) When the taps B and C were opened in order to pump off the gas and water vapour at the end of each experiment, the hydrogen, on account of its great power of diffusion compared with oxygen, passed rapidly through the condenser to the pump, leaving some of the oxygen behind condensed on the glass surfaces of the apparatus and dissolved in the water in the condenser. On carefully warming the glass this oxygen could be pumped out. The larger the amount of electrolytic gas produced the smaller the percentage of oxygen remaining on the glass surfaces. No perceptible quantity of oxygen was absorbed by the platinum wire, nor was hydrogen evolved from it when it was heated to incandescence.

This excess of hydrogen was always estimated and calculated as electrolytic gas.

The two following tables give the author's results.

The first consists of experiments carried out at almost constant temperature, and under as similar conditions as possible; the second gives the values obtained at a series of temperatures ranging from 1233° to 1637°.

TABLE I.

Temp. abs.	Percentage decomposition at 760 mm. pressure.	K.
1620°	0.05107	3.786
1621	0.05421	3.808
1620	0.04965	3.774
1620	0.04686	3.749
1620	0.05102	3.786
1620	0.05116	3.787
1620	0.04808	3.760
1620	0.04605	3.741
1620	0.05014	3.778
1620	0.05037	3.780
1620	0.05111	3.786
		Mean = 3.776

TABLE II.

Temp. abs.	Percentage decomposition at 760 mm. pressure.	K.
1233°	0.001087	3.768
1273	0.001864	3.783
1413	0.008366	3.768
1413	0.00906	3.803
1464	0.01658	3.857
1487	0.01612	3.753
1598	0.04992	3.849
1598	0.04963	3.844
1605	0.04719	3.802
1611	0.04996	3.806
1614	0.05263	3.818
1625	0.05855	3.830
1628	0.05716	3.809
1628	0.05628	3.803
1633	0.05825	3.805
1636	0.0591	3.797
1637	0.06118	3.810
		Mean=3.806

The third column in these tables contains the values for the equilibrium constant calculated from the percentage decomposition.

Three formulæ have been derived thermodynamically by Nernst and Wartenburg for calculating this constant K. The first was published in the *Göttingen Nachrichten* in 1905, and in a somewhat simplified form was used by Langmuir:—

$$\log \frac{2(100x)^3}{(2+x)(1-x)^2} = 11.51 - \frac{25030}{T} + 2.65 \log \frac{T}{1000} - 0.00055(T-1000);$$

neglect x compared to 1, and divide by 3,

$$\log x = 3.83 - \frac{8343}{T} + 0.88 \log \frac{T}{1000} - 0.00018(T-1000).$$

T is the absolute temperature, x the percentage decomposition. The constant K has therefore the value 3.83. At high temperatures it is necessary to substitute for x the value

$$\frac{x}{\sqrt[3]{\left(1 + \frac{x}{2}\right)(1-x)^2}}.$$

The experimental value for K determined by Langmuir from this formula is 3.79, which represents a difference of about 7 per cent. in the amount of decomposition from that

observed by Nernst and Wartenburg, for an alteration of 0.01 in the value of K makes 2.3 per cent. change in x .

The author has employed the same formula in calculating his values for K , and obtained the mean value for all his experiments of 3.791, which is identical with that of Langmuir.

The values in Table II. are, however, probably more correct than those in Table I., since they cover a temperature range of 400°.

The two other formulæ proposed by Nernst and Wartenburg were published three years ago (*Zeit. Physik. Chem.* 1906, li. p. 534).

The first of these,

$$\log \frac{2x^3}{\left(2 + \frac{x}{100}\right)\left(1 - \frac{x}{100}\right)^2} = 11.38 - \frac{25030}{T} + 2.40 \log \frac{T}{1000} - 0.00016(T-1000),$$

gave results which, when compared with those obtained by their experiments, were somewhat too high at high temperature, but were in good agreement at 1370°.

The second formula,

$$\log \frac{2x^3}{\left(2 + \frac{x}{100}\right)\left(1 - \frac{x}{100}\right)^2} = 11.46 - \frac{25030}{T} + 2.38 \log \frac{T}{1000} - 1.38 \times 10^{-4}(T-1000) - 0.685 \times 10^{-7}(T^2-1000^2),$$

gave results which were too high at about 1370°, but became more and more in agreement with the experimentally determined values as the temperature increased. If in both these formulæ x compared to 1 is neglected, and then the whole formula is divided by 3, the constant K has the value 3.79 or 3.82; and as one of these formulæ gives too high results where the other gives too low, and *vice versa*, it is evident that if the experimental values of Nernst and Wartenburg are correct, the constant must lie between them. Their mean 3.805 will be seen to be practically the same as that from the experiments given in Table II.

The values obtained by Löwenstein (*Zeit. Physik. Chem.* 1906, liv. p. 715) for the decomposition of water vapour at very high temperatures (up to 1968° abs.) are in agreement with the values calculated for $K=3.80$.

As these experiments were performed in a totally different manner from those of Nernst and Wartenburg, or Langmuir,

they confirm 3.80 as the most probable value for the equilibrium constant of water vapour, hydrogen, and oxygen.

An attempt was made to determine the lowest temperature at which water vapour could be decomposed by a hot platinum wire.

For this purpose, the taps B and C were both opened and a steady stream of water vapour was drawn through the bulb A while the temperature of the wire was gradually raised. The wire was first heated to 770°, and then raised about 25° C., each half hour. No oxygen or hydrogen was collected by the pump until the temperature was between 1023° and 1043°, when very small quantities of electrolytic gas (containing an excess of hydrogen) were obtained.

At this temperature the percentage decomposition is only about 28×10^{-6} per cent., so the greater quantity of the electrolytic gas produced would dissolve in the condensed water.

The fact that it was found possible to collect gas at so low a temperature as 1023° shows that the decomposition of the water vapour must begin at a considerably lower temperature, probably a few degrees above that at which oxygen and hydrogen would combine under similar conditions.

The University, Manchester.

LXVII. *Condensation of the Radium Emanation.* By
E. RUTHERFORD, F.R.S., Professor of Physics, University
of Manchester*.

RUTHERFORD AND SODDY first showed in 1903 that the radium emanation condensed from the gases with which it was mixed at a temperature of about -150° C. At that time only small quantities of radium preparations were available, so that the partial pressure of the emanation with the gases with which it was conveyed was exceedingly small. Notwithstanding the very minute quantity of emanation present, the temperatures of complete condensation and of complete volatilization were found to be sharply marked, and did not differ from each other by more than a few degrees. Some evidence was obtained that the emanation had a vapour-pressure like an ordinary gas.

This property of condensation of the emanation in liquid air has proved invaluable in all later researches as a means of separating the emanation from the inactive gases with which it is mixed.

* Communicated by the Author. A preliminary account of the results was communicated as a letter to 'Nature,' Feb. 18, 1909.

Using large quantities of radium, Sir William Ramsay and Cameron observed that the emanation, condensed in a glass tube surrounded by liquid air, could be gradually removed by continuous pumping, indicating that the emanation exerted a sensible vapour-pressure even at that low temperature. The writer has found that the rate of removal of the emanation by pumping increases rapidly as the temperature of the emanation-tube approaches the temperature of complete volatilization of the emanation.

The temperature of condensation, viz. -150°C ., found by Rutherford and Soddy corresponded to the liquefaction-point of the emanation under a very low pressure. If the emanation behaves like an ordinary gas, the temperature of initial condensation should rise with increase of pressure of the emanation. It was consequently of interest to examine how the condensation-point of the emanation varied with pressure, and to fix its boiling-point under atmospheric pressure.

Special experimental methods are necessary in order to determine the vapour-pressure of the very small volume of emanation available. It has been shown in the experiments of the writer * that the volume of pure emanation from one gram of radium in equilibrium is about 0.6 cubic mm. at normal pressure and temperature. This is in good agreement with the calculated value, viz. 0.585 cubic mm., which has been deduced by Rutherford and Geiger †. In the present experiments, the amount of emanation, available after the process of purification, was equivalent to the equilibrium amount from 140 mgs. of radium. Taking the calculated volume of the emanation, this corresponds to a volume of pure emanation of 0.082 cubic mm.

In order to obtain a column of gas of several centimetres length at atmospheric pressure, it was consequently necessary to employ capillary tubes of fine bore. In the experiments recorded later, glass capillary tubes were employed of diameter varying between about 0.05 mm. and 0.15 mm.

After purification of the emanation in the manner described in a previous paper ‡, the emanation was allowed to expand into a vertical glass reservoir and then compressed by raising the mercury into the capillary tube fixed at the top. This tube of length nearly 20 cms. was bent twice at right angles, so that the free end of length about 8 cms. was vertical and dipped downwards. The end of the capillary, of external

* Phil. Mag. Aug. 1908.

† Proc. Roy. Soc. A. lxxxi. p. 162 (1908).

‡ Rutherford, Phil. Mag. Aug. 1908.

diameter about 1 mm., was immersed in liquid pentane contained in a small unsilvered Dewar cylinder. The temperature of the bath could be varied by circulating liquid air through a glass U-tube placed in the liquid. The temperature of the bath, which was kept well stirred, was determined by means of a nickel-iron thermo-junction in series with a D'Arsonval galvanometer. The deflexions on the scale were calibrated by keeping one junction in a freezing-mixture and immersing the other successively in (1) a paste of solid carbon-dioxide and ether ($-78^{\circ}2$), (2) boiling ethylene ($-103^{\circ}5$), and (3) liquid air whose percentage of oxygen was determined.

As the complete purification of the emanation is a long and tedious process, and it is difficult to keep it pure over a wide range of observations, many of the experiments were made with emanation of about 50 to 60 per cent. purity. The true volume of the emanation present was determined by comparing its γ ray activity by means of an electroscope with that of a standard radium preparation, assuming that the true volume of the emanation from one gram of radium is 0.585 cubic mm. The actual volume occupied by the emanation and impurities was measured in the capillary at atmospheric pressure. By comparison of the calculated with the observed volume, the percentage of impurity was determined and also the correction to be applied to the observed pressure to give the true partial pressure of the emanation.

In the experiments on the condensation-point of the emanation for pressures above 5 cms., a capillary tube of mean diameter .05 mm. was used. The cross-section of the capillary was found to be elliptical in shape, the axes being .048 mm. and .052 mm. respectively. In this capillary the emanation from 100 mgrs. of radium would occupy a length of 3.0 cms. It was found necessary to use such a fine bore in order to be able to cool down the end of the capillary to the temperature of the pentane bath and yet to keep the mercury column from freezing.

The point of condensation was found in most cases to be well defined. At the moment of condensation a brilliant phosphorescent point of light due to condensed emanation appeared at the extreme end of the capillary. The temperature of the pentane bath was noted at the moment the phosphorescent point disappeared. This corresponded to the temperature of initial condensation at that particular pressure of the emanation.

Preliminary experiments made in this way showed that the condensation temperature of the emanation like that of all

gases rose with increase of pressure of the emanation. If the emanation started condensing at a particular temperature, lowering of the pressure at once caused a rapid volatilization of the condensed emanation.

A number of experiments were made of the initial condensation-point of the emanation at atmospheric pressure. This was found to be about -65°C. , the temperature of the pentane bath being measured, both by the thermo-couple and a pentane thermometer, and the pressure being kept constant. This fixes the true boiling-point of the emanation as -65°C. or 208° absolute. Since the emanation gives out heat, the temperature of the inside of the capillary was no doubt slightly higher than that of the pentane bath. The experiments are not, however, of sufficient precision to introduce small corrections of this kind.

A number of experiments were made on the vapour-pressure of the emanation, using a paste of solid carbon dioxide and ether to give a constant temperature ($-78^{\circ}\cdot 2\text{C.}$). As it was difficult to view the end of the capillary through the opaque paste, it was found convenient to remove rapidly the refrigerant, and observe at the moment of removal whether the emanation was condensed in the tube. It was found in this way possible to fix the point of initial condensation with considerable accuracy. An increase of one per cent. in the pressure was sufficient to cause a transition from no condensation to well-marked condensation.

The mean of experiments showed that the emanation commenced to condense at $-78^{\circ}\cdot 2$ at a pressure of 23 cms. It was found that the amount of impurity with the emanation had no influence on its condensation-point when the true partial pressure of the emanation was deduced in the manner previously discussed. An interesting effect was noticed in these experiments. On removal of the refrigerant, occasionally some of the paste adhered to the capillary. The rapid evaporation of this caused a local lowering of the temperature sufficient to cause a marked condensation of the emanation at points on the tube, even though no condensation was observed in the tube when in the bath itself.

The following table gives the vapour-pressure of the emanation at various temperatures.

Vapour-Pressure.	Temperature.
76 cms.	-65°C.
25 "	-78°
5 "	-101°
$\cdot 9$ "	-127°

A number of experiments have been made at still lower pressures, but the results of these will be reserved for a later paper. The vapour-pressure curve of the emanation is similar in general respects to that of carbon dioxide although, as we have seen, the boiling-point of the emanation is somewhat higher.

An unexpected effect was observed in these experiments which is still under examination. It was found that when the pressure was kept constant, the emanation did not all condense when once the condensation had started, but a considerable raising of the pressure was necessary to produce complete condensation. I do not think this effect could be ascribed to slowness of diffusion of the emanation in the capillary tube. It appeared as if the emanation were not homogeneous and that some of the emanation condensed at a lower temperature than the remainder. An investigation * in conjunction with Mr. Tuomikoski has lent support to this point of view. It has been found that, on condensation of the emanation, the uncondensed part which is pumped off has on the average a slower rate of decay than the part condensed. A more complete determination of the vapour-pressure curve of the emanation is withheld until this point has been more completely examined.

Experiments with Liquefied Emanation.

If the emanation in the capillary tube at about atmospheric pressure is plunged suddenly in a refrigerant well below the temperature of initial condensation, the emanation is condensed locally at several parts of the tube, probably at points where the glass is thinnest. The liquid emanation causes an intense greenish coloured phosphorescence on the walls of the tube. If the liquid emanation is concentrated in the bottom of the capillary, apart from the local phosphorescence of the glass, it appears practically colourless when viewed by a microscope by transmitted light. A very different effect is observed when the emanation is condensed in liquid air at the bottom of the capillary. A few seconds after the beginning of condensation, the emanation shows a reddish tinge and rapidly becomes orange-coloured. It is difficult to be certain whether the emanation under these conditions is in the liquid or solid state. The term "liquid emanation" will be used for convenience in describing the effects observed. If kept in liquid air, the condensed emanation viewed under its own light in a microscope retains its colour unchanged

* A preliminary account of this work was communicated to Manch. Lit. and Phil. Soc. March 23, 1909

with time. Viewed with the naked eye, the end of the capillary containing the liquid at first appears rose-coloured, but in the course of an hour or two becomes greenish. This is no doubt due to the increased local phosphorescence of the glass, due to the radiation from the active deposit formed by the condensed emanation. If the emanation is volatilized its colour instantly disappears, showing that the colour is a property of the liquid or solid emanation at low temperature when bombarded by its own α particles.

The extreme end of the capillary where it had been drawn off was conical in shape. With care, the whole of the liquid could be concentrated in this glass cone. This was most simply done by applying a pad of cotton-wool soaked in liquid air to the end of the capillary. The moment the emanation was all condensed, the capillary was plunged into liquid air. Under these conditions, the volume of the capillary occupied by the liquid could be examined at leisure by a microscope. The colour of the condensed emanation made it comparatively easy to locate its distribution. After several trials, the whole of the emanation was liquefied in the extreme tip of the capillary and did not occupy a length of more than a fifth of a millimetre. Knowing the diameter of the capillary the volume occupied by the liquid could be estimated approximately. This volume was certainly not greater than 1.2×10^{-4} cubic millimetre.

The amount of emanation in the capillary corresponded to 100 milligrams of radium, and was consequently equal to about .06 c.mm. We thus see that the volume of the liquid emanation in liquid air was certainly not greater than 1/500 of the volume of the gas at normal pressure and temperature. Taking the emanation as a monatomic gas of atomic weight 222, it can readily be calculated from the above data that the density of liquid emanation is not less than 5. No doubt if a much finer capillary were used in which to condense the emanation, a more accurate estimate could be made.

It has so far not been found possible to determine directly the density of the emanation on account of the very small quantity available for experiment. Its atomic weight can, however, be deduced with considerable confidence from radioactive considerations. It has recently been shown by Rutherford and Royds* in a decisive experiment that the α particle is an atom of helium. Since the atom of the emanation is derived from the radium atom (atomic weight 226) by the expulsion of an α particle, its atomic weight should be 222. The absence of combining properties of the emanation

* Phil. Mag. Feb. 1909.

indicates that it is an inert gas, and similar in that respect to the group of monatomic gases. Taking the view that it is monatomic, the emanation is the heaviest gas known with a density 111 times that of hydrogen.

For purposes of comparison, the atomic weight, boiling-point, and density of liquid of the heavier monatomic gases are given below.

	Argon.	Krypton.	Xenon.	Radium Emanation.
Atomic Weights	39.9	82	128	222
Absolute Boiling-point...	86°.9	121°.3	163°.9	208°
Density of liquid at } Boiling-point ... }	1.212	2.155	3.52	5?

It is seen from the above table that the boiling-point of xenon is about a mean between that of krypton and the emanation. From the increase of density of the liquid with atomic weight, it might reasonably be expected that the density of liquid emanation should be about 6—a result, as we have seen, not inconsistent with experiment. In a similar way, it is possible to form some idea of the probable critical pressure and temperature of the emanation.

I desire to express my thanks to the Radium Commission of the Vienna Academy of Sciences for the loan of the radium preparation which has made this and other work on the emanation possible.

LXVIII. *On the Self-Demagnetizing Factor of Bar Magnets.*
By SILVANUS P. THOMPSON, *D.Sc., F.R.S.,* and E. W. MOSS*.

[Plate XV.]

THIS paper consists of three parts :—(i.) A discussion of the significance and definition of the self-demagnetizing factor of magnets in general, and of bar-magnets in particular; (ii.) a redetermination of the values of the self-demagnetizing factor for bar-magnets of circular section; (iii.) determination of the values of the self-demagnetizing factor for bar-magnets of rectangular cross-sections of various proportions.

PART I.—PRELIMINARY. ON THE SIGNIFICANCE AND DEFINITION OF THE SELF-DEMAGNETIZING FACTOR.

Between any two magnet-poles, whether they are regarded as points, or as regions over which there is a surface-distribution of magnetism, there are magnetic forces. In

* Communicated by the Physical Society: read February 26, 1909.

the space between any two point-poles the intensity of the magnetic field that is due to these poles, at any point in the line joining them, is expressed by the equation :

$$\mathcal{H}_x = \frac{m_1}{(a+x)^2} - \frac{m_2}{(a-x)^2};$$

where the respective strengths of poles are m_1 and $-m_2$; a the half of the distance between them, and x the distance of the point in question from the mid-point between them. The value of this expression in no way depends on the material in the space between the poles, whether non-magnetic or magnetic, or actually magnetized in any manner.

If m_1 and $-m_2$ are numerically equal, the expression becomes :

$$\mathcal{H}_x = 2m \frac{a^2 + x^2}{(a^2 - x^2)^2}.$$

At the mid-point, under the same condition, the intensity has the minimum value of

$$\mathcal{H}_{\min.} = 2m \div a^2.$$

If the space between the two point-poles be regarded as occupied by a thin, cylindrical, uniformly-magnetized steel magnet the ends of which constitute the point-poles in question, then these equations will be the expressions for a self-produced magnetic field acting in a direction which opposes the actual magnetism of the magnet, and tending to demagnetize it. Each portion of the filiform magnet will be acted upon by a demagnetizing field, strongest towards the poles, weakest at the middle. The supposed uniform magnetization of the magnet will of course be unstable. If it were produced, even for a moment, there would at once be a retrocession of a portion of the magnetization from the ends, with a new distribution of the polarity. On the supposition that the middle part of the rod retains still its full flux, the retrocession of the pole would shorten the effective length of the magnet, diminishing the magnetic moment, but increasing any self-demagnetizing internal action. This tendency to produce a retrocession of the pole may operate to different degrees according to whether the bar consist of soft iron, or hard tungsten steel. In either case the retreat of the pole can be only incomplete; because if we suppose the pole to have actually retreated by any given amount—for example 1 centimetre—the end piece of that length will now be subjected to the magnetizing action of the rest of the bar, and will be remagnetized up to a certain point, namely, such that the

reaction of the magnetism of this piece is equal to the magnetizing action of the whole of the rest of the bar, less the demagnetizing reaction of the bar as a whole. The inevitable result is a distributed pole. It cannot remain concentrated at one point, on the end ; it must redistribute itself along the bar with a distribution determined by the conditions of equilibrium at every point.

Also the middle piece of the bar will not be exempt from influence, it, too, must diminish its inherent magnetism, because even in weak fields the magnetism of the hardest steel is subject to cyclical changes ; and because any retrocession of the poles is, *pro tanto*, productive of an increase in the self-demagnetizing force at the middle. Only in cases where this self-demagnetizing force at the middle is less than that which suffices to produce an irreversible change in the magnetism of the steel, that is only in cases where the bar is very long in proportion to its cross-section, can the action at the middle be regarded as negligible.

It is clear then, in general, that for every bar-magnet there will be a self-demagnetizing action the value of which, at the middle of the bar, depends, for a given intensity of magnetization, on the length of the bar relatively to its cross-section, on the permeability of its parts, and on the distribution of its surface-magnetism. Owing to the circumstance that with every kind of steel the permeability is neither constant, nor stands in any simple or even single-valued relation to the flux-density, any calculation of the actual polar distribution for rods or bars is exceedingly complicated and indeed impracticable.

As is well-known, the one and only form of magnet that is practicable for calculation is that of the ellipsoid, the properties of which are that for any and every value of the permeability, and when placed in any uniform field, the surface magnetism is so distributed that the magnetic force which this distribution of polarity exerts in the interior is uniform at every point within. Hence the internal demagnetizing force everywhere within is constant ; the resultant field at every point of the interior (if the structure is homogeneous and isotropic) is also constant, and the internal flux-density cannot but be uniform.

Du Bois and others have determined by experiment the demagnetizing actions of cylindrical rods of various dimensions, and have compared them with ellipsoids of revolution of similar dimensional proportions.

In the case of ellipsoids, it is natural to compare the value of the intensity of the self-demagnetizing force with the value

of the internal magnetization \mathcal{I} , because both of these are uniform throughout the interior. For an ellipsoid of revolution of given axial proportions, whether highly or only slightly magnetized, both \mathcal{H}_d the self-demagnetizing force, and \mathcal{I} , are proportional to one another. By definition \mathcal{I} is the quotient of the magnetic moment by the volume. For a given size of equatoreal cross-section of the prolate ellipsoid, the magnetic moment and the volume are both proportional to the axial length. But for ellipsoids of given equatoreal section and of different lengths, the self-demagnetizing force \mathcal{H}_d (for a given \mathcal{I} , or a given m) does not follow any simple function of the axial length. For small changes of length it is nearly proportional to the inverse square of the axial length, but is accurately expressible only in terms deducible from a rather troublesome elliptic integral. Maxwell and Du Bois (following F. Neumann) have given the general formulæ. But because both \mathcal{H}_d and \mathcal{I} are for an ellipsoid of given ellipticity proportional to one another, it was quite natural to regard the quotient of the former by the latter—that is to say the amount of self-demagnetizing force per unit of intrinsic magnetization—as a sort of natural coefficient, and to recognize it as a *self-demagnetizing factor*. Du Bois (following Maxwell) assigns to it the symbol N . It has a definite value for ellipsoids of revolution of any assigned ellipticity. Thus for an ellipsoid of equatoreal diameter 1 and axial length 10, the value of N is 0.2549 whatever the degree of magnetization. Thus if an ellipsoid of this form be magnetized so that \mathcal{I} has the value 100 c.g.s. units, the self-demagnetizing force within the ellipsoid will everywhere have the value of 25.49 gauss. Denoting the dimension-ratio of axial length l to equatoreal diameter d by the symbol $m = l \div d$ (in Du Bois' notation), then $m^2 N$ varies from 25.49, when $m = 10$, to 80 when $m = 1000$. (See Du Bois, *The Magnetic Circuit*, p. 41.)

But, if we now compare the case of the ellipsoid with that of the cylindrical bar, we find that the matter is not so simple. For with the bar, as stated above, \mathcal{H}_d is by no means uniform throughout the interior, neither is \mathcal{I} . The former has its minimum at the middle point of the axis, while the latter has its maximum at the equatoreal section of the bar. To compute the value of \mathcal{H}_d at the middle point (or at any other) is impossible without knowing the law of surface distribution, and this depends on too many conditions to be of service. But the nett value of \mathcal{H}_d for the entire bar can be easily determined by comparing the B - \mathcal{H} curve of the bar (found

by experiment) with the \mathcal{B} - \mathcal{H} curve of a ring (or infinitely long rod) of the same iron, and taking the difference of the values of \mathcal{H} for some assigned value of \mathcal{B} . On the other hand, values of \mathcal{I} can be found by experiment, either magnetometrically, giving the mean value, or ballistically, giving either maximum or mean according as whether the exploring coil on the bar is wound over its whole length or over its equatoreal zone only. The ratio $\mathcal{H}_d \div \mathcal{I}$ so deduced may still be called the *self-demagnetizing factor*, and values found for rods of different dimension-ratios.

Magnets of other forms, for example the slit toroid, or anchor-ring with a gap in it, and the horse-shoe magnet with parallel limbs of given proportions, will likewise have self-demagnetizing factors of their own, dependent on their geometry and on the distribution of their polarities. With them also, neither \mathcal{H}_d nor \mathcal{I} will have constant values at all points within the substance of the magnet; and for each form therefore the term "self-demagnetizing factor" bears a significance different from that which it possesses for the ellipsoid of revolution or for the cylindrical bar.

All previous writers have defined the term *dimension-ratio* as applied to a bar as the ratio between its length l and the diameter d of its circular section. But when we come to deal with forms of cross-section other than circular, it is inconvenient to use this mode of expression. For if we were dealing with a flat bar of breadth b , the curve for self-demagnetizing factors in terms of the ratio $l \div b$ would not be comparable with those for cylindrical bars in terms of $l \div d$. The preferable way, when such comparison has to be made, is to state a dimension-ratio, for bars of all and every form of section, in terms of the ratio which is borne by the length to the square-root of the area of section. The ratio $l \div \sqrt{A}$, we accordingly propose to denote by the symbol λ . For any given bar we have the relation $\lambda = \pi \times 1.128$.

Part II.—EXPERIMENTAL. ON THE VALUES OF THE SELF-DEMAGNETIZING FACTOR FOR BAR-MAGNETS OF CIRCULAR SECTION.

Several investigators, including Ewing, Fromme, Holz, and Ascoli, have written on the factor of self-demagnetization of cylindrical bar-magnets, and have given experimental values for bars having different ratios of length to diameter. The best-known results are those published by Du Bois, who has compared the values obtained with those for ellipsoids

having similar axial ratios. More recently, Riborg Mann obtained a series of values slightly higher than those obtained by Du Bois, who has accepted them as more correct than his own figures. Ewing's observations ranged over rods the lengths of which varied from 300 diameters down to 50 diameters. Du Bois' results go from a dimension-ratio of 1000 down to one of 10; those of Riborg Mann from one of 300 down to 5. The magnitude of the outstanding discrepancies may be indicated by stating the values found by different observers for the self-demagnetizing factor N for cylinders having a dimension-ratio of 50. For rods of this proportion Du Bois found $N = 0.0162$; Riborg Mann $N = 0.01825$. For the ellipsoid of revolution having the same axial ratio of 50, Du Bois and Riborg Mann agree in assigning the value 0.0181, and presumably the true value for the cylinder is less than that figure. Greater discrepancy is found for shorter cylinders. For a dimension-ratio of 10 Du Bois gives $N = 0.2160$, while Riborg Mann gives $N = 0.25500$.

To clear up, if possible, such discrepancies a research was undertaken in the laboratory at the Technical College, Finsbury.

The bars used were cut from two long rods of best Swedish iron carefully annealed, and for comparison a ring was forged from the same material. To each and all of the rods the same diameter was given, namely, 1.128 cm., in order that each might have a cross-section of precisely 1 sq. cm. After being turned down to approximate size they were annealed, and then finally turned to the precise size required.

The magnetizing coil used to magnetize the rods was a long coil wound on a brass tube 91.4 cm. in length and 4.75 cm. in external diameter. It was carefully overwound with 5800 turns of wire of No. 20 s.w.g., in seven layers. With this coil a very uniform field could be produced of any desired intensity up to $\mathcal{H} = 255$. The uniformity of the field between the ends of this coil was tested by means of a short coil of somewhat smaller diameter, wound on a turned bobbin of hard fibre, of a size fitted to slide inside the brass tube. The wires of this smaller coil were connected with a ballistic galvanometer, the throw of which was observed when the current in the long magnetizing coil was reversed. The field was found to be sensibly uniform for a length of 60 cm.; while the longest specimen of iron was only 40 cm. There was therefore no need to apply any corrections for non-uniformity of field.

The ballistic method was also used for determining the

magnetization of the bars. On the middle of each bar was wound an exploring coil of 10 turns of very fine wire, the breadth of each such coil not exceeding 0.25 cm. The galvanometer was calibrated by the short coil previously mentioned, its dimensions being accurately known. The magnetizing current was measured by a standard commercial amperemeter, the readings of which were calibrated at regular intervals of time by a Crompton potentiometer.

Each specimen was mounted on a carrier by means of which it could be inserted centrally in the middle of the long magnetizing coil. The galvanometer calibration having been effected, a test was made of each bar by subjecting it to a series of reversals in fields varying from $\mathcal{H}=20$ to $\mathcal{H}=255$, the throws of the galvanometer being noted; and for each bar a \mathcal{B} - \mathcal{H} curve was then plotted.

A similar curve having been plotted from the tests made on the ring, the values of the demagnetizing intensity of field \mathcal{H}_d , due to the self-demagnetizing action of the poles of each bar, could then be calculated, for any value of \mathcal{B} , by taking the abscissa, corresponding to that ordinate, in the curve for that bar, and subtracting the corresponding abscissa in the curve for the ring.

Let the field due to self-demagnetization at the mid-point of any bar, for any given flux-density \mathcal{B} , be called \mathcal{H}_d . Let the total impressed field due to the magnetizing coil be called \mathcal{H} ; and let the impressed field required in the ring to produce the same given value of \mathcal{B} be called \mathcal{H}_r . Then

$$\mathcal{H}_d = \mathcal{H} - \mathcal{H}_r.$$

Then since, by definition, the self-demagnetizing factor N has the value

$$N = \mathcal{H}_d \div \mathcal{I},$$

and

$$\mathcal{I} = \frac{\mathcal{B} - \mathcal{H}}{4\pi},$$

we get

$$N = \frac{4\pi \mathcal{H}_d}{\mathcal{B} - \mathcal{H}}.$$

Fig. 1 (Pl. XV.) gives the \mathcal{B} - \mathcal{H} curves for our rods, the dimension-ratios of which varied from 35.6 to 2.66. These curves were sensibly straight lines up to $\mathcal{B}=12,000$, or as high as the curves could be carried. The value $\mathcal{B}=10,000$ was

chosen for the calculation of the self-demagnetizing force and deduction of the self-demagnetizing factor, except for the very short rods in which lesser values of \mathcal{B} were alone available.

Fig. 2 gives as the final result the curve exhibiting the values of the self-demagnetizing factors found, for rods of different lengths, the corresponding values found by Du Bois and by Riborg Mann being added for comparison.

It will be seen (1) that our values are throughout lower than those found by either of these experimenters; (2) that we have carried the determinations down to shorter rods

TABLE I.—Demagnetizing Factors for Cylindrical Bars.

$\frac{l}{d}$	$\frac{l}{\sqrt{\text{Area}}}$	DEMAGNETIZING FACTORS.			
		Cylinder.			Ellipsoid of Revolution.
		Du Bois.	Riborg Mann.	Thompson & Moss.	
2.66	3.0	1.2	
3.55	4.0	0.83	
4.44	5.0	0.618	
5.0	5.64	...	0.6800	0.53	
5.34	6.0	0.483	
6.66	7.5	0.3518	
8.86	10.0	0.233	
10.0	11.28	0.216	0.2550	0.198	0.2549
10.67	12.0	0.18	
13.3	15.0	0.1287	
15.0	16.92	0.1206	0.1400	0.108	0.135
17.72	20.0	0.0826	
20.0	22.56	0.0775	0.08975	0.069	0.0848
25.0	28.2	0.0533	0.06278	0.049	0.0579
26.6	30.0	0.0438	
30.0	33.84	0.0393	0.04604	0.036	0.0432
35.6	40.0	0.0255	
40.0	45.12	0.0238	0.02744	0.0223	0.0266

N.B.—Figures in italics are values got by interpolation.

than those examined by either of them; (3) that the discrepancies between their results and ours are smaller as the dimension-ratios are larger.

The fact that our values are throughout lower than those of Du Bois and Riborg Mann is doubtless due to the circumstance that they used a magnetometric method, whilst we have returned to the ballistic method of Ewing. The values of \mathcal{S} which they employ are the mean values deduced

from the magnetic moment, and are presumably mean values throughout the length of the bar, whilst our values of \mathcal{J} are the values deduced from the action of an exploring coil wound round the equator of each bar, and presumably measure the maximum value of \mathcal{J} . As the self-demagnetizing action of a bar depends on neither the mean value, nor the maximum value of \mathcal{J} , as we have seen, but on a mean that is impossible to calculate unless the actual surface distribution of the magnetism is known, it appeared to us preferable to take the value of \mathcal{J} that can be ascertained with precision at the place where the self-demagnetizing force has its minimum, namely the centre of the bar.

One point of criticism on Riborg Mann's results may be permitted us. To give us confidence in our results, we have throughout used substantial bars of 1.128 cm. in diameter, and have raised the lengths. Riborg Mann used a single cylinder of iron 11.850 cm. in length, originally of a diameter 1.526 cm., therefore of a dimension-ratio of 7.76. This he turned down successively to smaller and smaller diameters until he reached a diameter of 0.237 cm., giving a dimension-ratio of 50. How he contrived to turn so thin a wire is remarkable. It would have a sectional area of only 0.0561 sq. cm. Further, while his cylinder was 11.850 cm. in length, his magnetizing coil was only 30 centimetres long and 4 cm. in diameter. The ends of his rod were therefore at points only $2\frac{1}{4}$ diameters distant from open ends of the coil, where therefore the value of the field would differ by some $2\frac{1}{2}$ per cent. from the value of the uniform field at the middle of the coil.

Part III.—EXPERIMENTAL. ON THE VALUES OF THE SELF-DEMAGNETIZING FACTOR FOR BAR-MAGNETS OF RECTANGULAR CROSS-SECTIONS OF VARIOUS PROPORTIONS.

We are not aware that any previous investigator has determined the self-demagnetizing factor for square bars or flat bars of rectangular section such as are often used in magnetic work.

A priori we should expect the self-demagnetizing factors to be less than for bars of equal section of circular form and equal length; since the greater perimeter of the rectangular forms is magnetically equivalent to giving to the end parts a polar expansion, reducing the reluctance of the air-paths of the external magnetic flux, and so bettering the magnetic circuit. And such has proved to be the case.

The experiments were made in exactly the same manner
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as those for the bars of circular section. Rectangular rods of the softest Swedish iron of various proportions were procured, and reduced by milling-cutter to the required form, so as in every case to have a sectional area of 1 square centimetre; the ratios of breadth to thickness being respectively 1:1; 2:1; 4:1; 6:1, and 10:1. From each of these rectangular rods pieces were cut of lengths of 10, 8, 6, 5, 4, and 3 centimetres respectively. In all 35 different ones were examined. For each of these a *B-H* curve was plotted; and the self-demagnetizing-factors were deduced as before.

In figs. 3, 4, 5, 6, 7, and 8 these various curves are plotted; and in fig. 9 the final results are summed up by plotting the several demagnetizing-factors as functions of λ the ratio of the length to the square-root of the area of section.

Table II. gives numerically the values of the self-demagnetizing factors obtained for various ratios of breadth b to thickness t of the cross-section, and also for various values of λ . The individual bars were carefully gauged for breadth and thickness, and the slight discrepancies (never exceeding 1 per cent. of the intended ratio) were allowed for; but being small they occasioned no difference in the plotting.

TABLE II.—Demagnetizing Factors for Bars of different lengths and equal Sectional Area, having Rectangular Sections from 10:1 to 1:1.

$\frac{l}{\sqrt{A}}=3.$		$\frac{l}{\sqrt{A}}=4.$		$\frac{l}{\sqrt{A}}=5.$		$\frac{l}{\sqrt{A}}=6.$		$\frac{l}{\sqrt{A}}=8.$		$\frac{l}{\sqrt{A}}=10.$	
$b/t.$	N.	$b/t.$	N.	$b/t.$	N.	$b/t.$	N.	$b/t.$	N.	$b/t.$	N.
10.03/1	0.828	10/1	0.586	10.05/1	0.44	10/1	0.354	10.2/1	0.2358	10/1	0.178
5.95/1	0.925	5.96/1	0.66	5.99/1	0.488	6/1	0.3885	5.96/1	0.264	5.99/1	0.19
3.98/1	1.02	3.99/1	0.726	3.98/1	0.528	3.96/1	0.415	4/1	0.28	3.98/1	0.206
2.0/1	1.098	2/1	0.775	2/1	0.575	2/1	0.448	2/1	0.3	2/1	0.22
1.492/1	1.13	1.5/1	0.7980	1.49/1	0.59	1.5/1	0.4645	1.5/1	0.3075	1.495/1	0.224
0.9914/1	1.13	0.99/1	0.8	0.993/1	0.59	0.996/1	0.465	0.993/1	0.224

It will be noticed that the Table records values also for bars having the ratio of 1.5:1; but no curves are given for this ratio, as they were practically the same as those for square

bars. In plotting the $B-H$ curves for these particular bars, it was possible in two cases only to distinguish the curves from those for the square bars, and in these two cases the difference was extremely small.

For equal values of the ratio of l to \sqrt{A} , it was found in general that the self-demagnetizing factor, for bars having a sectional ratio of 2 to 1, was about 93 per cent. of that for bars of square section; while for flat bars, having a sectional ratio of 10 to 1, the value of the self-demagnetizing factor went down to about 75 per cent. of that for bars of square section.

LXIX. *The Absorption of Röntgen Rays.* By C. G. BARKLA, M.A., D.Sc., Lecturer in Advanced Electricity, and C. A. SADLER, M.Sc., Oliver Lodge Fellow, University of Liverpool*.

THE results of experiments that have been made by a number of investigators on the absorption of X-rays are so complicated by a variety of conditions, and frequently appear so inconsistent, that few general conclusions can be drawn from them.

The heterogeneity of the beams used not only masks any peculiarity in the phenomena connected with a particular constituent, but makes exact comparison between the results of different experimenters, and even of the same experimenter, impossible.

In addition to this, as recent investigations have shown†, there are peculiarities in the absorption phenomena which are intimately connected with certain phenomena of secondary radiation; and a knowledge of these is necessary in order to classify and explain the former.

Through our investigations on the secondary X-rays emitted by substances subject to X-rays, we have been enabled to use almost perfectly homogeneous beams, and have become acquainted with the character of the secondary radiation emitted by many elements.

It has been found that each of the elements Cr, Fe, Co, Ni, Cu, Zn, As, Se, Ag, when subject to a suitable primary beam of X-rays, emits an almost perfectly homogeneous beam of

* Communicated by the Authors.

The expenses of this research have been partially covered by a Government Grant through the Royal Society.—C. G. B.

† "Homogeneous Secondary Röntgen Radiations," Barkla & Sadler, Phil. Mag. Oct. 1908, pp. 550-584.

X-rays *, the penetrating power of which is characteristic of the element emitting it. As these penetrating powers vary considerably—the radiation from Cr being very soft and that from Ag fairly penetrating—the above metals furnish us with a series of nine homogeneous beams which can conveniently be used for accurate investigations of many X-ray phenomena. The variation in penetrating power is shown by the following values for the coefficient of absorption in aluminium of each radiation, the coefficient λ being defined by the equation $I = I_0 e^{-\lambda x}$.

Radiator.	Coefficient of Absorption in Al.	
	(ρ = density of Al.)	
Cr	136	ρ
Fe	88.5	ρ
Co	71.6	ρ
Ni	59.1	ρ
Cu	47.7	ρ
Zn	39.4	ρ
As	22.5	ρ
Se	18.9	ρ
Ag	2.5	ρ

Phenomena of Transmission.

Before attempting to make accurate experiments on the absorption of these radiations by various elements, it is necessary to know something of the phenomena accompanying the transmission of X-rays through absorbing substances.

The secondary X-ray phenomena, which have an important bearing on experiments on absorption, may be stated briefly as follows:—

When a beam of X-rays is transmitted through any substance, secondary X-rays of the same penetrating power are emitted by that substance in all directions. The

* Mixed with the homogeneous radiation is an exceedingly weak scattered radiation of the same penetrating power as the primary radiation; but the ionization produced by this is usually negligible in comparison with that of the homogeneous radiation—in certain cases not more than 1 per cent. The radiation from Ag was transmitted through Al in order to get rid of most of the scattered radiation.

The evidence that these rays are X-rays is overwhelming. Some of it has been given in the paper on "Homogeneous Secondary Röntgen Radiations."

distribution of this radiation has, in all the cases investigated, been found to be approximately that expected from a theory of scattering on the æther-pulse theory.

In addition to this, many elements—possibly all when subject to a suitable primary radiation—emit a homogeneous X-radiation which is characteristic of the element emitting it, its penetrating power being independent of that of the primary radiation exciting it.

A primary radiation excites this homogeneous radiation only when it—the primary—is of more penetrating type than the homogeneous radiation.

Regarding the distribution of this type of radiation, it has been shown by one of us* that when a polarized beam of Röntgen radiation is incident on Fe, Cu, Sn, Pb, &c., the homogeneous secondary radiation which is emitted is equally intense in directions in and perpendicular to the plane of that polarization—that is, the intensity of secondary radiation in any direction is independent of the position of the plane of polarization of the exciting primary beam.

Again, the homogeneous radiation from these metals has been found equally intense in a direction perpendicular to the direction of propagation of the primary, and one almost opposite to that of primary propagation †.

It is thus evident, and it has since been further verified, that the homogeneous radiation is equally intense in all directions.

Now the energy of this secondary radiation is so great that in many experiments it would produce enormous errors if not taken into consideration. In measuring the intensity of a beam of X-rays transmitted through an absorbing substance and proceeding in the original direction of propagation, it is therefore necessary either to so arrange the apparatus that the effect of the secondary rays may be neglected or to make a correction for it.

It was also important to ascertain if the beam emerging from the absorbing plate and proceeding in the original direction of propagation was identical in properties with the incident beam, or if it was transformed in any manner.

A few simple experiments showed that all the phenomena observed when a homogeneous beam of X-rays was transmitted through a metal plate placed before the detecting electroscope could be explained qualitatively simply by considering the superposition of the secondary radiation

* Barkla, "Polarized Röntgen Radiation," *Phil. Trans. A.* vol. cciv. 1905, pp. 467–478, and later papers.

† *Phil. Mag.* Feb. 1908, pp. 288–396.

on the primary radiation, diminished in intensity by transmission.

All the phenomena observed may be divided into two groups:—

(1) When a radiation was transmitted through an element whose characteristic secondary radiation was of an equally or more penetrating type, or through an element from which this type of radiation has not been observed, the emergent radiation was identical with the incident radiation.

(2) When a radiation was transmitted through an element whose characteristic radiation was of more absorbable type, the emergent radiation differed from the incident beam in penetrating power.

The following are a few examples of these types of transmission. The first three exhibit no change after transmission, the last three a considerable change.

Radiation.	Substance through which the radiation was transmitted.	Percentage Absorption by transmission.	Substance used to test absorbability.	Percentage Absorption.
Cu	—	0	Cu (·00067 cm.)	25·4
	Cu	74	„	25·3
Cu	—	0	Zn (·00262 cm.)	69·7
	Zn	97	„	68·2
Ag	Al	16	Al (·0208 cm.)	13·8
	„	72	„	14
	„	92	„	13·8
	„	97	„	14·2
Ag	—	0	Fe (·00315 cm.)	30·8
	Fe	51	„	37
	„	79·7	„	36·1
Ag	—	0	Al (·0208 cm.)	14·5
	Fe	51	„	21·7
Cu	—	0	Al (·0104 cm.)	72
	Fe	97·2	„	85·6

From these and many other instances it became evident that only in those cases of transmission in which the primary radiation was able to stimulate a homogeneous secondary radiation was there any appreciable difference between the penetrating powers of the incident and emergent beams. It remained, however, to show that this change was simply due to the superposition of the homogeneous secondary

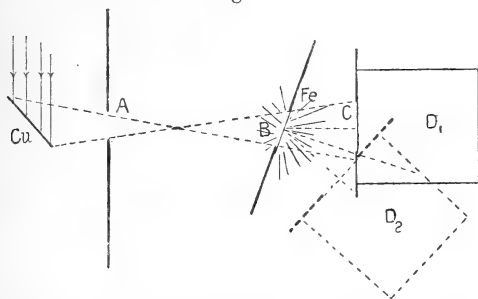
radiation on the primary, and that the primary was itself unaltered.

As an example, Ag radiation, after the scattered rays had been sifted out, was absorbed to the extent of 14·5 per cent. by a sheet of Al 0·0208 cm. in thickness; but after transmission through Fe—51 per cent. being absorbed—the transmitted radiation was absorbed by 21·7 per cent. by the same sheet of Al. It thus appeared much softer. It was seen, however, that this emergent radiation still consisted of an untransformed Ag radiation with a more easily absorbed radiation superposed. For when the transmitted radiation was again passed through a sheet of Al 0·0208 cm. thick, the easily absorbed Fe radiation was absorbed completely and the transmitted radiation was again practically pure Ag radiation, being absorbable to the extent of 14·4 per cent. by the same sheet of aluminium.

In order to ascertain if the constitution of the emergent beam in such a case could be quantitatively accounted for, the following simple experiment was made :—

A thin sheet of Fe was placed in the path of the homogeneous radiation from Cu, so that the angle of incidence of the central ray was about 20° (fig. 1).

Fig. 1.



An electroscope capable of being rotated about a vertical axis through the centre of the Fe sheet B (set in a lead screen) was placed successively in the two positions D_1 and D_2 . In position D_1 it received the direct beam after passing through the thin Fe sheet, and a portion of the secondary radiation emerging from the iron sheet, the central ray being inclined at an angle of about 20° to the normal; in position D_2 it received only the secondary radiation which emerged from the sheet at the same angle on the other side of the normal.

The two secondary beams then suffered approximately the same absorption on emerging from the iron plate.

The ratio of the ionizations in the electroscope in positions

D_1 and D_2 was 177:100, after correction for the radiation from air and the normal ionization. Thus, according to the simple theory of the superposition of the homogeneous secondary radiation on the untransformed transmitted radiation, $\frac{77}{177}$ of the ionization in position D_1 was produced by the transmitted copper radiation, while $\frac{100}{177}$ was the result of the radiation from the iron plate. But the absorption of the radiation entering the electroscope in position D_1 was found to be such as would be obtained if the mixture had contained $\frac{96}{177}$ of iron radiation. Thus the ratio of intensities of the iron radiation in the two directions BD_1 and BD_2 was 96:100.

This is as near equality as could be expected from the nature of the experiment. We conclude, therefore, that the observed phenomena are due simply to the homogeneous radiation uniformly distributed, superposed on the transmitted radiation, and that the radiation proceeding in the original direction is untransformed.

Now, as in experiments on the absorption of such radiations, beams of considerable cross-section must be used to produce measurable effects, an electroscope placed behind the absorbing plate, unless very distant from that plate, must receive a portion of the secondary radiation from that plate. Thus without correction the transmitted beam would appear more intense than it actually is. The correction might have been determined experimentally in the manner shown above. A simple calculation, however, gives us the constitution of the beam received by the electroscope; and consequently the correction that must be applied to determine the true diminution in energy of the primary beam in terms of quantities which may be easily determined.

Let I_0 be the intensity of a parallel beam of homogeneous radiation incident normally on a plate of absorbing material of thickness t ,

then $I = I_0 e^{-\lambda_1 x}$ gives the intensity I at a depth of x ,

where λ_1 is the coefficient of absorption in the substance.

An electroscope placed in the path of the transmitted beam beyond the absorbing plate (fig. 2) receives the beam, together with a portion of the radiation scattered by the plate, and of the secondary homogeneous radiation characteristic of the material of the plate.

It may easily be shown that the former—the scattered radiation—may be neglected in most, and possibly in all, of the cases considered.

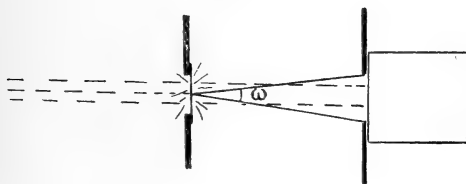
If A is the area of the primary beam, the energy of the secondary radiation emitted by a thin layer of thickness dx in unit time $= kIA dx$,

where k is a constant depending on the character of the incident radiation and on the material of the plate. It will be called the transformation coefficient*.

But of this the fraction $\frac{\omega}{4\pi} e^{-\lambda_2(t-x)}$ enters the electroscope,

where ω is the average solid angle subtended by the aperture into the measuring electroscope at points in this layer, and λ_2 is the coefficient of absorption of the secondary radiation in the substance of the plate itself, when the effect of obliquity of the secondary rays and the absorption in air and the thin window may be neglected.

Fig. 2.



The total energy of secondary radiation entering the electroscope per second

$$\begin{aligned} &= \int_0^t \frac{\omega}{4\pi} kIA e^{-\lambda_2(t-x)} dx \\ &= \frac{\omega}{4\pi} kAI_0 e^{-\lambda_2 t} \int_0^t e^{(\lambda_2 - \lambda_1)x} dx \\ &= \frac{\omega k}{4\pi} \frac{AI_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \end{aligned}$$

\therefore $\frac{\text{Energy of Secondary Radiation entering electroscope}}{\text{Energy of Primary Radiation entering electroscope}}$

$$= \frac{\frac{\omega k}{4\pi} \frac{AI_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})}{AI_0 e^{-\lambda_1 t}}$$

* This coefficient is so called by analogy with the absorption coefficient, $\frac{dE}{dx} = -kI$, where $-dE$ is the energy emitted by a thin layer of unit area and thickness dx in unit time.

But the ionizations produced by these beams are not proportional to their energies, as they have different penetrating powers. If i_1 and i_2 represent the relative numbers of ions produced by the two beams, if of equal intensity and cross-section, in a thin layer of air,

$$\text{then } \frac{\text{Secondary Ionization}}{\text{Primary Ionization}} = \frac{\omega k}{4\pi(\lambda_2 - \lambda_1)} \frac{i_2}{i_1} \left[1 - e^{-(\lambda_2 - \lambda_1)t} \right]$$

All possible cases can be divided into three classes :—

(1) When a homogeneous primary radiation passes through a plate of an element whose characteristic radiation is of equal or greater penetrating power, no appreciable secondary radiation of this type is emitted, i. e. $k=0$. The secondary radiation characteristic of the absorbing element is not then present in the transmitted radiation, and this exhibits no special powers of penetration.

This type of transmission occurs when the radiation from Cr passes through C, Mg, Al, Fe, Ni, Cu, &c., or when Cu radiation passes through Cu, Zn, Ag, &c.

(2) When a homogeneous primary radiation passes through a plate of an element whose characteristic radiation is of more absorbable type and more absorbable *in the plate itself* than is the primary radiation, then the secondary radiation is excited, k is finite and $\lambda_2 - \lambda_1$ is positive.

In this case, as the thickness of the absorbing plate is increased, the beam received by the electroscope gradually approaches a final constitution in which the ratio of the energies of secondary and primary radiations is

$$\frac{\omega}{4\pi} \frac{k}{(\lambda_2 - \lambda_1)}.$$

Such conditions are obtained when a radiation *much* more penetrating than the radiation characteristic of an element passes through that element, as when radiation such as is emitted by Ag is transmitted through Fe, Ni, Cu, Zn, &c.

(3) When a homogeneous radiation passes through an element whose characteristic radiation is of more absorbable type, but more penetrating to the element emitting it, k is again finite and $(\lambda_2 - \lambda_1)$ is negative.

In this case, as the thickness of the absorbing plate is increased, the ratio of the energy of secondary radiation to that of the primary radiation increases indefinitely; that is, the primary radiation becomes ultimately transformed into radiation characteristic of the absorbing substance.

Such a transformation occurs when a radiation like that characteristic of Se passes through Zn, Cu, Ni, Fe, &c., or the radiation characteristic of Cu is transmitted through Fe.

Absorption Experiments.

In the experiments which were made to determine the true absorption of homogeneous beams, the radiator R, used to produce the beam, was placed at a distance of about 7 centimetres from a rectangular aperture A_1 (3 cm. \times 2 cm.) in a lead screen. A second lead screen with a similar rectangular aperture A_2 was placed at a distance of 7 centimetres from the other, and behind the second aperture was placed the thin paper and aluminium face of the electroscope used to measure the intensity of the beams.

The absorbing plates were placed across the aperture A_1 .

As the radiation from R necessarily produced a diverging beam, only a portion of the energy passing through A_1 passed through the aperture A_2 , consequently the secondary radiation from the plate produced greater proportional effects than that calculated for a parallel beam. The ratio of the ionizations due to the secondary and primary beams had to be multiplied by a factor which expresses the ratio of the energy passing through aperture A_1 to that passing through aperture A_2 when no absorbing plate intervened. This was found to be about 4.9.

Thus

$$\frac{\text{Secondary Ionization}}{\text{Primary Ionization}} = 4.9 \times \frac{\omega}{4\pi} \times \frac{k}{\lambda_2 - \lambda_1} \frac{i_2}{i_1} \left[1 - e^{-\overline{\lambda_2 - \lambda_1} t} \right]$$

$$= 4.9 \times .01 \times k \frac{i_2}{i_1} \frac{1 - e^{-\overline{\lambda_2 - \lambda_1} t}}{\lambda_2 - \lambda_1} \text{ approximately.}$$

Now, $k \frac{i_2}{i_1}$ may be got by direct experiment, for $\frac{k}{\lambda_1} \frac{i_2}{i_1}$ is merely the fraction of the energy of primary radiation absorbed which is transformed into or which appears as secondary radiation—as measured by the ionizations produced in a thin film of air.

$\left(k \frac{i_2}{i_1} \right)$ has been determined by one of us by direct experiment*; λ_2 was determined by using the absorbing substance as a radiator also, and so observing the absorption of the radiation characteristic of the substance by the same substance—this, of course, required no correction for secondary

* C. A. Sadler: "Transformations of Röntgen Rays," Phys. Soc. Lond. April 23, 1909.

radiation; λ_1 was determined approximately without the application of the correction term.

Thus as an example—

When As radiation was transmitted through Cu,

$$\begin{aligned}\left(\frac{k i_2}{i_1}\right) &= 368 \times \frac{128.9}{60.7} \\ \lambda_1 \text{ or } {}_{\text{As}}\lambda_{\text{Cu}}^* &= 169 \times \rho_{\text{Cu}} \\ \lambda_2 \text{ or } {}_{\text{Cu}}\lambda_{\text{Cu}} &= 53 \times \rho_{\text{Cu}} \\ \therefore (\lambda_2 - \lambda_1) &= -116 \times \rho_{\text{Cu}} = -1038.\end{aligned}$$

$$\begin{aligned}\therefore \frac{\text{Secondary Ionization}}{\text{Primary Ionization}} &= 4.9 \times .01 \times 368 \times \frac{128.9}{60.7} \times \frac{1}{-1038} \left[1 - e^{(1038)(.00093)} \right] \\ &= .06 \text{ approximately.}\end{aligned}$$

The ionization in the electroscope due to the primary beam when the absorbing plate of Cu was in position was thus found to be about 6 per cent. less than the observed ionization. Similar corrections were made in the other cases of transmission in which a secondary radiation characteristic of the absorbing substance was set up. The maximum correction affected the absorption coefficient by about 4 per cent.

A correction was also necessary in a few cases for the presence in the radiation from Cr of the weak heterogeneous scattered radiation. Though in most experiments the effect of this was inappreciable, when the absorbing plates used absorbed over 90 per cent. of radiation, the much more penetrating scattered rays having been only slightly absorbed appeared in much higher proportion than normally, and introduced an error.

By absorbing the radiation from Cr by various thicknesses of Al and comparing the absorption coefficients calculated from these absorptions, the error introduced in the case of the higher absorptions was obtained. A curve was plotted giving the error corresponding to any particular absorption. This correction was then applied to the calculated values of the absorptions by C, Ag, and Sn—substances which absorb similarly to Al. In no other case could the error, due to the same cause, have been more than 1 or 2 per cent. †, and that in one or two cases only. The values finally obtained for the coefficients are given in the following table. As the

* ${}_{\text{As}}\lambda_{\text{Cu}}$ means the coefficient of absorption of the radiation from As by Cu.

† The absorptions by Pt and Au of the radiation from Cr are excepted. These were too high to be relied upon. The value given in the table for Pt was obtained not by experiment, but by calculation from the observed ratio between the absorption coefficients.

value of λ varies as the density of the absorbing element, the values of the more fundamental quantity $\frac{\lambda}{\rho}$ are tabulated, ρ being the density of the absorbing element. The quantity $\frac{\lambda}{\rho}$ may be called the mass coefficient of absorption.

TABLE I.
Mass absorption coefficients $\left(\frac{\lambda}{\rho}\right)$.

Radiator.	ABSORBER.										
	C	Mg.	Al.	Fe.	Ni.	Cu.	Zn.	Ag.	Sn.	Pt.	Au.
Cr	15.3	126.5	136	103.8	129	143	170.5	580.5	713.7	[516.8]	[507+]?
Fe	10.1	80	88.5	66.1	83.8	95.1	112.5	381	472	340	367
Co	7.96	63.5	71.6	67.2	67.2	75.3	91.5	314	392	281	306
Ni	6.58	51.8	59.1	314	56.3	61.8	74.4	262	328	236	253
Cu	5.22	41.4	47.7	268	62.7	53.0	60.9	214	272	194	210
Zn	4.26	34.7	39.4	221	265	55.5	50.1	175	225	162.5	178.2
As	2.49	19.3	22.5	134	166	176	203.5	105.3	131.5	105.7	106.1
Se	2.04	15.7	18.9	116.3	141.3	149.8	174.6	87.5	112	93.0	100.0
Ag41	2.2	2.5	17.4	22.7	24.3	27.1	13.3	16.5	56.5	61.4

Fig. 3 (p. 750) exhibits the relation between the mass coefficients of absorption in a number of elements and the mass coefficient of absorption in Al—the former being plotted as ordinates and the latter as abscissæ.

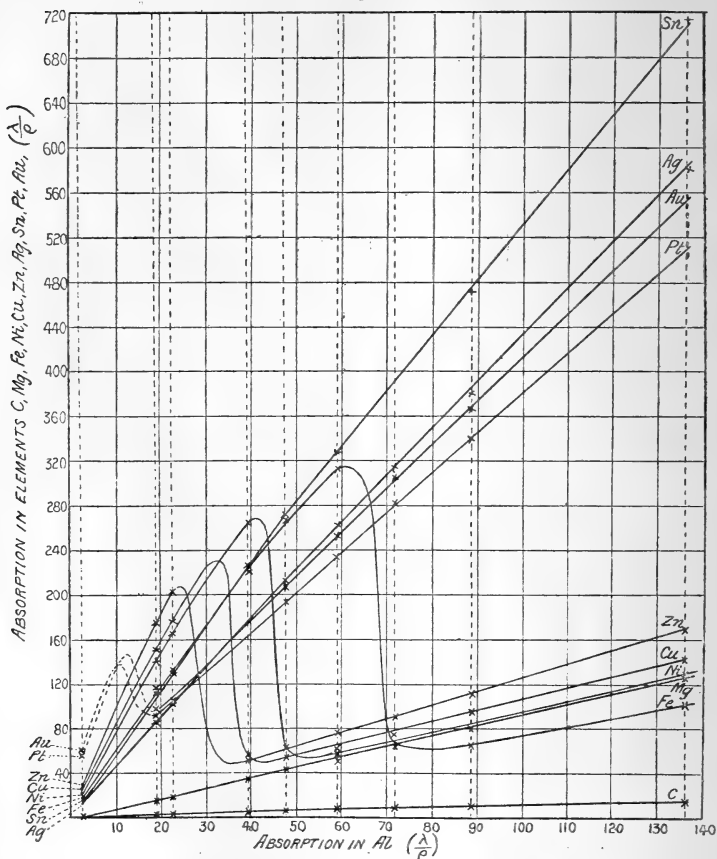
Discussion of Results.

In studying these results it is necessary to know something of the secondary rays emitted by the various absorbing substances.

C, Mg, and Al are elements from which a characteristic secondary radiation has not yet been observed. They certainly do not emit such a radiation in appreciable intensity when subject to ordinary beams of X-rays, unless it be in the form of exceedingly soft radiation, much softer than that of chromium, and such as would be absorbed in a thin layer of air. There is, however, the possibility, if not the probability, of the emission of such a radiation from all elements.

Fe, Ni, Cu, and Zn are among the elements from which a homogeneous characteristic radiation is emitted in considerable intensity when the primary beam is of ordinary penetrating power. Ag emits a homogeneous radiation when

Fig. 3.



subject to a primary more penetrating than this radiation, that is more penetrating than any radiation here used. Sn emits a radiation which on the whole appears slightly more absorbable than the silver radiation, but although it has not been investigated minutely, there is strong reason to believe this consists of two homogeneous radiations, a very soft one, and one either just more absorbable or just more penetrating than that characteristic of Ag.

Pt and Au have been found to emit a radiation which,

from preliminary experiments, appears slightly more absorbable than that from As. Neither has, however, been examined exhaustively.

On comparing the absorption coefficients, it is seen that if we consider simply those absorptions which do not result in the production of a secondary radiation characteristic of the absorbing substance, the ratio of the coefficients of absorption of any radiation by two elements is approximately a constant, whatever be the penetrating power of the primary radiation experimented upon.

Thus if $_{\text{X}}\lambda_{\text{A}}$ denote the coefficient of absorption of a certain radiation X in a substance A,

$$\frac{_{\text{X}}\lambda_{\text{A}}}{_{\text{X}}\lambda_{\text{B}}} = \frac{_{\text{Y}}\lambda_{\text{A}}}{_{\text{Y}}\lambda_{\text{B}}} = \frac{_{\text{Z}}\lambda_{\text{A}}}{_{\text{Z}}\lambda_{\text{B}}}$$

approximately, provided the radiations X, Y, Z do not excite the characteristic radiations in the absorbing substances A, B, and C.

Table II. gives the ratio of the coefficient of absorption in each absorbing substance to the coefficient of absorption in Al for each radiation used in these experiments. The relationship is, however, more clearly shown by the curves in fig. 3.

TABLE II.

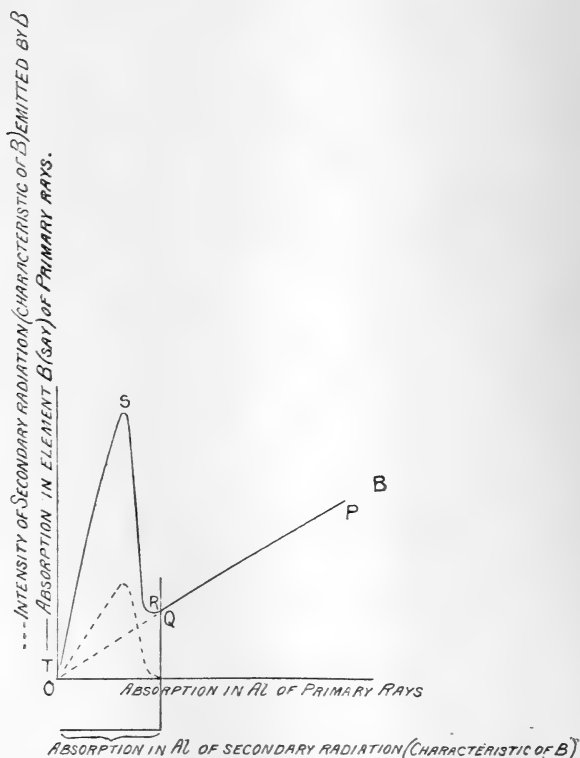
Radiator.	ABSORBERS.									
	$\frac{\text{C}}{\text{Al}}$	$\frac{\text{Mg}}{\text{Al}}$	$\frac{\text{Fe}}{\text{Al}}$	$\frac{\text{Ni}}{\text{Al}}$	$\frac{\text{Cu}}{\text{Al}}$	$\frac{\text{Zn}}{\text{Al}}$	$\frac{\text{Ag}}{\text{Al}}$	$\frac{\text{Sn}}{\text{Al}}$	$\frac{\text{Pt}}{\text{Al}}$	$\frac{\text{Au}}{\text{Al}}$
Cr	·112	·930	·763	·949	1·051	1·254	4·27	5·25	[3·8]	3·73+?
Fe	·114	·904	·747	·947	1·074	1·271	4·31	5·33	3·84	4·14
Co	·111	·887	·928	·939	1·052	1·278	4·38	5·47	3·92	4·27
Ni	·111	·876	5·31	·953	1·046	1·259	4·44	5·55	3·99	4·28
Cu	·109	·866	5·62	1·314	1·111	1·277	4·48	5·70	4·06	4·40
Zn	·108	·881	5·61	6·7	1·408	1·272	4·44	5·71	4·12	4·52
As	·111	·857	5·95	7·29	7·82	9·045	4·68	5·84	4·69	4·71
Se	·108	·831	6·15	7·42	7·93	9·24	4·63	5·93	4·92	5·29
Ag	·164 *	·88	6·96	8·80	9·72	10·84	5·32 *	6·60 *	22·6	24·6

* See remarks later.

It is also seen that in the cases of absorption which result in the emission of the secondary radiation characteristic of the absorbing substance there is considerable deviation from this relationship, the absorption being invariably greater by a considerable amount than would be produced if the simple proportionality still held. There is thus a special absorption and a very large one invariably connected with the emission of the secondary radiation which is characteristic of the absorbing element.

The point is so important that one curve will be treated in detail as typical of all substances. Beginning with a very soft homogeneous primary radiation, the absorption in a substance B (say) is considerable. As the radiation is made more penetrating the absorption in aluminium and in B diminish proportionately, as shown by the portion of the

Fig. 4.



curve PQ (fig. 4). This proportionality continues until the primary radiation is just as penetrating as the secondary

radiation characteristic of the substance B. When the incident radiation is made more penetrating than the secondary radiation characteristic of the absorbing substance B, the absorption first ceases to diminish as rapidly as the absorption in Al, then it increases along RS. At the same time the secondary radiation characteristic of B begins to be emitted, then rapidly increases. As the primary radiation becomes still more penetrating, the absorption in B begins to diminish as along ST, and the intensity of the secondary radiation diminishes at the same rate as the ionization produced by the primary radiation in air, which is probably at approximately the same rate as the absorption in air and consequently in Al, over the range of penetrating powers experimented upon.

The portion of the ordinates above PO may be regarded as representing the absorption in the substance B which is connected with the emission of the secondary radiation characteristic of B. We may thus divide the absorption coefficient into two parts, one bearing an approximately constant ratio to the corresponding quantity for any absorbing substance, and the other the part connected with the emission of the characteristic secondary radiation.

A few features require special mention. The greatest deviations from the law of proportionality occur in those cases in which the energy scattered is a considerable fraction of the total energy. It has been calculated that in the light elements the portion of $\frac{\lambda}{\rho}$ due to scattering is of the order .2.

It is seen here, however, that if we subtract .16 from $\frac{A_g \lambda_C}{\rho}$, $\frac{A_g \lambda_{Mg}}{\rho}$, and $\frac{A_g \lambda_{Al}}{\rho}$, there is strict proportionality within the limits of experimental error, between the coefficients of absorption in C, Mg, and Al.

Again, the apparent departure in the cases of $A_g \lambda_{Ag}$ and $A_g \lambda_{Sn}$ is much diminished by a similar consideration.

There appears, however, to be a slight residual change in the ratios in the cases of Ag and Sn absorptions, and possibly very slight in the cases of absorption by Pt and Au, but the energy of the radiation scattered in these two cases has not been determined.

We may therefore conclude that the proportionality spoken of is a very accurate one through a big range of penetrating powers, if we subtract the energy scattered from that absorbed before determining the absorption coefficient.

Without yet entering into the discussion of any theory, let
Phil. Mag. S. 6. Vol. 17. No. 101. May 1909. 3 E

us consider how much of the energy absorbed has been accounted for. One of us in a previous paper * has shown that when X-rays within the range of penetrating powers experimented upon are transmitted through a substance of atomic weight less than sulphur, the energy of the radiation scattered is independent of the penetrating power of the radiation, and depends merely on the quantity of matter traversed.

It was shown that if I be the intensity of a beam passing through air under atmospheric conditions,

$$\frac{dI}{dx} \text{ due to scattering} = -0.0024 I, \text{ approximately,}$$

$$\therefore \frac{1}{\rho} \frac{dI}{dx} \text{ due to scattering} = -0.2 I$$

for all light elements.

If by analogy with the absorption coefficient we call the portion of $\left(-\frac{1}{I} \frac{dI}{dx}\right)$ due to scattering the scattering coefficient s , we get $\frac{s}{\rho}$ for elements of atomic weight less than sulphur to be 0.2.

This law does not hold, however, for elements of higher atomic weight. The value of $\frac{s}{\rho}$ is much more difficult to determine in these owing to the usual admixture of the characteristic secondary radiation. Only in the case of Ag has the scattered radiation been measured after complete elimination of the other type of radiation. In this case it was found to be about 6.5 times that found from light elements. This would give for $\frac{s}{\rho}$ about 1.3.

In copper, however, an estimate of the quantity of scattered radiation mixed with the homogeneous secondary rays was made and found to be about twice as great as for the light elements, when a penetrating primary radiation was used.

On comparison of the scattering coefficients with the total absorption coefficients it will be seen that only when the rays are fairly penetrating and when absorption takes place in light atoms does the scattering account for a large fraction of the loss of energy of the primary beam. Thus in the case of the transmission of Ag radiation through C, nearly half of the total loss of energy is due to scattering. Below are given

* Barkla, "Energy of Secondary Röntgen Radiation," *Phil. Mag.* May 1904, pp. 553-560.

approximate values for the fraction $\frac{\text{energy scattered}}{\text{energy absorbed}}$ for the most penetrating and the most absorbable beams used in these experiments.

I. Moderately penetrating radiation (from Ag):—

Absorbing Substance.	$\frac{\lambda}{\rho}$	$\frac{s}{\rho}$	$\frac{s}{\lambda} = \frac{\text{Energy scattered.}}{\text{Total energy absorbed.}}$
C . . .	409	2	5
Al . . .	25	2	08
Cu . . .	243	4	016
Ag . . .	133	15?	11

II. Very soft radiation (from Cr):—

C . . .	153	2	013
Al . . .	135	2	0015
Cu . . .	143	4?	003
Ag . . .	5805	15	0026

It has been shown in a previous paper* that the energy of the secondary radiation characteristic of an absorbing element is in many cases probably very great. In a certain case the ionization produced by these rays from Cu was about 300 times that produced by the scattered rays from an equal mass of light elements. If we assume as an approximation that the absorptions of two different beams in air are proportional to the ionizations produced in air, we are led to the conclusion that in this case the energy of the homogeneous radiation was 45 times the energy of scattered radiation and about $\frac{1}{3}$ of the total absorption, or more than $\frac{1}{3}$ of the special absorption connected with the emission of these homogeneous rays.

It was also shown in the paper referred to that the secondary radiation excited in one of these substances (Cu) was proportional to the ionization produced by the primary beam in a thin film of air—or proportional to what may be called the coefficient of ionization in that substance—when the primary was beyond a certain penetrating power.

We may thus express this:—

$${}_x k_{\text{Cu}} \propto {}_x i_{\text{air}};$$

but, on the assumption stated,

$${}_x i_{\text{air}} \propto {}_x \lambda_{\text{air}}, \quad \therefore {}_x k_{\text{Cu}} \propto {}_x \lambda_{\text{air}}.$$

* Barkla and Sadler, Phil. Mag. Oct. 1908, pp. 550-584.

But ${}_x\lambda_{\text{air}} \propto {}_x\lambda_{\text{Al}}$,

from the results of the experiments on absorption ;

$$\therefore {}_xk_{\text{Cu}} \propto {}_x\lambda_{\text{Al}}.$$

This leads us to the conclusion that the energy of secondary radiation produced by a primary radiation in passing through a thin sheet of copper is proportional to the absorption of the primary radiation in a thin sheet of aluminium, provided this primary radiation is beyond a certain penetrating power.

Though the assumption made has not been strictly justified, it probably gives us an approximation to the truth.

One of us has made more detailed investigation of the energy of this type of radiation based on the same assumption*.

Energy is also emitted by the absorbing-substance in the form of corpuscular radiation.

Thus in at least three forms is energy re-emitted.

Besides these, a portion of the energy absorbed must be spent in the process of ionization, and possibly some is transformed directly into heat.

The results of these experiments are of such wide application and affect so many phenomena connected with X-rays, that it is impossible to enter into a detailed discussion of their bearing on the results of investigations on X-rays. The simplification that results is frequently enormous. It is, however, desirable to say something of the absorption of a heterogeneous beam of X-rays such as is commonly experimented upon.

When such a heterogeneous beam is transmitted through any element X, there is of course (1) a selection of the rays of the more absorbable type, (2) a special selection of those rays of greater general penetrating power than the secondary radiation characteristic of X—*i. e.* those able to stimulate the secondary radiation in X, and (3) an emission of secondary rays which are of more absorbable type than the radiations which produced them, but which may be much more penetrating to the element X and to elements whose characteristic radiations are more penetrating than that characteristic of X.

All of these factors contribute to the change in the character of the beam resulting from transmission.

By a proper choice of radiations and absorbing substances it becomes a simple matter to arrange experiments in which there appears (1) no change in transmission, (2) change to more penetrating type, (3) change to more absorbable type,

* C. A. Sadler, "Transformations of Röntgen Rays," Phys. Soc. Lond., April 23, 1909.

without even considering the secondary rays which are superposed on the primary.

A few words are also necessary regarding the rays emitted by an X-ray tube. As the substance of the anticathode of an X-ray tube is subject to a very intense X-radiation, it is the source of an intense secondary radiation. The primary radiation proceeding towards the surface of incidence of the cathode rays, unless very "soft," excites the secondary radiation, which is usually more penetrating to the anticathode and produces greater ionization in the air outside. The primary radiation penetrating further into the anticathode also excites an intense secondary radiation which proceeds in all directions. Half of this is therefore directed towards the surface, and as it usually suffers much less absorption than the primary radiation producing it, it emerges with little loss of intensity and is superposed on the primary beam.

Even if the primary rays were all produced in the surface-layer of molecules, the secondary radiation would produce ionizations comparable with those produced by the true primary rays: as the primary rays cannot be produced absolutely at the surface, the proportion of secondary rays must increase rapidly with an increase at the depth at which they are produced. Thus the "soft" radiations experimented upon by Mr. Kaye (*Phil. Trans. A.* vol. 209, pp. 123-151) are what might have been expected from our previous experiments on secondary rays, and many of the properties which he records are the properties of the secondary rays previously published by us. A comparison of some of the absorption coefficients reveals their unmistakable identity.

In the majority of experiments on X-rays, however, the radiation has already passed through a comparatively thick sheet of glass and has been robbed of a large proportion of the secondary rays.

These considerations also afford an explanation of the fact that the beam of X-rays proceeding from a "soft" tube is much more completely polarized than one from the same tube when harder. In the former case there are less secondary X-rays, as well as less X-rays produced by secondary corpuscular rays.

They also show why the more penetrating portion of a heterogeneous beam from an X-ray tube is the more polarized. The penetrating portion contains the true primary radiation in larger proportion.

[*Note.*—All our experiments on the behaviour of nickel have confirmed our previous conclusion—that in all phenomena

connected purely with X-rays with which we are experimentally acquainted, nickel behaves as a normal element of atomic weight 61.3. This is true of (1) the absorption by nickel unconnected with the production of X-rays, (2) the absorption connected with the emission of secondary X-rays, (3) the character of the secondary rays emitted, (4) the intensity of the secondary X-rays emitted]

Theory.

If an attempt be made to account for these results on Prof. Bragg's neutral pair theory, we are at once led into difficulties. The pairs constituting the homogeneous secondary radiation characteristic of a particular element must have been either in the primary radiation or in the atoms of the element subject to that radiation. If we make the former assumption we have to account for pairs varying enormously in velocity being scattered in all directions with *one* velocity—a velocity characteristic of the atom. If, on the other hand, we assume that the pairs were originally in the atom, we must account for their ejection with one velocity by assuming some disruption to occur in the atom. The former of the two assumptions is inconceivable, and it is against the latter—the disruption theory—that much of Prof. Bragg's argument has been directed. If, however, we change the point of view and assume disruption, we must account for:—All the pairs transmitted through a substance being undiminished in velocity; those moving with a velocity greater than a critical velocity being stopped in greater proportion than those moving more slowly; those moving still more quickly being stopped in smaller proportion; and the equality of the velocity required to produce instability with that acquired by another pair due to the instability set up in the atom. Assumptions might possibly be made to satisfy any individual result, but in the combination we find the difficulties insuperable.

The results of these experiments are, however, in their general nature what would be expected on the æther-pulse theory, as shown in previous papers*.

A more detailed discussion of this theory will be given when several points have been further tested.

It is possible that all the substances experimented upon also emit a very easily absorbed secondary radiation, and that the absorptions which have been observed in the case of very "soft" radiations have been accompanied by the emission of such radiations.

* Phil. Mag. Oct. 1908, pp. 550-584, and *Jahrbuch der Radioaktivität und Elektronik*, iii. pp. 246-324.

Summary.

By the use of homogeneous beams of X-rays varying considerably in penetrating power the laws governing the absorption of X-rays have been investigated. The results of experiments may be briefly stated as follows :—

For very absorbable radiations the ratio between the coefficients of absorption in any two elements remains approximately constant while the penetrating power of the radiation varies greatly.

This law, however, does not hold when the radiation transmitted through an element is made more penetrating than the secondary radiation characteristic of that element; this, again, is the condition necessary to the production of that secondary radiation.

As the primary radiation is made more penetrating than this secondary radiation, the absorption in this particular element *increases*, first slowly, then very rapidly. The secondary radiation—homogeneous and characteristic of the element—begins to be emitted, first feebly, then in considerable intensity. The absorption increases to many times the absorption experienced by a somewhat “softer” radiation.

When the primary radiation is made still more penetrating, the absorption again begins to decrease and the intensity of secondary radiation decreases at the same rate as the ionization produced by the primary radiation in a thin film of air.

In the case of the transmission of a penetrating radiation through elements of low atomic weight, the energy of the primary beam which appears as secondary radiation of the same penetrating power as that of the primary producing it—scattered radiation—is a considerable fraction of the total energy absorbed.

In other cases investigated the fraction is small.

Evidence points to the conclusion that in the case of transmission of primary rays which are most efficient as producers of the secondary rays in any element, the energy of this secondary radiation is a considerable fraction of the energy absorbed.

Investigation has failed to reveal any transformation of X-rays by transmission, other than the transformation into secondary rays proceeding in all directions. The laws discovered are sufficient to account for a great variety of apparently complicated results recorded by many investigators. The authors are not aware of any peculiar phenomenon of absorption which cannot be explained in terms of these laws.

[In all experiments on absorption, both the normal absorption of "soft" rays and the special absorption connected with the emission of secondary rays, and on secondary rays, both as regards intensity and character, nickel behaves as a normal element of atomic weight about 61.3.]

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Liverpool.

LXX. *On the Distribution of Thorium in the Earth's Surface Materials.* By J. JOLY, F.R.S.*

RECENT observations have shown that the emanation of thorium exists in the atmosphere to an extent which is difficult to account for unless a very considerable quantity of the element thorium is distributed in the surface rocks and soils. Thus Bumstead (*Am. Journ. Sc.* July 1904) and Dadourian (*Am. Journ. Sc.* Jan. 1905) in New Haven, Conn., Blanc (*Phil. Mag.* March 1907) at Rome, and Wilson (*Phil. Mag.* Feb. 1909) at Manchester, have found that of the total active deposit gathered upon a negatively charged wire exposed for many hours to the air, a considerable part—it may be even a major part—exhibits the properties of the active deposit derived from the emanation of thorium. Blanc estimates that at Rome from 50 to 70 per cent. of the whole activity is due to thorium; and Wilson infers from his observations at Manchester that there must be about seven times as much thorium as uranium in the surface soils. On the other hand, Gockel (*Le Radium*, Jan. 1909) finds that at Zermatt the emanation of thorium is almost entirely absent.

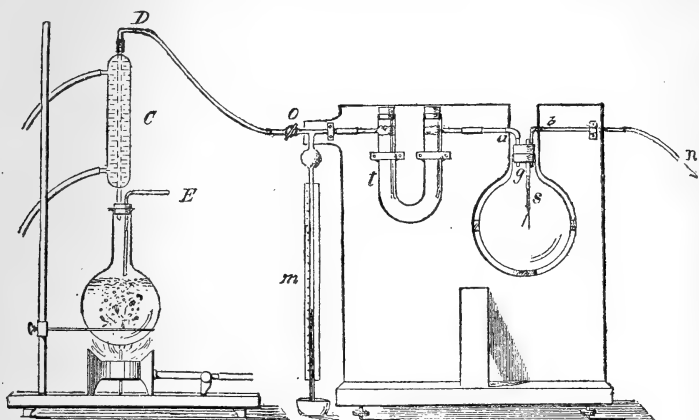
It is a question of much importance to geological science to decide how far these observations refer to purely local conditions, or whether they indicate a general prevalence of thorium in rocks.

Method of Measurement.

The material under investigation is brought into solution by the aid of reagents (*e. g.* hydrochloric acid and the carbonates of soda and potassium) which are themselves shown to be free from thorium by the method to be now described. The solution is boiled in a flask (see figure, p. 761) to which the steam is returned by an attached condenser. A preliminary boiling, lasting from 20 to 30 minutes, is required to completely expel any accumulated emanation of radium. A small quantity of powdered talc must always be put into the solution in order to secure uniformity of ebullition.

* Communicated by the Author.

A slow current of air, entering the flask at E, carries the evolved emanation of thorium to the electroscope. On its way the air passes through the drying-tube, *t*, containing phosphoric anhydride; the current being steadily maintained by a water-suction pump working under a constant head.



The velocity of flow is not such as to interfere with the observation of the gold leaf. A thin copper tube, with finely perforated walls, diffuses the entering air current; and the tube by which it is withdrawn reaches nearly to the bottom of the electroscope. These latter details are not shown in the figure.

In order to reproduce on each occasion the same velocity of air-flow, a stopcock, O, and a manometer tube, *m* (which dips into a vessel of oil), are placed between the condenser and the drying-tube. The stopcock serves to introduce an obstruction to the current of air, so that a certain rarefaction (indicated by the height to which the column of oil is drawn in *m*) must be established to the right of the obstruction in order that the flow may take place. By this arrangement variations in the resistance of the drying-tube will not affect the indications of the manometer. In a more recent form of the apparatus the U drying-tube is replaced by a shorter straight tube, and a second very short tube, containing granulated calcium chloride, is placed between the stopcock and the condenser. This latter tube serves to protect the stopcock from moisture.

The period of the emanation of thorium being only 54 seconds, it is evidently of importance to secure its rapid evolution from the solution and its immediate transfer to the electroscope. The first condition seems secured by the process of ebullition. The second is best attained by an air-flow

which is neither too fast nor too slow. If too slow a large part of the emanation must be transformed before it attains the electroscope. If too fast it is swept through the electroscope in so highly dilute a condition that, again, its effects must be enfeebled. There must be, therefore, some most favourable rate at which the air-flow should take place. With the arrangements described we evidently have the adjustment of the current under control. To find the most favourable current we work with solutions containing small, known amounts of thorium; first reading the normal rate of discharge of the electroscope while the air-current is flowing, and then again when the emanation is entering on the air-current. It will be found that the normal rate of discharge is but little affected, if at all, by small variations in the rate of air-flow. We can in this manner not only determine the most favourable air-current, but we can standardize the readings of the electroscope.

The electroscope I have used in the observations given below has a voluminar capacity of about 450 ccs. The gold leaf is about 2.5×0.2 cm. in dimension, and the electrostatic capacity is such that the radium constant is 0.8×10^{-12} grams radium per scale-division per hour.

The following experiments will show the degree of sensitiveness and consistency attained by this method of measuring small quantities of thorium. I am by no means satisfied that the best rate of transfer of the emanation has yet been attained. The thorium salt used is the nitrate as supplied by Kahlbaum.

				Gain in Scale- Divisions per hour.
6.6×10^{-4} grams. Th in	450 ccs. distilled water		22.5
3.3 " " "	1400 " sea-water		11.3
1.7 " " "	1600 " "		6.3
1.7 " " "	1500 " rock-solution		7.6

Under different and less favourable conditions of air-current the following comparative observations were made on (a) thorium nitrate of Kahlbaum, (b) thorium oxide of Kahlbaum, and (c) a specimen of thorium oxide kindly given to me by Mr. Moss; a specimen which had been some years in his possession.

				Gain in Scale- Divisions per hour.
3.3×10^{-4} grams. Th (a) in	1000 ccs. water		6.9
6.8 " " "	(b) " 1000 " "		14.0
8.8 " " "	(c) " 1400 " "		19.0

It would seem, from the foregoing results, that the readings of the electroscope are not dependent on the volume of the solution containing the thorium, and consequently we must

infer that the evolution of the emanation by the ebullition is so rapid and complete that the whole of it, sensibly, is brought into the space above the solution. It is probable that the sensitiveness would be enhanced if the air withdrawn from the electroscope was circulated again through the flask; a modification which does not appear to present any great difficulties. Another important deduction from the experiments cited above is that the presence of considerable quantities of substances in solution does not appear to affect the evolution of the emanation. The "rock-solution" contained 10 grams of a lava and the accompanying reagents; the result quoted is obtained as a difference on the thorium content of the rock. These inferences have been substantiated by many other experiments carried out for purposes of calibration.

Under the conditions affording the values given in the first table, we see that a gain of one scale-division per hour in the rate of discharge of the electroscope corresponds to 2.8×10^{-5} grams of thorium in the solution. It is quite possible under ordinarily favourable conditions to be sure of a change of one scale-division per hour by reading the normal rate of discharge both before and after the test of a solution. If, now, we deal with 50 grams of rock, the determinable quantity of thorium is about 0.6×10^{-6} grams per gram of rock. In order to make a comparison with the average amount of uranium in rocks we may assume the case of a rock having 4×10^{-12} grams of radium per gram. Remembering that 3.7×10^{-7} grams of radium are in equilibrium with one gram of uranium, we find that the amount of uranium present is 1.2×10^{-5} grams. It follows that if there is as much thorium as uranium in rocks, we may evaluate the mass of thorium to a fractional part or deal with quantities of material much less than 50 grams.

Preliminary Results.

The readings of the electroscope can be satisfactorily standardized only by use of a thorium mineral of known composition. The following few determinations, which are based on the emanating power of Kahlbaum's thorium nitrate, may therefore require some amendment later on. As commercial salts have been found to possess an activity which for the quantity of thorium present is less than that observed in minerals, the correction on the present results would be one of reduction.

The solutions used had been in many cases originally prepared for radium determinations. The quantity of dissolved rock is, therefore, small. The readings were, however, perfectly definite and unmistakable. The major limit recorded in

some cases, where no positive result was obtained, is arrived at by dividing the smallest detectable amount of thorium, under the conditions of the experiment, by the number of grams of the substance ; in the case of the sea-waters by the number of cubic centimetres.

		Thorium in grams per gram.
Lava, Vesuvius, 1906, 10 grams		3.6×10^{-5}
" " 1895-99, 10 grams		3.1 "
" " 1832, 10 grams		3.4 "
" " 1794, 10 grams		0.9 "
" " 1631, 10 grams		2.5 "
" St. Helena, 10 grams		0.9 "
Gneiss, St. Gothard Tunnel, 8.38 grams		1.9 "
Mica-schist (sedimentary), St. Gothard Tunnel, 9.1 grams		1.8 "
Gneissose granite, St. Gothard Tunnel, 10 grams		2.1 "
Gneiss, Simplon Tunnel, 10 grams		1.6 "
Diabase, Fifeshire, 10 grams	less than	0.3 "
Shale, Moffat Dale, 45 grams		0.5 "
Granite, Co. Wicklow, 20 grams		0.9 "
Carboniferous Limestone, Armagh, 22.5 grams ...	less than	0.2 "
Marsupites Chalk, 32 grams	less than	0.9 "
Red Clay, N. Pacific, 2.4 grams	less than	1.3 "
" Central Pacific, 10 grams		0.5 "
Radiolarian Ooze, Central Pacific, 6.3 grams	less than	0.5 "
Manganese Nodule, S. Pacific, 12 grams		0.2 "
Sea-Water, Coast of Dublin, 1400 ccs.	less than	2.1×10^{-8}
" S. Atlantic, 1790 ccs.	less than	1.6 "
" " 1500 ccs.	less than	2.0 "
" Indian Ocean, 2800 ccs.		1.1 "

The number of observations is as yet too small to merit any full discussion or justify definite conclusions. But the fact that every rock examined, with three exceptions, was found to contain thorium, lends considerable support to the view that it is a generally prevalent constituent. The quantities observed in some cases are such as might afford an explanation of the local occurrence of thorium emanation in the atmosphere, although the general average of the few rocks examined is hardly such as to account for a general prevalence of the emanation to the extent to which it has been observed in particular localities, unless thorium is breaking up faster than is generally assumed. It may turn out, however, that the surface soils are richer in thorium than the rocks from which they are derived. The results on the Vesuvian lavas seem to accord well with Blanc's observations on the abundance of thorium-emanation in the atmosphere at Rome, the leucitic lavas entering extensively into the surface-geology of the district. The quantities of thorium observed in some of these materials exceeds the average uranium content of rocks about three times. It is true that the lavas in question possess a very unusual radium richness, so that the local

evolution of the radium-emanation could not be assumed to be subordinate in amount to thorium-emanation. However, it is to be supposed that the much shorter-lived emanation of thorium would be more local in its effects than the radium-emanation, which, breaking up slowly, has time to be diffused over a wider area.

The amount of thorium in sea-water is small, and until an observation was made on a large quantity of water, only negative results were obtained. A thorium salt added to sea-water gives an immediate precipitate (which is cleared by an acid). It may be that the poverty in thorium of the oceanic waters and the underlying sediments is not a true indication of the thorium content of the rocks, but is to be referred to chemical effects consigning the thorium to the littoral deposits.

The geological bearing of the distribution of thorium cannot be discussed without a knowledge of the rate of transformation of that element. According to Bragg (Phil. Mag. June 1906) this element is breaking up at one-fifth the rate of uranium. If this is so, and taking into account the lesser emission of α particles by the derivatives of thorium, the indications of the results given above assign to thorium a relatively unimportant position compared with uranium as a heat-producing agent in the rocks. On the other hand, if the rates of transformation are about equal (Soddy, Phil. Mag. Oct. 1908), the importance of thorium might be comparable with that of uranium, the quantities of the two elements in the rocks not appearing to differ greatly in amount.

LXXI. *On Induced Stability.* By ANDREW STEPHENSON*.

1. NOT only may a single pivoted body be maintained in the position of unstable equilibrium by vertical oscillation of the pivot, but furthermore a number of freely jointed links may be similarly maintained.

In the simplest case, when there are only two links, the conditions of stability are readily found. With the usual notation the equations of motion in the neighbourhood of the vertical are

$$\begin{aligned} \{ (h_1^2 + k_1^2 + \mu l_1^2) D^2 - (g + 2an^2 \cos nt)(h_1 + \mu l_1) \} \theta + \mu l_1 h_2 D^2 \phi &= 0, \\ \{ (h_2^2 + k_2^2) D^2 - (g + 2an^2 \cos nt)h_2 \} \phi + l_1 h_2 D^2 \theta &= 0, \end{aligned}$$

where μ is the ratio of the mass of the second, or upper, rod

* An addition to a paper under the above title, Feb. 1908. Communicated by the Author.

to that of the first, or lower, and α is half the amplitude of the applied motion. Writing these equations

$$[aD^2 - \{g + \alpha n^2(e^{int} + e^{-int})\}] \theta + bD^2 \phi = 0,$$

$$[pD^2 - \{g + \alpha n^2(e^{int} + e^{-int})\}] \phi + qD^2 \theta = 0,$$

we have two particular solutions, each of the form

$$\theta = \sum_{-\infty}^{\infty} A_r e^{(c+rin)t}, \quad \phi = \sum_{-\infty}^{\infty} B_r e^{(c+rin)t},$$

where

$$\left. \begin{aligned} \{-g + a(c+rin)^2\} A_r - \alpha n^2 (A_{r-1} + A_{r+1}) + b(c+rin)^2 B_r &= 0 \\ \{-g + p(c+rin)^2\} B_r - \alpha n^2 (B_{r-1} + B_{r+1}) + q(c+rin)^2 A_r &= 0 \end{aligned} \right\} \quad (r)$$

The set of conditional equations, (r), determines c and the relative values of the coefficients. When α is small the terms diminish rapidly from A_0 and B_0 , and when α approaches the limit zero, αn remaining finite, we obtain

$$(ap - bq)^2 c^4 + \{2(\alpha n)^2 (a^2 + p^2 + 2bq) - g(a+p)(ap - bq)\} c^2 + 4(\alpha n)^4 - 2(\alpha n)^2 g(a+p) + g^2 (ap - bq) = 0.$$

The roots of this quadratic in c^2 are real for all values of αn . For stability the quantities

$$\begin{aligned} &2(\alpha n)^2 (a^2 + p^2 + 2bq) - g(a+p)(ap - bq) \\ \text{and} \quad &4(\alpha n)^4 - 2(\alpha n)^2 g(a+p) + g^2 (ap - bq) \end{aligned}$$

must be positive. Thus stability is always ensured by making the frequency of the applied motion sufficiently large. For two equal rods, each of length l , the condition is

$$(\alpha n)^2 > 0.683lg.$$

It may be noted for the sake of comparison that for a single rod of length $2l$, for stability $(\alpha n)^2 > \frac{2}{3}lg$.

2. In the case of a chain of three uniform rods, each of length l , we obtain the c equation by a method similar to the preceding:—

$$26(cl)^6 + 9(2594\mu - 41lg)(cl)^4 + 9\{154156\mu^2 - 11572\mu lg + 112(lg)^2\}(cl)^2 + 81\{27040\mu^3 - 12480\mu^2 lg + 570\mu(lg)^2 - 5(lg)^3\} = 0,$$

where $\mu = \frac{1}{3}(\alpha n)^2$, and α is small.

The roots of this equation in c^2 are real for all values of αn . They are negative if

$$(\alpha n)^2 > 1.79lg,$$

the condition for stability.

March, 1909.

LXXII. *On the Treatment of Electrodynamics.**To the Editors of the Philosophical Magazine.*

GENTLEMEN,—

IN the February number of your Magazine Dr. Lehfeldt, in an article on Electrodynamics, speaks of the highly artificial nature of the ordinary method of calculating the induction inside a magnetic material. I have always felt this, and for some years have abandoned the "crevasse" method in favour of the following.

Consider an area of one square centimetre drawn in the interspace between the molecular magnets and at right angles to the direction of magnetization, then from the definition of intensity of magnetization (I) it follows in the usual way that the resultant amount of N pole on one side of this area and of S pole upon the other side is I . Each unit of N pole gives rise to 4π lines which cross the area to the S side, and hence the induction within the material due to the arrangement of its elementary magnets is $4\pi I$. Adding this to the field H due to other sources we obtain

$$B = H + 4\pi I.$$

The following proof of the work done in taking a specimen through a hysteresis cycle may be of interest, as it depends upon first principles.

If m is the magnetic moment of a molecular magnet and θ the angle its axis makes with the direction of magnetization, $I = \sum m \cos \theta$ for 1 c.c., and $\sum m \sin \theta = 0$. On causing m to rotate through an additional angle $d\theta$ the work done, with or against, the existing field H is equal to $Hm \sin \theta d\theta$, and for 1 c.c. the total work is $\sum Hm \sin \theta d\theta = H \sum m \sin \theta d\theta$.

$$\begin{aligned} \text{but since } I &= \sum m \cos \theta, \\ dI &= d\sum m \cos \theta \\ &= -\sum m \sin \theta d\theta. \end{aligned}$$

$$\therefore \text{Work done} = H dI.$$

The integration for a cycle upon the H - I diagram follows in the ordinary way.

The above proofs are independent of any assumption as to the nature of the ultimate magnetic particle.

S. G. STARLING.

Municipal Technical Institute,
West Ham, E.
Feb. 5, 1909.

LXXIII. *Notices respecting New Books.*

Outlines of Physical Chemistry. By G. SENTER, Ph.D., B.Sc.
London: Methuen & Co. 1909. Price 3s. 6d.

THIS excellent textbook is intended as an elementary introduction to the subject of Physical Chemistry. The subject has now grown so large that it is extremely difficult to cover the whole range in a single small book; but Mr. Senter has succeeded in covering a large part of it. The parts upon which most stress has been laid are those dealing with the modern theories of solutions, the principles of chemical equilibrium, electrical conductivity, and electromotive force. The picture presented by this simple introduction is one of the amazingly rapid development which the subject has undergone. Many of the views given here were only accepted (at any rate by the purely physical school) after very considerable opposition; and even now are accepted only as working hypotheses. This opposition was by no means unreasonable, as will be realized by all those who are watching the more recent modifications which these hypotheses are undergoing. We refer in the main, of course, to the theory of solutions; and we think we detect the signs of change in the fact that our author does not claim the validity of the physico-chemical theories of solution (put forward by van't Hoff and others) with the same dogmatism that we usually meet with. It is now realized by *all* that the solvent is not the mere idle medium whose sole function is to provide a space in which the molecules of the solute can move about. Its complete function is still unknown, and sufficient is said here for an elementary volume in regard to rival theories. We do not doubt that there will be considerable modification of all existing views before many years are out. Meanwhile there are huge groups of phenomena which can all be placed together and coordinated on the basis of thermodynamic arguments, which latter are independent of any theory of the mechanism of the processes concerned. If we may indicate a preference, it is that, while not ignoring mechanical theories, chief stress should *for the present* be laid on those facts and phenomena which can so be coordinated. Since Dr. Senter does lay very considerable stress on this side of the subject, we may wish him all good speed. We do not hesitate in thinking that this book supplies a real want. It is simple, so that the beginner will understand almost all of it; at the same time it does not shirk difficulties, though it often has to leave them not entirely removed. Lastly, its price is remarkably low, so that a student has no reason to avoid purchasing it.

Department of Commerce and Labor, Coast and Geodetic Survey.
O. H. TITTMANN, Superintendent. *United States Magnetic Tables and Magnetic Charts for 1905.* By L. A. BAUER, Inspector of Magnetic Work and Chief of Division of Terrestrial Magnetism. Washington: 1908. Pp. 154. Large octavo, with 7 charts.

THE volume consists mainly of tables, comprising observed values of the magnetic elements and the values thence deduced for the epoch January 1, 1905. Data are given for over 4000 land stations—including fully 800 outside the borders of the United States itself—and for more than 200 stations at sea. There are also voluminous tables showing the secular change in the elements throughout the United States, going back in a good many cases to 1750. The results are embodied in 7 charts. The first 5 show the isogonals, isoclinals, and isomagnetics (of horizontal, vertical, and total intensity). The sixth shows the curves that would be traced on the earth's surface by one always travelling in the direction in which the compass-needle points. The last chart contains two types of curves, illustrating the secular change in the elements. Though the epoch selected is January 1, 1905, use is made of observational data as recent as midsummer 1907. The charts are of large size and aim at showing all local irregularities of importance.

LXXIV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 332.]

December 16th, 1908.—Prof. W. J. Sollas, LL.D., Sc.D., F.R.S.,
President, in the Chair.

THE following communication was read:—

‘On the Igneous and Associated Sedimentary Rocks of the Tourmakeady District (County Mayo).’ By Charles Irving Gardiner, M.A., F.G.S., and Prof. Sidney Hugh Reynolds, M.A., F.G.S. With a Palæontological Appendix by Frederick Richard Cowper Reed, M.A., F.G.S.

January 13th, 1909.—Prof. W. J. Sollas, LL.D., Sc.D., F.R.S.,
President, in the Chair.

The following communications were read:—

1. ‘On Labradorite-Norite with Porphyritic Labradorite.’ By Prof. Johan H. L. Vogt, F.M.G.S.

This paper deals with a rock occurring at Napp Farm, on Flakstadö, off the northern coast of Norway. It contains 23 per
Phil. Mag. S. 6. Vol. 17. No. 101. May 1909. 3 F

cent. of labradorite-phenocrysts, in a crystalline groundmass made up of a more acid plagioclase, hypersthene, diallage, and titanomagnetite, with small quantities of biotite, and very little spinel, apatite, and pyrite. A little secondary hornblende and some garnet are present, but few other secondary products. Olivine is conspicuously absent.

The plagioclase-phenocrysts are more acid in their outer zones, and the groundmass plagioclase is still more acid as determined optically and by specific gravity and analysis. Analyses are tabulated of the bulk of the rock and the groundmass, and of the separated plagioclases and the magnetite. From these the relative proportions of the constituents are calculated, and the formula of the feldspars determined; also the titanomagnetite proves to be a mixture of magnetite with ilmenite.

The order of crystallization is found to be:—(1) Phenocryst plagioclase; (2) plagioclase with magnetite; and (3) plagioclase, magnetite, pyroxenes. The plagioclase-phenocrysts started to form at points from 8 to 18 centimetres apart, and when they had grown to some size a second crop of smaller crystals grew at nearer points; the magnetite was much aggregated round the plagioclase-phenocrysts, growing together with the plagioclase of the groundmass; and finally the remaining magma crystallized. The order of crystallization is found to be consonant with the physico-chemical laws applying to the phase liquid-solid. Graphic representations are given to illustrate the order of crystallization of a ternary system of plagioclase, magnetite, and pyroxene; and it is shown that the separation of such a system would yield first the formation of plagioclase-phenocrysts, secondly the solidification of a magnetite-plagioclase eutectic, and finally an eutectic of plagioclase, magnetite, and pyroxene. In considering the equilibrium between the solid and the liquid albite-anorthite phase, it is found that equilibrium must have been maintained sufficiently long for the phenocrysts to acquire a composition different from the first-formed crystals, but that eventually the equilibrium broke down—a matter possibly connected with the size of the phenocrysts in relation to the viscosity of the solution. The history of events in connexion with the pyroxenes was probably similar to that in the case of the plagioclases, but this portion of the subject is not fully worked out. The temperature-interval of crystallization is estimated to have been between about 1400° and 1000°.

The author considers that this investigation establishes that the processes of crystallization in a magma may be explained in all details according to physico-chemical laws.

2. 'On the Genus *Loxonema*, with Descriptions of New Proterozoic Species.' By Mrs. Jane Longstaff (*née* Donald), F.L.S.

January 27th, 1909.—Prof. W. J. Sollas, LL.D., Sc.D., F.R.S.,
President, in the Chair.

The following communications were read:—

1. 'The Conway Succession.' By Miss Gertrude L. Elles, D.Sc.

In this area the author has found a complete succession of strata, from Llandeilian up to Salopian date. The divisions proposed are as follows:—

			Zone of
SALOPIAN.	Benarth Flags & Grits	{ <i>Cyrtograptus symmetricus</i> . <i>Monograptus riccartonensis</i> . <i>Cyrtograptus murchisoni</i> .
VALENTIAN.	{ Gyffin Shales. (300 feet.)	{	{ <i>Monograptus crenulatus</i> . <i>Monograptus crispus</i> . <i>Rastrites maximus</i> . <i>Monograptus sedgwicki</i> . <i>Monograptus gregarius</i> . <i>Mesograptus modestus</i> .
	(Conway Castle Grits. (150 feet.)		
ASHGILLIAN.	{ Deganŵy Mudstones. (30 feet.)	{	{ <i>Phacops mucronatus</i> & <i>Dicellograptus anceps</i> .
CARADOCIAN.	{ Bodeidda Mudstones. (350 feet.)	{	{ <i>Trinucleus</i> -Beds.
LLANDEILIAN.	{ Cadnant Slates (Upper).	{	{ <i>Dicranograptus clingani</i> . <i>Climacograptus wilsoni</i> .
	{ Cadnant Slates (Lower). (310 feet.)	{	{ <i>Dicranograptus brevicaulis</i> & <i>Mesograptus multidentis</i> . <i>Climacograptus peltifer</i> .
CONWAY MOUNTAIN VOLCANIC SERIES.	{	{	{
	(Upper or Coetmor Ash-Group.		
	Upper Brecciated Lava-Group. (710 feet.)		
	Lower or Bodlondeb Ash-Group. (60 feet.)		
	Lower Banded Lava-Group. (1400 feet.)		

After references to the literature and an account of the landscape and structure of the district, the beds are described in ascending order, lists of fossils being given from the more important exposures. The sections show that there is no break whatever in the sequence between the Ordovician and the Silurian rocks in the district. Petrological notes on the chief igneous rocks are given, and then a detailed comparison is established between the rocks of this area and those of South Wales, the Rhayader and Tarannon districts, Lakeland, the South of Scotland, and Pomeroy. The Conway Mountain Volcanic Series appears to be equivalent to the Borrowdale volcanic rocks of the Lake District, and the Cadnant Slates and Bodeidda Mudstones equivalent to the Upper *Dicranograptus*-Shales, *Trinucleus*-Beds, and Sholeshook Limestone of South Wales, the Sleddale and Roman Fell Groups of Lakeland, and the Upper Glenkiln and Lower Hartfell of the South of Scotland. The Deganŵy Mudstones are paralleled with the Redhill Beds and the Ashgill Shales. Close comparison is

possible between the graptolitic zones of the Gyffin Shales and corresponding beds at Rhayader, Tarannon, in the Lake District, and the South of Scotland. Finally, the Benarth Flags are compared with the Brathay Flags, the Riccarton Beds, and beds above and including *Cyrtograptus murchisoni* in the Tarannon district.

2. 'The Depth and Succession of the Bovey Deposits.' By Alfred John Jukes-Browne, B.A., F.G.S.

The total thickness of the Tertiary Beds in the Bovey Basin has never yet been ascertained, because no boring has yet reached the bottom of the basin in which they lie; and no one has yet attempted to make out a stratigraphical succession from the sections exposed in the numerous clay-pits.

Some years ago, however, Messrs. Candy & Co., of the Heathfield Potteries, put down a boring which reached a depth of 526 feet from the surface. Having obtained some particulars concerning the beds traversed by this boring, the author is able to discuss the succession of the Bovey deposits, so far as they have been explored. The following is a generalized description of the strata seen in the Heathfield pit, and penetrated by the boring from the bottom of that excavation:—

	<i>Thickness in feet.</i>
Superficial deposits	about 20
Beds of clay and sand, with occasional beds of lignite	250
Beds of lignite and clay, with one of sand	36
Beds of lignite, with thin layers of clay	220½
	<hr/>
	526½

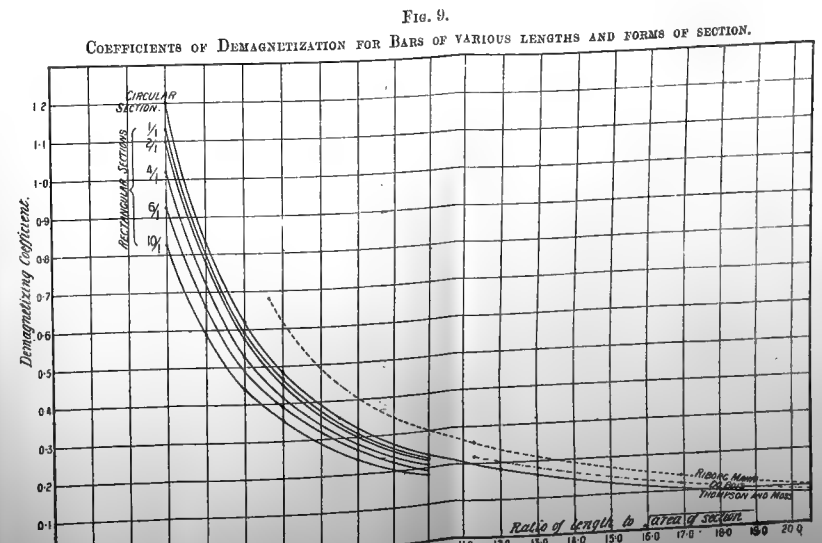
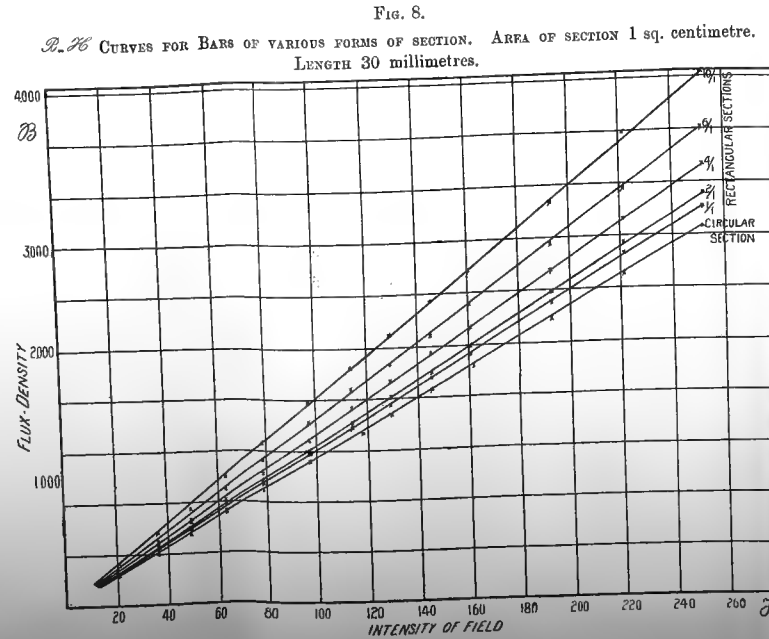
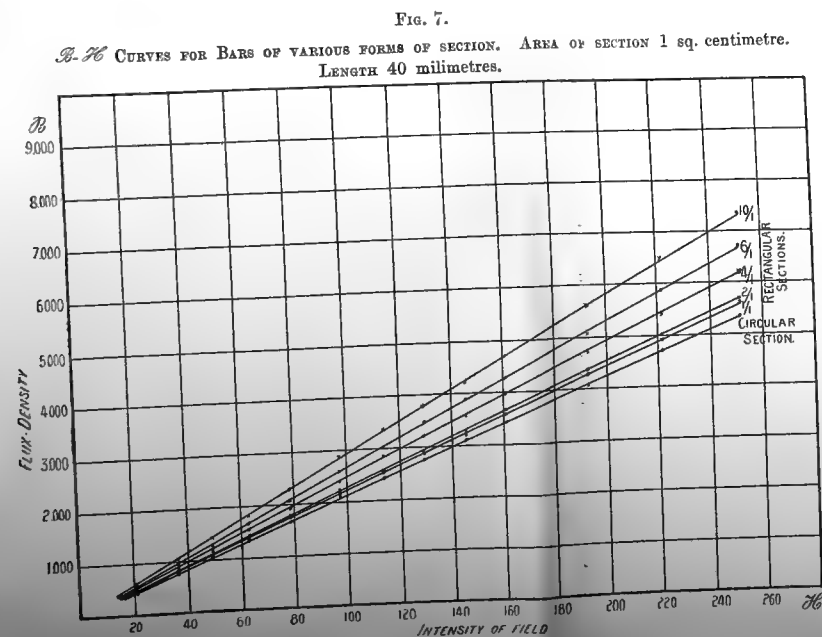
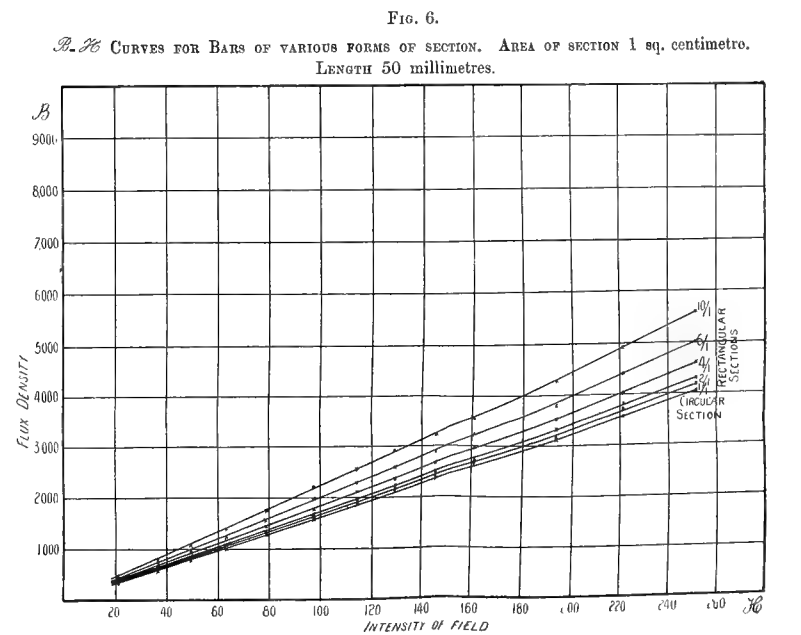
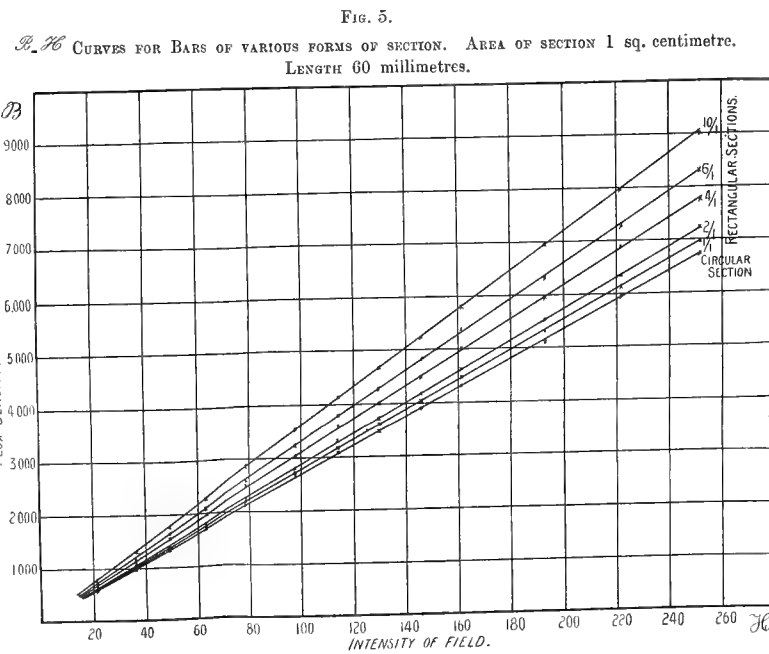
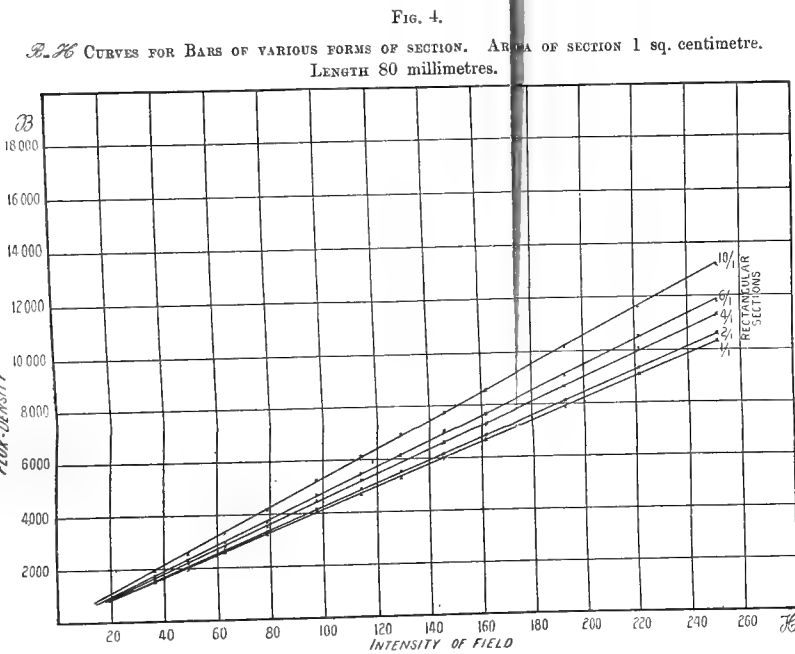
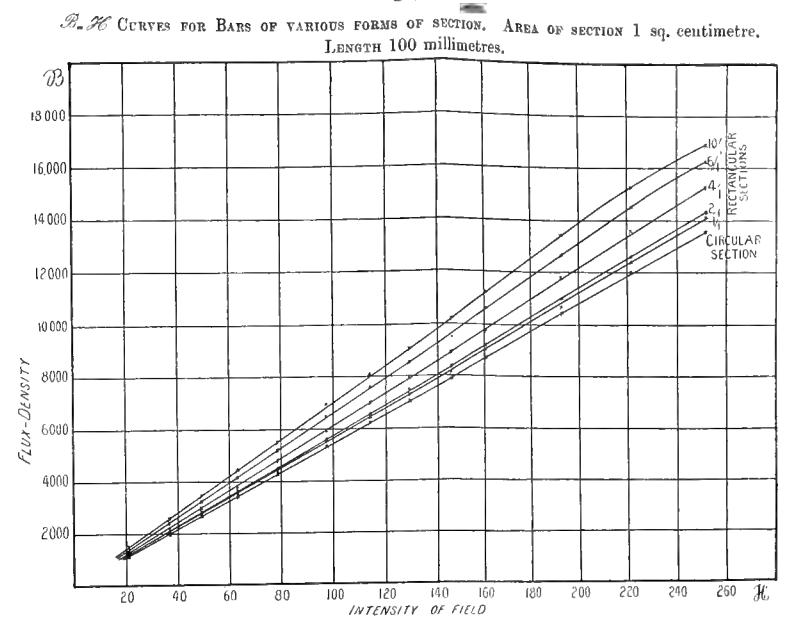
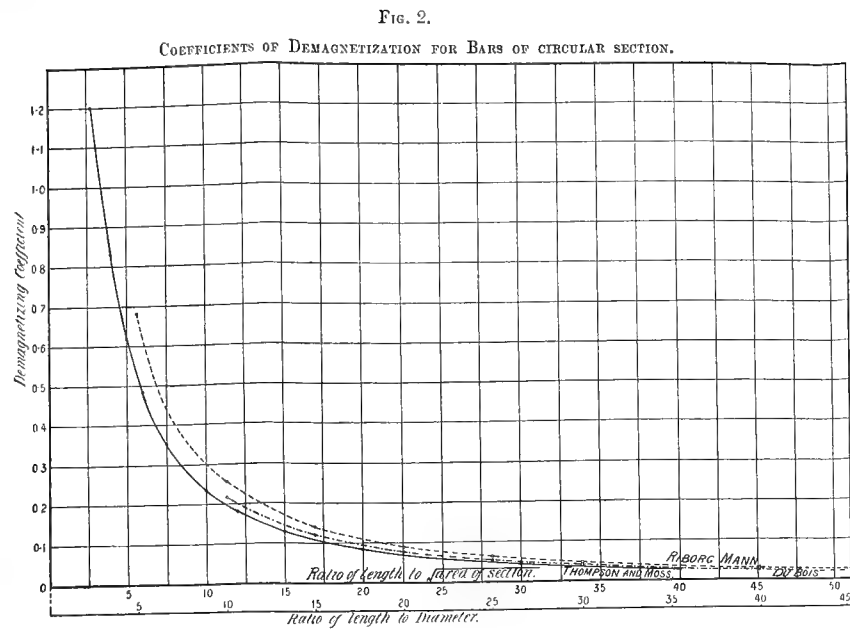
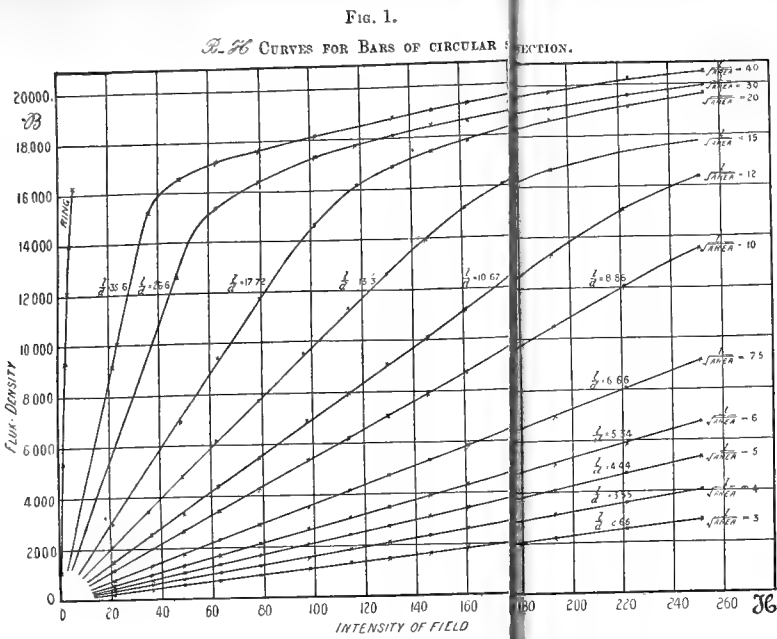
The author confirms the conclusion arrived at by Pengelly in 1861, with regard to the relative age of the beds exposed in the 'old coal-pit' south-east of Bovey Tracey and those proved in a boring to the east of it. From all the data mentioned, and assuming the actual base of the Tertiary deposits to be not more than 30 feet below the bottom of the Heathfield boring, he estimates the total thickness of the 'Eocene' beds to be about 613 feet.

The Bovey Basin itself is regarded as a tectonic basin or post-Eocene pericline, and not as a lake-basin; although, during the deposition of the higher part of the series, it may have formed part of a large lacustrine or lagoon area, extending over the greater part of East Devon.

The author dissents from Heer's view of the manner in which the lignites were formed, discusses the identification of some of the plants, and concludes that the lignites which form the mass of the lower beds represent the growth and decay of successive swamp-forests, similar to that of the Great Dismal Swamp of Virginia at the present day.

Assuming that these lower beds are of Eocene age, and contemporaneous with the Bournemouth Beds of the Hampshire Basin, the author points out that nothing has yet been proved with regard to the higher beds, which may be of Bartonian, or even of Oligocene age.

SECTION. AREA OF SECTION 1 sq. centimetre.
OF D millimetres.





INDEXED

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1909.



LXXV. *The Motion of Electrons in Solids. Part I.—Electric Conductivity, Kirchhoff's Law and Radiation of Great Wave-length.* By J. H. JEANS, M.A., F.R.S., Professor of Applied Mathematics in Princeton University*.

1. **T**HE Electron-theory of metals has been developed by Drude, Thomson, Lorentz, and others. The work of all these writers postulates a motion consisting of free paths and collisions. These and other conceptions imported from the Kinetic Theory of Gases have led very successfully to a general interpretation of many of the phenomena of solids, but cannot be expected to lead to accurate quantitative results. I have therefore thought it would be of value to try to develop a theory which shall be free from all suppositions for which there is not direct numerical justification. Such a theory, if it can be constructed, ought to lead not only to predictions of the nature of the phenomena with which it is concerned, but also to an exact evaluation of the quantities involved in these phenomena.

Electric Conductivity.

2. In a conductor which is under the influence of no externally-impressed electric force, we suppose that there are N electrons per unit volume, moving with the velocities

* Communicated by the Author.

assigned to them by Maxwell's law *. At a temperature T given by $RT = \frac{1}{2h}$, the law of distribution of velocities is

$$N \sqrt{\frac{h^3 m^3}{\pi^3}} e^{-hm(u^2+v^2+w^2)} du dv dw. \quad \dots \quad (1)$$

If an electric force X is brought into play parallel to Ox , this distribution will immediately be altered. Each electron will acquire momentum parallel to Ox at a rate Xe , but this gain in momentum will be held in check by a perpetual transfer of momentum between each electron and all the molecules by which it is influenced at any instant.

Under law (1), the average value of u is zero. Under the new law, the average of u will have some value u_0 , different from zero. Corresponding to any value of u_0 , there is a current i_x parallel to Ox of amount

$$i_x = Neu_0. \quad \dots \quad (2)$$

3. A brief calculation will show that for all values of X with which we shall be concerned, u_0 is small in comparison with the average numerical values of u . We may perfectly legitimately neglect squares of u_0 , and, in particular, in calculating any quantity which has ultimately to be multiplied by u_0 , we may assume formula (1) to give the distribution of velocities.

4. We proceed now to calculate an expression for the transfer of momentum between electrons and molecules. We shall do this first, for simplicity, upon the supposition that the motion consists of free paths and collisions. Afterwards we shall find a perfectly general expression.

Consider an electron approaching a molecule with a velocity of components u, v, w . It will describe a curved orbit having its free path before collision and its free path after collision as asymptotes. Its loss of momentum at collision will depend on u, v, w , on the orientation of the molecule, and on the position in which the free path before collision meets a perpendicular plane through the centre of the molecule.

We know that all positions are equally likely for this latter point, and if the solid is isotropic all orientations are equally likely for the molecule. On averaging, we can find the probable loss of momentum at collision. Clearly this average

* It is important to remember that Maxwell's law gives the partition of velocities for particles in collision or acted on by any field of force, as well as when on a free-path. The existence of a field of force alters the distribution of density, but not of velocities (*cf.* the author's 'Dynamical Theory of Gases,' p. 78).

loss of momentum, regarded as a vector, can have no direction except that of the initial velocity u, v, w : its amount will depend on the constants of the electron and molecules and on $u^2 + v^2 + w^2$, but not on u, v , and w separately. We can accordingly suppose that the original velocity u, v, w is reduced by collision to

$$(1-\alpha)u, \quad (1-\alpha)v, \quad (1-\alpha)w,$$

where α depends on $u^2 + v^2 + w^2$ and constants only.

In a time dt which is large compared with the time of a collision, the loss to the momentum of the N electrons is therefore of the form

$$N\gamma u_0 dt, \quad N\gamma v_0 dt, \quad N\gamma w_0 dt,$$

where u_0, v_0, w_0 are average values of u, v, w and γ depends on $u^2 + v^2 + w^2$ and constants only.

We accordingly have the equation

$$\frac{d}{dt}(Nmu_0) = NXe - N\gamma u_0, \quad . \quad . \quad . \quad (3)$$

provided the interval of time dt is taken to be large compared with the time of collision.

5. Let us now carry out the corresponding calculation without assuming the existence of free paths. We fix our attention on all the electrons of which the velocity-components at a given instant $t = 0$ lie within a small range $du dv dw$ surrounding the values u, v, w . If there were no externally applied electric force, the law of distribution of these electrons in space would be

$$NAe^{-2h\chi} dx dy dz \quad . \quad . \quad . \quad (4)$$

where χ is the potential energy of an electron at the point x, y, z and A is such that

$$A \iiint e^{-2h\chi} dx dy dz = 1,$$

the integral being taken throughout a unit volume. With an electric force X acting, the law will be different from this by terms of the order of $\frac{u_0}{u}$, but these terms may be neglected whenever their retention would lead to ultimate terms of the order of $\frac{u_0^2}{u^2}$.

Assuming (4) to be the law of distribution of these electrons in space, we calculate the values of u, v, w after time τ , corresponding to all initial positions of the electrons,

and average according to the law (4). In this way we find the average velocity components u' , v' , w' after time τ . The vector u' , v' , w' must, from symmetry, be in the same direction as u , v , w : its amount will depend on $u^2 + v^2 + w^2$ and on the constants of the matter and the electrons.

Thus we must have

$$u' = uf(\tau, u^2 + v^2 + w^2), \text{ \&c. ;}$$

and hence, on averaging for all values of u , v , w , we find that if u_0' , v_0' , w_0' are the values of u_0 , v_0 , w_0 after time τ ,

$$u_0' = u_0\phi(\tau, h). \quad . \quad . \quad . \quad . \quad (5)$$

If we could evaluate ϕ , this equation would be the generalization, in integral form, of equation (3) with $X=0$.

We can obtain some information as to the function ϕ . We notice first that at time $t=0$,

$$m \frac{du}{dt} = - \frac{d\chi}{dx};$$

so that, on averaging, the value of $\frac{du_0}{dt}$ is zero, and hence $\phi(\tau, h)$, for small values of τ , is of the form $1 + \tau^2 f(h)$.

Next we consider the form of the function for large values of τ . Let τ be so large compared with the time of encounter with a molecule, that the velocities of those electrons which originally had velocities u , v , w may be regarded as distributed at random. Then, if u_0'' , v_0'' , w_0'' are the values of u_0 , v_0 , w_0 after a time 2τ , we shall have, in addition to equation (5),

$$u_0'' = u_0\phi(2\tau, h); \quad u_0'' = u_0'\phi(\tau, h), \text{ \&c.,}$$

whence it follows that

$$\phi(\tau) = Ae^{-B\tau},$$

and equation (6) can be put in the form

$$\frac{du_0}{dt} = -Bu_0$$

provided dt is sufficiently large. This equation is exactly of the form of (3) with $X=0$. It follows that the general equation, no matter what the nature of the motion of the electrons, is of the type of equation (3).

6. We now return to equation (3), which with a suitable value for γ will express the relation between X and u_0 in any motion in which the quantities do not change too rapidly with the time.

In the steady state in which a steady current i_x is maintained by an electric force X , the left-hand member must vanish, so that we have

$$u_0 = \frac{Xe}{\gamma}$$

or by equation (2),

$$i_x = \frac{Ne^2}{\gamma} X \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

giving Ohm's law. The conductivity for steady currents, which we shall call κ , is given by

$$\kappa = \frac{Ne^2}{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we replace u_0 and γ in equation (3) by their values as given by equations (2) and (7), we obtain

$$\frac{di_x}{dt} = \frac{Ne^2}{m} \left(X - \frac{i_x}{\kappa} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

7. Corresponding to a periodic impressed force

$$X = X_0 \cos pt,$$

the solution of this equation is

$$i_x = \kappa X_0 \cos (pt - \epsilon) \cos \epsilon, \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\tan \epsilon = \kappa p \frac{m}{Ne^2}.$$

We can readily calculate the rate at which energy is dissipated by the resistance. In time dt and in volume $dx dy dz$, an amount of electricity $i_x dt dy dz$ falls through a potential-difference $X_0 \cos pt dx$, its gain in momentum being nil. The work dissipated is accordingly

$$i_x X_0 \cos pt dx dy dz dt,$$

so that the loss of energy per unit volume per unit time is

$$\begin{aligned} i_x X_0 \cos pt &= \kappa X_0^2 \cos \epsilon \cos (pt - \epsilon) \cos pt \\ &= \kappa X_0^2 (\cos^2 \epsilon \cos^2 pt + \cos \epsilon \sin \epsilon \cos pt \sin pt). \end{aligned}$$

The average value of this, averaged over a number of complete periods, is

$$\frac{1}{2} \kappa X_0^2 \cos^2 \epsilon = \frac{1}{2} X_0^2 \frac{\kappa}{1 + \frac{\kappa^2 p^2 m^2}{N^2 e^4}}.$$

If c is the conductivity for currents of frequency p , this expression must be equal to $\frac{1}{2} c X_0^2$, and hence we must have

$$c = \frac{\kappa}{1 + \frac{\kappa^2 p^2 m^2}{N^2 e^4}} \quad \dots \quad (10)$$

8. For the same quantity c , Sir J. J. Thomson, by a different method, obtains

$$c = \kappa \left(\frac{\sin \frac{1}{2} p t}{\frac{1}{2} p t} \right)^2, \quad \dots \quad (11)$$

where t is the time of description of a free-path, and is connected with κ by the relation

$$\kappa = \frac{1}{2} \frac{N e^2}{m} t.$$

If we eliminate t between this and (11) we get as Thomson's value for c ,

$$c = \kappa \left(\frac{\sin \frac{\kappa p m}{N e^2}}{\frac{\kappa p m}{N e^2}} \right)^2 = \kappa \left(1 - \frac{1}{3} \frac{\kappa^2 p^2 m^2}{N^2 e^4} + \dots \right).$$

The divergence between this and formula (10) can, I think, be traced to the fact that Thomson's system is, so to speak, kinematical and not dynamical. In a dynamical system the time of a free-path must depend on the velocity with which the path is described.

Propagation of Light.

9. Let X be the intensity parallel to Ox , measured in electromagnetic units, and i_x the corresponding component of convection-current, connected with X by equation (8). The total current parallel to Ox is

$$\frac{K}{4\pi C} \frac{dX}{dt} + i_x, \quad \dots \quad (12)$$

so that this is equal to $\frac{1}{4\pi} \left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right)$.

For waves of frequency p , we can take both X and i_x proportional to e^{ipt} , so that from equation (8),

$$i_x = \frac{1}{ip \left(\frac{1}{\kappa} + \frac{m}{N e^2} ip \right)} \frac{dX}{dt}, \quad \dots \quad (13)$$

and the total current (12) becomes

$$\frac{K'}{4\pi C} \frac{dX}{dt},$$

where

$$K' = K + \frac{4\pi C}{ip\left(\frac{1}{\kappa} + \frac{m}{Ne^2}ip\right)} \quad \dots \quad (14)$$

The analysis for the propagation of light is now identical in form with that for the propagation of light in a non-conducting medium of inductive capacity K' .

10. It will be seen that the equations arrived at in this way are exactly identical with those given by Drude, although reached in a different manner. Our original differential equations (3) and (8) we found to be true only when the infinitesimal interval dt could be supposed at least as great as the time of a collision. Thus our equations will be true only for light of period much greater than the time of a collision.

Number of Free Electrons per Unit Volume.

11. As Thomson has noticed*, the experiments of Hagen and Rubens†, combined with a formula expressing the variation of conductivity with frequency, will give us information as to the value of N . The velocity of an electron is about 10^7 , the radius of an atom about 10^{-8} cm. The time of collision is therefore probably about 10^{-15} sec.; for light of wave-length $\lambda = 4\mu$, the time of vibration would be about thirteen times the time of collision. It would therefore appear to be legitimate to use our formula for light of wave-length 4μ and greater, but probably not for wave-length much less than this.

Rubens and Hagen denote by C_λ the product of $(100-R)$ (where R is the reflecting power for wave-length λ) and the square root of κ , the conductivity for infinite wave-length. They denote by C'_λ the quantity $\frac{36.5}{\sqrt{\lambda}}$. According to a theory which neglects the variation of conductivity with frequency, C'_λ ought to be equal to C_λ . According to our equations C'_λ ought to be equal to $(100-R)\sqrt{c}$, so that we ought to have

$$\left(\frac{C_\lambda}{C'_\lambda}\right)^2 = \frac{\kappa}{c} = 1 + \frac{\kappa^2 p^2 m^2}{N^2 e^4}, \quad \dots \quad (15)$$

so long at least as λ is not less than about 4μ .

* 'The Corpuscular Theory of Matter,' p. 84.

† Phil. Mag. vii. p. 165 (1904).

This equation neglects resonance, and that this neglect is not justifiable is clear from the circumstance that in many of Rubens and Hagen's experiments the value of C_λ/C'_λ is less than unity. We shall avoid this difficulty, in so far as it can be avoided, by a method due to Schuster*. If we allow for resonance, equation (15) must be replaced by

$$\left(\frac{C_\lambda}{C'_\lambda}\right) = 1 + \frac{\kappa^2 p^2 m^2}{N^2 e^4} - S,$$

where S is a term required by resonance, which, from the principle of conservation of energy, can be shown to be necessarily positive. It follows that the true value of N is necessarily less than the value calculated from equation (15).

12. No metal seems to be sufficiently regular in its optical behaviour in the infra-red to justify us in neglecting S altogether. One of the most regular is platinum. For this Hagen and Rubens find †

$$\left(\frac{C_\lambda}{C'_\lambda}\right)^2 = 2.0, 1.17 \text{ at } \lambda = 4\mu, 8\mu \text{ respectively.}$$

The corresponding upper limits for N are 2.3×10^{23} and 2.7×10^{23} respectively. To raise the temperature of 2.3×10^{23} electrons by 1°C . requires .75 calorie, while to raise the temperature of 1 c.c. of platinum by 1°C . requires only .69 calorie. Thus the greatest value for N permitted by the known specific heat of the substance is 2.1×10^{23} .

For gold, which is less regular in its optical behaviour, the upper limit given for N by equation (13) is 6×10^{23} , while the greatest value consistent with the known specific heat of gold is 1.9×10^{23} .

The substance for which our equations give the smallest upper limit for N is steel. From the reflecting power at $\lambda = 8\mu$, the upper limit for N is $.96 \times 10^{23}$, while the specific heat of steel admits a value as great as 3×10^{23} . But our theory has taken no account of the magnetic properties of the metals concerned, so that probably the upper limit obtained for N on the supposition that $\mu = 1$ for steel is untrustworthy.

Except for steel, there seems to be no substance for which the upper limit for N is not greater than the upper limit already set by the specific heat of the substance. In so far as our calculations assign a value for N at all, these values are in general agreement with the law derived by Schuster ‡

* Phil. Mag. vii. p. 154 (1904).

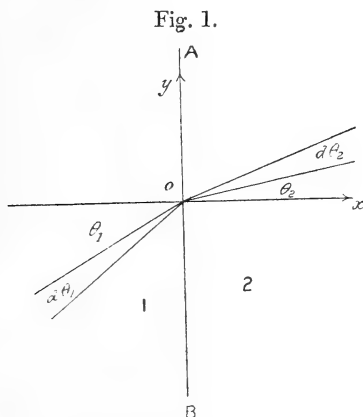
† I have used the numerical values given in Table II. p. 165, of the paper of Hagen and Rubens.

‡ Phil. Mag. vii. p. 151 (1904).

from the application of Drude's equations to visible light : "the number of free electrons in a metal is equal to the number of atoms, or exceeds that number not more than three times." Or, if n is the number of atoms per unit volume, $N = pn$, where p lies between 1 and 3. Assuming as we do that the electrons have energy appropriate to the temperature of the body, the law of Dulong and Petit sets the upper limit 2.2 to the value of p for all metals, and a still smaller upper limit for those metals for which the atomic heat is less than 6.5 (*e. g.* for platinum, atomic heat = 6.29, corresponding limit for p assigned by specific heat is 2.1, our limit for p is 2.3, Schuster's limit for p is 1.91). When the limiting value for p is found to be close to the limit allowed in this way by the specific heat, the inference is that almost all the heat-energy of the substance may reside in its free electrons. In such a case (if any such exists) the atoms must form an almost stationary network of obstacles through which the electrons move. We shall return to this later (§ 23).

Kirchhoff's Law.

13. Let medium 1 be air and medium 2 be metal. Let a beam of radiation of frequency p be incident on the boundary AB of the media, bounded by cones of angles θ_1 and $\theta_1 + d\theta_1$ in medium 1, and passing into medium 2 between cones θ_2 and $\theta_2 + d\theta_2$.



If K_1, K_2 are the inductive capacities of the media, and C the velocity of radiation *in vacuo*, we have the equations

$$\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} = \frac{K_1}{C} \frac{dX}{dt}, \text{ \&c.}$$

in air : and in the metallic medium

$$\begin{aligned}\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} &= \frac{K_2}{C} \frac{dX}{dt} + 4\pi i_x \\ &= \frac{K_2'}{C} \frac{dX}{dt}, \text{ \&c.} \quad . \quad . \quad . \quad (16)\end{aligned}$$

where K_2' is given by equation (14).

The velocity V_1 of radiation in medium 1 is given by $V_1^2 = C^2/K_1\mu_1$. We introduce an angle θ_2' and a velocity V_2' given by

$$\frac{\sin^2 \theta_2'}{\sin^2 \theta_1} = \frac{\frac{1}{K_2' \mu_2}}{\frac{1}{K_1 \mu_1}} = \frac{V_2'^2}{V_1^2} \quad . \quad . \quad . \quad (17)$$

Clearly both $\sin \theta_2'$ and V_2' will be complex, but their ratio is real.

The exponential through which the time and space-coordinates enter in the radiation in medium (1) may be taken to be

$$\exp. ip \left(t - \frac{1}{V_1} (x \cos \theta_1 + y \sin \theta_1) \right);$$

the corresponding factor in medium (2) will be

$$\exp. ip \left(t - x \frac{\cos \theta_2'}{V_2'} - y \frac{\sin \theta_2'}{V_2'} \right) \quad . \quad . \quad . \quad (18)$$

Remembering that $\frac{\sin \theta_2'}{V_2'}$ is real, we see that it will be possible to introduce two new quantities θ_2 and V_2 , both real, such that

$$\begin{aligned}\frac{\sin \theta_2'}{V_2'} &= \frac{\sin \theta_2}{V_2}; \\ \frac{\cos \theta_2'}{V_2'} &= \frac{\cos \theta_2}{V_2} - i\epsilon.\end{aligned}$$

The exponential (18) now becomes

$$e^{-\epsilon p x} \exp. ip \left(t - \frac{1}{V_2} (x \cos \theta_2 + y \sin \theta_2) \right),$$

and we now see that θ_2 and V_2 are the angle of refraction and velocity of the radiation in medium 2.

14. We introduce a new complex quantity u_{12} , associated with radiation passing from medium 1 to medium 2, and

given for vibrations polarized in the plane of incidence by

$$u_{12}^2 = \frac{K_2'}{\mu_2} \frac{\mu_1 \cos^2 \theta_2'}{\cos^2 \theta_1}, \quad \dots \quad (19)$$

and for vibrations polarized perpendicular to the plane of incidence, by

$$u_{12}^2 = \frac{\mu^2}{K_2'} \frac{K_1 \cos^2 \theta_2'}{\cos^2 \theta_1} \cdot \dots \quad (20)$$

Then if R_{12} denotes the coefficient of reflexion of radiation incident at angle θ_1 in medium (1), it can be shown in the usual way * that

$$R_{12} = \left| \frac{1 - u_{12}}{1 + u_{12}} \right|^2,$$

where $|\alpha + i\beta|$ denotes the modulus of $\alpha + i\beta$, namely $\sqrt{\alpha^2 + \beta^2}$.

For the coefficient of reflexion of radiation incident at angle θ_2 in medium (2), we have similarly

$$R_{21} = \left| \frac{1 - u_{21}}{1 + u_{21}} \right|^2,$$

where u_{21} is given by equations similar to (19) and (20), but having K_1, μ_1, θ_1 interchanged with K_2', μ_2, θ_2' . It is at once

obvious that $u_{21} = \frac{1}{u_{12}}$, and from this it follows that $R_{12} = R_{21}$.

It should be noticed that in this equation R_{12} is evaluated for light incident at an angle θ_1 , but R_{21} for light incident at an angle θ_2 .

15. Let media 1 and 2 be filled with radiation such that the energy per unit volume of radiation of frequency between p and $p + dp$ is $E_1 dp$ in medium 1 and $E_2 dp$ in medium 2.

The stream of energy which falls onto the boundary at an angle between θ_1 and $\theta_1 + d\theta_1$ per unit time is

$$\frac{1}{2} E_1 V_1 \cos \theta_1 \sin \theta_1 d\theta_1 dp,$$

so that the amount transmitted from medium 1 to medium 2 is

$$\frac{1}{2} E_1 V_1 dp \int (1 - R_{12}) \cos \theta_1 \sin \theta_1 d\theta_1. \quad \dots \quad (21)$$

Similarly the amount transmitted from 2 to 1 is

$$\frac{1}{2} E_2 V_2 dp \int (1 - R_{21}) \cos \theta_2 \sin \theta_2 d\theta_2. \quad \dots \quad (22)$$

In a state in which there is an equal exchange of energy

* Jeans, 'Electricity and Magnetism,' p. 523.

between the two media these expressions must be equal. Now we have, from equation (17),

$$\frac{\sin \theta_1}{V_1} = \frac{\sin \theta_2'}{V_2'} = \frac{\sin \theta_2}{V_2},$$

and hence

$$\frac{\cos \theta_1 \sin \theta_1 d\theta_1}{V_1^2} = \frac{\cos \theta_2 \sin \theta_2 d\theta_2}{V_2^2}.$$

Since R_{12} is the same function of θ_1 as is R_{21} of θ_2 , it follows that

$$\frac{1}{V_1^2} \int (1 - R_{12}) \cos \theta_1 \sin \theta_1 d\theta_1 = \frac{1}{V_2^2} \int (1 - R_{21}) \cos \theta_2 \sin \theta_2 d\theta_2, \quad \dots \dots (23)$$

so that expressions (21) and (22) will be equal if

$$E_1 V_1^3 = E_2 V_2^3. \quad \dots \dots (24)$$

This is a known result of which we have now obtained a proof by electrodynamical, without thermodynamical, principles.

16. The energy in a cavity in a body of any kind being of the form

$$E_1 = f(T, p), \quad \dots \dots (25)$$

it follows that the energy in the interior of the body itself must be of the form

$$E_2 = \left(\frac{V_1}{V_2} \right)^3 f(T, p), \quad \dots \dots (26)$$

so that the partition of energy inside matter can be determined in terms of black-body radiation. Using this value for E_2 , the total emission per unit area per unit time from the surface of the body, by equation (22),

$$\begin{aligned} &= dp f(T, p) \frac{V_1^3}{2V_2^2} \int (1 - R_{21}) \cos \theta_2 \sin \theta_2 d\theta_2 \\ &= dp f(T, p) \frac{1}{2} V_1 \int (1 - R_{12}) \cos \theta_1 \sin \theta_1 d\theta_1, \end{aligned}$$

by equation (23). Since θ_1 is the angle the issuing radiation makes with the normal, the limits for θ_1 are from 0 to $\frac{\pi}{2}$. The coefficient of absorption A for light incident on the

body is given by

$$A = \frac{\int_0^{\frac{\pi}{2}} (1 - R_{12}) \cos \theta_1 \sin \theta_1 d\theta_1}{\int_0^{\frac{\pi}{2}} \cos \theta_1 \sin \theta_1 d\theta_1} \\ = 2 \int_0^{\frac{\pi}{2}} (1 - R_{12}) \cos \theta_1 \sin \theta_1 d\theta_1,$$

so that the stream of issuing radiation is

$$\frac{1}{4} A f(T, p) dp.$$

Putting $A=1$ we pass to the case of an ideal perfectly black body, and find that the stream of issuing radiation must be $\frac{1}{4} f(T, p) dp$.

Hence, in general the stream issuing from any body is A times the stream issuing from a black body, as required by Kirchhoff's law.

We have derived Kirchhoff's law as a consequence purely of electron theory. It appears that the law is true quite independently of whether the æther is in equilibrium with matter or not: in fact the law is seen to be entirely independent of thermodynamic conditions of all kinds. It is consequently illegitimate to draw any thermodynamical inferences from the fact that Kirchhoff's law is observed to be true in nature.

Emission of Radiation.

17. If charges e, e', \dots move with velocities $\mathbf{u}, \mathbf{u}', \dots$ there is no radiation if $\sum e \frac{d\mathbf{u}}{dt} = 0$. If this condition is not satisfied, let $E\mathbf{U}$ be the vector which is the resultant of the vectors $e\mathbf{u}, e'\mathbf{u}', \dots$. Then

$$-E \frac{d\mathbf{U}}{dt} + \sum e \frac{d\mathbf{u}}{dt} = 0,$$

so that the radiation from E, e, e', \dots is nil. Hence the radiation from e, e', \dots moving with velocities $\mathbf{u}, \mathbf{u}', \dots$ is equal to and identical with that of a single charge E moving with velocity \mathbf{U} . In this result it has to be supposed that the distances apart of the charges e, e', \dots are small compared with the wave-length of the emitted radiation, or at least of that part of it with which we are concerned.

Let us take an element of volume dv which contains a great number of electrons $N dv$, and let us suppose for the present that the linear dimensions of dv are small compared with the wave-length of the light with which we are concerned. Let these electrons have velocities of components

$$u, v, w; u', v', w' \dots,$$

and let the mean of these components, as before, be u_0, v_0, w_0 .

Then we can replace the radiation from these electrons by the radiation from an electron of charge $Ne dv$ having velocity components u_0, v_0, w_0 . If f is the acceleration of this electron, the radiation from it in time t is*

$$\int_0^t \frac{2\mu}{3V} (Ne dv f)^2 dt.$$

The components of the total current i in the element dv are

$$Ne dv u_0, Ne dv v_0, Ne dv w_0,$$

so that we have

$$Ne dv f = \frac{di}{dt},$$

and the radiation in time t is

$$\frac{2\mu}{3V} \int_0^t \left(\frac{di}{dt} \right)^2 dt.$$

We can express $\frac{di}{dt}$ in a Fourier-series in the form

$$\frac{di}{dt} = \frac{1}{\pi} \int_{p=0}^{p=\infty} (A_p \cos pt + B_p \sin pt) dp,$$

where

$$\left. \begin{aligned} A_p &= \int_0^t \frac{di}{dt} \cos pt dt, \\ B_p &= \int_0^t \frac{di}{dt} \sin pt dt, \end{aligned} \right\} \dots \dots \dots (27)$$

* Here V is the velocity in the medium, and μ is the magnetic permeability. The result is easily obtained by modifying Larmor's calculation ('Æther and Matter,' p. 227).

The radiation is calculated by integrating over a sphere of large radius, so that we must imagine all absorbing and dispersing electrons removed from inside this sphere. Thus V is not the velocity in the actual medium, but in the medium when freed from absorption and dispersion. It is given by $V^2 = C^2/K\mu$, and becomes identical with V_2 , the actual velocity of waves in the medium, when $p=0$.

and now have*

$$\int_0^t \left(\frac{di}{dt}\right)^2 dt = \frac{1}{\pi} \int_0^\infty (A_p^2 + B_p^2) dp.$$

The total emission of the element dv in time t is accordingly

$$\frac{2\mu}{3\pi V} \int_0^\infty (A_p^2 + B_p^2) dp, \quad . \quad . \quad . \quad (28)$$

and in this expression the coefficient of dp represents the emission of radiation of frequency between p and dp .

Instead of (27) we may take

$$\begin{aligned} A_p &= \int_0^t \frac{di}{dt} \cos pt \, dt \\ &= \left| i \cos pt \right|_0^t + p \int_0^t i \sin pt \, dt; \end{aligned}$$

when t is large enough, the first term may be ignored, and we may take

$$A_p = p \int_0^t i \sin pt \, dt, \quad . \quad . \quad . \quad (29)$$

$$B_p = -p \int_0^t i \cos pt \, dt. \quad . \quad . \quad . \quad (30)$$

The calculation of the emission requires the evaluation of these integrals.

18. Let us first perform the calculation on the simplifying assumption that the time t can be divided into n equal free-path periods each of time τ , and let us suppose that at the end of each of these periods the velocities of the electrons are replaced by new velocities which have no reference to the old. We have, by equation (29),

$$\begin{aligned} A_p &= p \sum_{s=0}^{s=n-1} \int_0^\tau i \sin p(t+s\tau) \, dt \\ &= \sum_{s=0}^{s=n-1} 2i \sin p\left\{t + \left(s + \frac{1}{2}\right)\tau\right\} \sin \frac{1}{2}p\tau, \end{aligned}$$

so that, since there is no relation between the values of i on

* Rayleigh, Phil. Mag. [5] xxvii. p. 466.

the different free-path periods,

$$A_p^2 = \sum_{s=0}^{s=n-1} 4i^2 \sin^2 p\{t + (s + \frac{1}{2})\tau\} \sin^2 \frac{1}{2} p\tau$$

$$B_p^2 = \sum_{s=0}^{s=n-1} 4i^2 \cos^2 p\{t + (s + \frac{1}{2})\tau\} \sin^2 \frac{1}{2} p\tau,$$

and consequently

$$A_p^2 + B_p^2 = 4ni^2 \sin^2 \frac{1}{2} p\tau, \quad . \quad . \quad . \quad (31)$$

where i^2 now denotes the average value of i^2 in the different intervals.

We can easily evaluate i^2 . The x -component of i is given by

$$i_x = e(u + u' + \dots),$$

so that

$$i_x^2 = e^2(u^2 + u'^2 + \dots) = N dv e^2 \frac{RT}{m},$$

and hence

$$i^2 = 3RT \frac{Ne^2}{m} dv. \quad . \quad . \quad . \quad (32)$$

Substituting in expression (28) from equations (31) and (32), the emission in time t is found to be

$$\left(\int \frac{8\mu}{\pi V} \frac{\sin^2 \frac{1}{2} p\tau}{\tau} \frac{Ne^2}{m} RT dp \right) t dv. \quad . \quad . \quad . \quad (33)$$

This expression is proportional to t and to dv as it ought to be. We notice also that it vanishes if τ is either very great or very small, so that when the motion is regarded as made up of free-paths, the whole phenomenon of emission depends on the free-path being finite.

From the formula for the emission we can readily calculate the partition of radiant energy in the matter in the steady state in which emission and absorption are equal to one another. Let the energy in this state be

$$\int E_p dp$$

per unit volume. Then, as Thomson shows*, the absorption of energy in the element dv in time t must be

$$\left(\int \frac{4\pi C^2}{K} c E_p dp \right) t dv, \quad . \quad . \quad . \quad (34)$$

where c is the conductivity of the medium for currents of

* Phil. Mag. xiv. p. 223.

frequency p . The proper value to use for c when we suppose the motion made up of free-paths of equal duration is that given by Thomson, namely (in our notation)

$$c = 2 \frac{\sin^2 \frac{1}{2} p \tau}{p^2 \tau} \frac{Ne^2}{m}.$$

Using this value of c , and equating the emission (33) to the absorption (34), we obtain

$$E_p = \frac{p^2 RT K \mu}{\pi^2 C^2 V} = \frac{p^2 RT}{\pi^2 V^3}, \quad . \quad . \quad . \quad (35)$$

19. We next examine how this must be modified when we do not assume the existence of free-paths.

If t_1, t_2 are any instants within the interval from 0 to t , and if i_1, i_2 are the values of i at these instants, we have, by equation (29)

$$A_p = p \int_0^t i_1 \sin pt_1 dt_1 = p \int_0^t i_2 \sin pt_2 dt_2,$$

so that, by multiplication,

$$A_p^2 = p^2 \int_0^t \int_0^t i_1 i_2 \sin pt_1 \sin pt_2 dt_1 dt_2.$$

Similarly from equation (30)

$$B_p^2 = p^2 \int_0^t \int_0^t i_1 i_2 \cos pt_1 \cos pt_2 dt_1 dt_2,$$

and by addition

$$\begin{aligned} A_p^2 + B_p^2 &= p^2 \int_0^t \int_0^t i_1 i_2 \cos p(t_1 - t_2) dt_1 dt_2 \\ &= p^2 \int_{t_1=0}^{t_1=t} \int_{t_2=0}^{t_2=t} i_1 i_2 \cos p(t_1 - t_2) d(t_1 - t_2) d\left(\frac{t_1 + t_2}{2}\right). \end{aligned} \quad (36)$$

Let the two instants t_1, t_2 be at equal intervals θ from their middle instant t' of which the value is $\left(\frac{t_1 + t_2}{2}\right)$. Let i' be the current at t' . The expectation of rate of change of i is (cf. equation (8))

$$\frac{di}{dt} = -\frac{Ne^2}{m\kappa} i, \quad . \quad . \quad . \quad (37)$$

of which the integral is

$$i = i_0 e^{-\epsilon t}, \quad . \quad . \quad . \quad (38)$$

where $\epsilon = \frac{Ne^2}{m\kappa}$. Hence, knowing the current i at time t' , the value of i_2 , the current at time t_2 , *i. e.* after an interval θ , is

$$i_2 = i' e^{-\epsilon\theta} + j,$$

where j is a quantity of which the "expectation" is zero. Similarly, since the motion is reversible, the value of i_1 is

$$i_1 = i' e^{-\epsilon\theta} + j',$$

where j' has expectation zero, and has no correlation with j . Putting in these values, equation (36) becomes

$$A^2 + B^2 = p^2 \iint i'^2 e^{-2\epsilon\theta} \cos 2p\theta d(2\theta) dt', \quad \dots (39)$$

the terms in j, j' and jj' being omitted because, as their average value is zero, they vanish on integration. The exponential $e^{-2\epsilon\theta}$ vanishes very rapidly as the interval 2θ between t_1 and t_2 increases—this is the mathematical expression of the fact that there is very little correlation between the values of i at intervals of time far apart. Hence we may integrate with respect to θ from $-\infty$ to $+\infty$, taking θ always positive in the exponential $e^{-2\epsilon\theta}$, and so obtain

$$\begin{aligned} A^2 + B^2 &= p^2 \int_{t'=0}^{t'=t} i'^2 \frac{2\epsilon}{\epsilon^2 + p^2} dt' \\ &= \frac{2p^2\epsilon}{\epsilon^2 + p^2} 3RT \frac{Ne^2}{m} t dv, \end{aligned}$$

by equation (32). Replacing ϵ by its value, this gives the emission of the element dv in the form (*cf.* equation (28))

$$\left(\int \frac{4\mu}{\pi V} \frac{\kappa}{1 + \frac{\kappa^2 p^2 m^2}{N^2 e^4}} p^2 RT dp \right) t dv. \quad \dots (40)$$

This, as before, must be equal to the absorption given by expression (34). The proper value for c is now that calculated for continuous motion, and given in equation (10). Thus the absorption must be

$$\left(\int \frac{4\pi C^2}{K} \frac{\kappa}{1 + \frac{\kappa^2 p^2 m^2}{N^2 e^4}} E_p dp \right) t dv,$$

and equating this to expression (41) we obtain

$$E_p = \frac{p^2 RT K \mu}{\pi^2 C^2 V} = \frac{p^2 RT}{\pi^2 V^3}, \quad (41)$$

which is exactly the value obtained before (equation (35)).

20. We notice that E_p depends on the structure of the medium only through the factor $1/V^3$. This is as it should be, by equation (26), to accord with the experimental fact that the radiation in a cavity is independent of the nature of the matter. From equations (25), (26), and (41), it follows that the radiation in the cavity must be given by

$$E_p = \frac{p^2 RT}{\pi^2 V_1^3}, \quad (42)$$

where V_1 is the velocity in the cavity.

If λ is the wave-length of radiation of frequency p we have $p\lambda = 2\pi V$ in metal, and $= 2\pi V_1$ in the cavity. The radiation $\int E_p dp$ in the metal, E_p being given by equation (41), accordingly transforms into

$$\int 8\pi RT \lambda^{-4} d\lambda, \quad (43)$$

when expressed in terms of λ , and the energy (42) in air transforms into exactly the same expression. Moreover, expression (43) expresses the partition of energy demanded by the law of equipartition*, both for the metal and the cavity.

21. When dealing with waves of shorter wave-length the principal modification which has to be made originates in the need for changing equation (38). Other modifications are needed which alter the radiation by an appreciable fraction of its whole amount, but it is easily seen that this particular modification changes the order of magnitude of the radiation†.

For, by § 5, when p is very great, we must replace equation (38), namely

$$i = i_0 e^{-\epsilon t},$$

by an equation of the form (*cf.* equation (5))

$$i = i_0 \phi(t),$$

in which $\frac{\partial \phi}{\partial t} = 0$ when $t = 0$.

* Phil. Mag. xvii. p. 231.

† These remarks apply only to natural radiation, and not to the radiation inside an ideal perfectly-reflecting enclosure. For this latter radiation it will be proved, in the second part of this paper, that equations (38) and (43) are true throughout the whole spectrum.

Thus in equation (39), instead of an integral of the form

$$\int e^{\pm 2\epsilon\theta} \cos 2p\theta d(2\theta), \dots \dots \dots (44)$$

we have an integral of the form

$$\int \phi(2\theta) \cos 2p\theta d(2\theta). \dots \dots \dots (45)$$

The value of the integral (44) is

$$\frac{2\epsilon}{\epsilon^2 + p^2},$$

but, in virtue of the fact that $\frac{d\phi}{dt} = 0$ when $t=0$, the value of the integral (45) falls off as $e^{-\beta p}$ (where β is independent of p) when p is very great*.

This is perfectly in accordance with observation, and it seems to be as far as the theory can be carried without introducing special laws of force between electrons and matter. This will be done in another paper.

Conclusion.

22. The result of § 20 seems to prove beyond reasonable doubt that there is equipartition of energy between the different vibrations of great wave-length, both in the interior of the matter and in a cavity in the matter.

Each vibration has the energy appropriate to a temperature T which has been introduced into our analysis as the temperature determined by the kinetic energy of the free electrons.

Two pieces of evidence identify this temperature T with what we call the temperature of the matter. There is first the evidence provided by the observed energy of radiation of great wave-length†; there is, secondly, the evidence provided by the observed energy of the electrons escaping from hot metals‡.

Thus we can say that the temperature of a solid is defined equally well either by

(i.) the mean energy of vibrations of great wave-length in its interior; or

(ii.) the mean kinetic energy of the free electrons in its interior.

* 'Dynamical Theory of Gases,' §§ 237-240.

† Lummer and Pringsheim, *Verhand. d. deutscher phys. Gessellschaft*, 1900, p. 163.

‡ Richardson and Brown, *Phil. Mag.* xvi. p. 353.

We have seen that these two definitions lead to the same temperature. The first definition explains at once the tendency for the temperatures of two bodies to equalise by heat-radiation; the second definition explains the tendency for their temperatures to equalise by conduction.

In a gas the temperature is defined in only one way, namely, by the mean energy of translation of its molecules. The question arises as to whether the temperature of a solid can also be defined in a similar way.

So far as present evidence goes, it seems as though this question must be answered in the negative, for the following reasons.

23. For almost all metals the atomic heat is, to within a few per cent., equal to 5.88, the value required if each atom had associated with it energy $3RT$ —*i.e.* twice the translational energy of a free electron or molecule of gas at temperature T .

If the energy of motion of the atom is governed by the temperature T , each atom must, on account of this motion, have associated with it energy exactly equal to $3RT$, half of this being contributed by its average kinetic energy ($\frac{3}{2}RT$), and half by its average potential energy ($\frac{3}{2}RT$). But it is difficult to imagine the atoms moving freely as regards translational motion without at the same time being set into rotation, and the energy of this rotation, if governed by the temperature T , would be $3RT$ per atom. Each atom would now have energy $6RT$ associated with it*.

Further, each atom has associated with it a number of free electrons of which the average, according to Schuster's table†, is about two per atom. This adds a further contribution $3RT$ to the energy to be associated with each atom.

Thus the energy per atom would, under these circumstances, seem to be about $9RT$, made up of three equal contributions of $3RT$ each from motion of atoms, rotation of atoms, and motion of electrons. The value permitted by the specific heats is uniformly $3RT$. The uniformity of this number indicates that we must attribute it to a similar origin in all substances.

It seems as if the only permissible view is that the $3RT$ is contributed by the motion of electrons. If we accept this, the rejection of the contribution $6RT$ from the atoms means that the energy of motion of these is very small in comparison

* If we take the *molecule* as unit, each molecule would have energy $6RT$, and therefore each atom, at least in diatomic substances, would have energy $3RT$, the required amount. But Dulong and Petit's law is not limited to diatomic elements, and seems to show conclusively that the *atom* must be taken as unit.

† Phil. Mag. vii. p. 155.

with what it would be if governed by the temperature. The atoms now form a framework of obstacles through which the electrons move. This framework is continually set into vibration by collisions with the electrons, but its motion is dissipated into radiation so rapidly that the atoms never acquire a motion comparable with temperature-motion.

This view compels us to suppose that each atom has always two electrons associated with it*. The whole of these electrons need not be free at all temperatures. In some substances which are poor conductors, only a few electrons may be free, and in raising the temperature a large part of the energy may be used in setting free new electrons. For such substances the atomic heat need not be near 5.88 , but it should approximate to this value as we reach temperatures at which all the electrons have become free. Weber's researches† show that such an asymptotic limit exists. Here, for instance, is the variation of the atomic heat of carbon (diamond):—

Temperature ...	0°	50°	100°	150°	200°	606°	806°	985°
Atomic heat ...	1.12	1.72	2.28	2.81	3.33	5.26	5.36	5.49

The energy of motion of the atoms, although small, need not be altogether negligible, so that the limit reached when all the electrons have been set free, may be somewhat greater than the exact value 5.88 . As the melting-point is approached, the energy of the atoms will increase greatly, so that this value may be much exceeded. Thus Pionchon‡ finds for the atomic heat of iron at 500° the value 9.84 .

24. This view of the matter is still not completely satisfactory, for it leaves it difficult to understand the process of conduction of heat between a solid and a gas in contact with it. It seems as if we must suppose the atoms to form into large clusters, which themselves take up the temperature-motion from the electrons and might convey it to the molecules of a gas in contact. The motion of these clusters as they are bombarded by the electrons, would be similar to that of the solid particles in the Brownian movements. The energy of motion of the clusters, namely, $3RT$ or $6RT$ for each, would only contribute imperceptibly to the value of the specific heat.

If this last conjecture is correct there would be another way of defining the temperature of a solid in addition to the two already given; we might define it by the mean kinetic energy of its molecular clusters.

January 7, 1909.

* Except perhaps in elements whose atomic weight has been calculated solely from the specific heat.

† Pogg. *Ann.* cliv. p. 575.

‡ *C. R.* cvi. p. 1344.

LXXVI. *The Molecular and some other Constants of the Inactive Gases.* By G. RUDORF, Ph.D., B.Sc.*

PART I.

THE monatomic gases are of particular interest from the standpoint of the kinetic theory, and the present paper has for its object the calculation of some of the molecular constants in various ways with a view to seeing how far the values thus obtained agree among themselves. It may be stated at the outset that the result is disappointing.

A. *Molecular Velocities.*

The kinetic theory of gases gives us the equations

$$p = \frac{1}{3} \rho G^2, \quad \dots \dots \dots \text{(I.)}$$

and

$$\Omega = \sqrt{\frac{8}{3\pi}} \cdot G, \quad \dots \dots \dots \text{(II.)}$$

where G is the velocity calculated from the mean kinetic energy (Clausius) and Ω the mean velocity according to Maxwell, p the pressure, and ρ the density.

For a pressure of 1 atm.,

$$p = 76 \times 981 \times 13 \cdot 595 \text{ dynes/cm}^2,$$

and

$$G = \sqrt{\frac{3 \cdot 041 \times 10^6}{\rho}}.$$

To calculate G and Ω then we require to know only the density ρ .

Helium.—Ramsay and Travers (Proc. Roy. Soc. lxii. p. 316 (1898)) give the value 1·98 for the density compared with oxygen (16). Taking the weight of 1 litre of oxygen as 1·4296 g.†, a value used by them in a later paper (*v. infra*), we get for the weight of 1 litre He 0·1769 g., and hence for the density 0·0001769. As the experimental data are not given in the paper, we must be content with this figure. The value 1·98 for the density compared with oxygen has since been amply confirmed, *e. g.* Olszewski, Kamerlingh Onnes, Schierloh, &c.

Neon.—From the experimental data published by Ramsay and Travers (Phil. Trans. cxvii. p. 47 (1901)), the density works out to $\rho = 0 \cdot 00089$.

[*Note.* Owing to an oversight, the volume of the bulb used in determining the density was given as 32·038 c.c. Sir Wm. Ramsay has kindly informed me that the correct value should be 32·697 c.c.]

* Communicated by Sir W. Ramsay, K.C.B., F.R.S.

† All figures relating to densities are referred to 0° C. and 760 mm. pressure.

Argon.—Using the experimental results of Ramsay and Travers (Proc. Roy. Soc. lxiv. p. 183 (1898)), the density of pure argon is $\rho=0\cdot001782$. Here again the volume of the bulb is incorrect. It should be 160·17 c.c. instead of 163·19 c.c. or of 163·17 in the Phil. Trans. paper.

Krypton and Xenon.—The densities of these gases, carefully fractionated, have been redetermined by Moore (J. C. S. xciv. p. 2181 (1908)), and from the data published by him we get for

Krypton..... $\rho=0\cdot003709$, and for
Xenon $\rho=0\cdot005842$

as compared with 0·003644 and 0·00572 originally obtained by Ramsay and Travers (Phil. Trans. l. c.).

We have now all the data necessary for the calculation of G and Ω :—

Density of Helium ... $\rho=0\cdot0001769$
 „ Neon $\rho=0\cdot00089$
 „ Argon ... $\rho=0\cdot001782$
 „ Krypton... $\rho=0\cdot003709$
 „ Xenon ... $\rho=0\cdot005842$

And using these values, we obtain the following results :—

Molecular Velocities.

	He.	Ne.	A.	Kr.	X.
G_0 cm./sec.	$13\cdot11 \times 10^4$	$5\cdot845 \times 10^4$	$4\cdot13 \times 10^4$	$2\cdot86 \times 10^4$	$2\cdot28 \times 10^4$
Ω_0 cm./sec.	$12\cdot08 \times 10^4$	$5\cdot385 \times 10^4$	$3\cdot806 \times 10^4$	$2\cdot64 \times 10^4$	$2\cdot10 \times 10^4$

Some previous calculations are tabulated below.

Author.	Reference.	HELIUM.			ARGON.		
		Density.	G.	Ω .	Density.	G.	Ω .
O. E. Meyer.	Die kinetische Theorie der Gase. 2te Aufl. p. 194 (1899).	0·000191	$11\cdot62 \times 10^4$	0·0017810	$3\cdot81 \times 10^4$
R. Schmidt.	Ueber die Diffusion von Argon und Helium. Diss. (1904).	0·00017697	$12\cdot077 \times 10^4$	0·0017822	$3\cdot806 \times 10^4$
J. H. Jeans.	The Dynamical Theory of Gases, p. 113 (1904).	1·98 ($O_2=16$)	$13\cdot11 \times 10^4$	19·9 ($O_2=16$)	$4\cdot13 \times 10^4$

B. Mean free paths.

The kinetic theory in its simplest form gives the following relation between the viscosity coefficient η , the density ρ , the mean free path L , and the molecular velocity G :—

$$\eta = \frac{1}{3} \cdot \rho L G,$$

and this is equivalent to

$$\eta = 0.3614 \rho L \Omega.$$

O. E. Meyer (*l. c.* p. 189), after introducing several correcting factors involving complicated integrals, arrived finally at the relation

$$\eta = 0.30967 \rho L \Omega,$$

and this we will use for our calculations of L . The values for ρ and Ω are given above, and we require, therefore, only the values of the viscosity coefficients η . These have been determined for helium and argon only.

Lord Rayleigh (*Proc. Roy. Soc. lxx. p. 198 (1896)*) found, relatively to air = 1, for

Argon 1.21, and for

Helium..... 0.96.

From these results, taking the absolute value of η_0 (air) = 0.000172, Meyer calculated

$$\eta_0 (\text{Argon}) = 0.000208,$$

$$\eta_0 (\text{Helium}) = 0.000165.$$

Schultze (*Ann. d. Phys. v. p. 140 (1901)*) found for Argon

$$\eta_0 = 0.0002104,$$

and for Helium (*Ann. d. Phys. vi. p. 302 (1901)*)

$$\eta_0 = 0.0001891.$$

This latter value is considerably higher than Lord Rayleigh's, but it has been confirmed quite recently by Schierloh, working with carefully purified gases. Prof. Dorn kindly communicated these results to me; I do not think they have yet been published. Schierloh found for

$$\text{Helium, } \eta_0 = 0.0001887, \text{ and for}$$

$$\text{Argon, } \eta_0 = 0.0002114.$$

Using these values of η , we obtain for the mean free path L_0 , using the above formula,

$$\text{for Helium, } L_0 = 2.85 \times 10^{-5} \text{ cm.,}$$

$$\text{and for Argon, } L_0 = 1.006 \times 10^{-5} \text{ cm.}$$

From Lord Rayleigh's values of η , and his own values of Ω , Meyer (*l. c.* p. 194) calculated for

$$\text{Helium, } L_0 = 2.40 \times 10^{-5} \text{ cm.,}$$

$$\text{and for Argon, } L_0 = 0.99 \times 10^{-5} \text{ cm.}$$

From Schultze's values of η Schmidt calculated for

$$\text{Helium, } L_0 = 2.857 \times 10^{-5} \text{ cm.,}$$

$$\text{and for Argon, } L_0 = 1.002 \times 10^{-5} \text{ cm.}$$

J. J. Thomson ('The Conduction of Electricity through Gases,' 2nd edit. p. 450) calculated L for helium from Lord Rayleigh's viscosity value and found 2.6×10^{-5} cm., but he does not give any further numerical details.

The viscosities of neon, krypton, and xenon have not been determined.

C. *Molecular Diameters.*

The mean effective diameter of the molecule (σ) can be calculated in a number of ways; and it is of interest to see whether the results thus obtained agree among themselves. There are two general methods:—

1. The method involving L ; and
2. The method involving N .

1. *The method involving L .*

From the kinetic theory, we have the relation

$$L = \frac{1}{\sqrt{2} \cdot \pi N \sigma^2};$$

and from this by multiplying through by σ and rearranging

$$\sigma = 6\sqrt{2} \left[\frac{\pi}{6} N \sigma^3 \right] L.$$

The quantity in the bracket, $\frac{\pi}{6} N \sigma^3$, is the volume occupied by the molecules themselves in unit gas volume, and if we denote this by V , we have

$$\sigma = 6\sqrt{2} V L. \quad . \quad . \quad . \quad . \quad . \quad (A)$$

We can get at V from

- (a) The density of the liquid as an upper limit;
- (b) The coefficient b of the van der Waals' equation;
- (c) The dielectric constant; and
- (d) The refractive index.

Of course this method of calculating σ is only applicable to helium and argon, as L is not known for the other gases.

Jeans (*l. c.* p. 250) by introducing a number of corrections for the persistence of velocities after collision, &c., deduced the formula

$$L = \frac{1.31}{\sqrt{2\pi N \sigma^2}}.$$

This would give

$$\sigma = 4.58 \sqrt{2VL} \dots \dots \dots (B)$$

(a) Upper limit of σ from the density of the liquid or solid.

It is assumed that in the liquid or solid state, the molecules are so closely packed together that the volume they inhabit is practically the volume they themselves occupy. This method was first used by Loschmidt. In this case V will be simply the ratio of the density of the gas to the density of the liquid. For helium, Kamerlingh Onnes (Communications from the Phys. Labor. Leiden, Nr. 108 (1908), and *Nature*, lxxviii. p. 370 (1908)) found the density of the liquid (d) to be 0.15. For argon the highest value recorded by Baly and Donnan (*J. C. S.* lxxxi. p. 907 (1902)) is 1.423. From these values we get the following for σ :—

	ρ .	d .	$V = \frac{\rho}{d}$.	L .	σ .
Helium ...	0.0001769	0.15	1.18×10^{-3}	2.85×10^{-5} cm.	28.5×10^{-8} cm.
Argon.....	0.001782	1.423	1.252×10^{-3}	1.006×10^{-5} cm.	10.7×10^{-8} cm.

(b) From the coefficient b of van der Waals' equation.

The coefficient b is theoretically four times the volume occupied by the molecules. For helium, Kamerlingh Onnes (*l. c.*) found $b = 0.0007$. For argon we can calculate b from the critical constants (*vide* the corresponding section *infra*), and find $b = 0.001347$.

	b .	$V = \frac{b}{4}$.	L .	σ .
Helium	0.0007	0.00018	2.85×10^{-5} cm.	4.35×10^{-8} cm.
Argon	0.001347	0.000337	1.006×10^{-5} cm.	2.87×10^{-8} cm.

(c) From the dielectric constant (K).

The Clausius-Mosotti theory of dielectrics gives

$$\frac{K-1}{K+2} = V.$$

For He, K has recently been determined by Hochheim (*Verh. d. d. phys. Ges.* x. p. 446 (1908)), who found $K=1.000074$. Using this value we get

$$V=0.0000247 \quad \text{and} \quad \sigma=0.597 \times 10^{-8} \text{ cm.}$$

The dielectric constant of argon is unknown.

(d) From the refractive index (μ).

The electromagnetic theory of light gives for $\lambda=\infty$, $K=\mu^2$; hence

$$\frac{\mu^2-1}{\mu^2+2} = V;$$

and as μ does not differ greatly from unity, we get

$$V = \frac{\mu^2-1}{\mu^2+2} = \frac{2(\mu-1)}{3}.$$

Burton (*Proc. Roy. Soc.* lxxx. p. 390 (1908)) has determined the dispersion of helium and argon. He found for

$$\text{Helium, } \mu = 1.00003478 + \frac{7.5 \times 10^{-16}}{\lambda^2},$$

and for

$$\text{Argon, } \mu = 1.0002792 + \frac{1.6 \times 10^{-15}}{\lambda^2}.$$

The value for He agrees well with that obtained by Cuthbertson and Metcalfe (*Proc. Roy. Soc.* lxxx. p. 411 (1908)):

$$\mu-1 = 0.0000347 \left(1 + \frac{2.4}{\lambda^2 10^{11}} \right).$$

We have, therefore,

$$\mu_{\lambda=\infty} \text{ for Helium} = 1.0000347, \text{ and}$$

$$\mu_{\lambda=\infty} \text{ for Argon} = 1.0002792.$$

Using these values we get

$$\sigma \text{ for Helium} = 0.559 \times 10^{-8} \text{ cm., and}$$

$$\sigma \text{ for Argon} = 1.59 \times 10^{-8} \text{ cm.}$$

Summary.

The following table summarises the results obtained in this section.

Values of σ from the relation $\sigma = 6\sqrt{2}VL$.

	$V = \frac{\rho}{d}$.	$V = \frac{b}{4}$.	$V = \frac{K-1}{K+2}$.	$V = \frac{\mu^2-1}{\mu^2+2}$.
Helium ...	28.5×10^{-8} cm.	4.35×10^{-8} cm.	0.597×10^{-8} cm.	0.559×10^{-8} cm.
Argon.....	10.7×10^{-8} cm.	2.87×10^{-8} cm.	1.59×10^{-8} cm.

Using Jeans' formula, these values must be reduced in the ratio 6 : 4.58, and this will give the following values for σ .

Values of σ from the relation $\sigma = 4.58\sqrt{2}VL$.

	$V = \frac{\rho}{d}$.	$V = \frac{b}{4}$.	$V = \frac{K-1}{K+2}$.	$V = \frac{\mu^2-1}{\mu^2+2}$.
Helium ...	21.8×10^{-8} cm.	3.32×10^{-8} cm.	0.455×10^{-8} cm.	0.426×10^{-8} cm.
Argon.....	8.16×10^{-8} cm.	2.19×10^{-8} cm.	1.21×10^{-8} cm.

2. The method involving N.

In the former section we put $\frac{\pi}{6} \cdot N\sigma^3 = V$; and consequently if we know N, the number of the molecules in unit gas volume under normal conditions of temperature and pressure, we can obtain another series of values of σ independently of the mean free path L.

The most accurate values of N, which, if Avogadro's law be true, is not dependent on the nature of the gas, are those obtained by methods which have no reference to the kinetic theory. From determinations of the charge on an ion, Thomson deduced the value $N = 3.6 \times 10^{19}$ per 1 c.c., and Wilson $N = 3 \times 10^{19}$ to 6×10^{19} . From his electromagnetic theory of radiation, Planck (*Ann. d. Phys.* iv. p. 564 (1901) and *Arch. Néerl.* (ii.) vi. p. 55 (1901)) deduced the value *

* Nernst (*Theoretische Chemie*, 5te Aufl. p. 430) gives Planck's value as 5×10^{19} , but I do not know where he has got it from. The reference given is to another paper. It might be pointed out that Sirk (*Ann. d. Phys.* xxv. p. 894 (1908)) has obviously copied this figure and the wrong reference, without first verifying either—always a risky proceeding.—G. R.

$N = 2.76 \times 10^{19}$, and pointed out that, if his theory be correct, this value must be exact and not simply an approximation. Quite recently this value has been confirmed by Rutherford and Geiger (Proc. Roy. Soc. lxxxi. p. 162 (1908)) by measuring the charge on an α particle. They found $N = 2.72 \times 10^{19}$, and give reasons for considering the previous determinations of Thomson and Wilson too high.

We will, in the present paper, take $N = 2.75 \times 10^{19}$ per 1 c.c., which is probably correct to 1 per cent.

We can now, as before, calculate σ from

- (a) The density of the liquid as an upper limit ;
- (b) The coefficient b of van der Waals' equation ;
- (c) The dielectric constant ;
- (d) The refractive index.

(a) From the density of the liquid.

The following table gives the values of V for all the gases except Neon, the density of which in the liquid state is unknown. The determination of d for He is by Kamerlingh Onnes (*l. c.*), for Argon by Baly and Donnan (*l. c.*), for Kr and X by Ramsay and Travers (Phil. Trans. *l. c.*). The values for the gaseous state (ρ) are those deduced above.

	ρ .	d .	$V = \frac{\rho}{d}$.
Helium	0.0001769	0.15	1.18×10^{-3}
Neon	0.00089	?	?
Argon	0.001782	1.423	1.252×10^{-3}
Krypton	0.003709	2.155	1.72×10^{-3}
Xenon	0.005842	3.52	1.66×10^{-3}

Taking $N = 2.75 \times 10^{19}$, we get the following values of σ .

	He.	Ne.	A.	Kr.	X.
$\sigma \times 10$ cm.	4.34	?	4.43	4.93	4.88

The values for He and A are considerably smaller than those obtained in the corresponding section above ; and it is rather noteworthy that the σ -values for He, A, Kr, and X are so little different from one another.

(b) From the coefficient b of van der Waals' equation.

As this coefficient represents approximately four times the volume of the molecules, we have

$$b = 4 \left(\frac{\pi}{6} N \sigma^3 \right),$$

whence

$$\sigma = \sqrt[3]{\frac{3}{2\pi} \cdot \frac{b}{N}}.$$

b can be calculated from the isothermals and also from the critical constants, although the values thus obtained do not, as a rule, agree very well with each other. For our purpose, it will be sufficient to use the b -values obtained from the critical constants. The relation is

$$b = \frac{1}{8 \cdot 273} \cdot \frac{T_k}{p_k}.$$

(p_k being in atmos and T_k in degrees absolute).

The following table gives the values of T_k , p_k , b and σ for all the gases except neon. Those for argon, krypton, and xenon were determined by Ramsay and Travers, those for helium by Kamerlingh Onnes.

	T_k .	p_k .	b .	σ .
Helium	About 5°	2 to 3	0·0007	$2 \cdot 3 \times 10^{-8}$ cm.
Neon	?	?	?	?
Argon.....	155°·6	52·9	0·001347	$2 \cdot 86 \times 10^{-8}$ cm.
Krypton.....	210°·5	54·3	0·001774	$3 \cdot 14 \times 10^{-8}$ cm.
Xenon	287°·75	57·2	0·002304	$3 \cdot 42 \times 10^{-8}$ cm.

(c) From the dielectric constant.

From the relation $\frac{K-1}{K+2} = \frac{\pi}{6} N \sigma^3$, we get, since K does not differ much from unity,

$$K-1 = \frac{\pi}{2} N \sigma^3.$$

For Helium $K = 1 \cdot 000074$; hence $\sigma = 1 \cdot 20 \times 10^{-8}$ cm.

(d) From the refractive index.

From the relation $K = (\mu_{\lambda=\infty})^2$, we get

$$\frac{\mu^2 - 1}{\mu^2 + 2} = \frac{\pi}{6} N \sigma^3,$$

whence

$$\mu - 1 = \frac{\pi}{4} N \sigma^3.$$

The refractivities $(\mu - 1)$ of the inactive gases compared with air = 1 have been determined by Ramsay and Travers (Phil. Trans. *l. c.*) and from these the refractive indices can be calculated, using $\mu_{\text{air}} = 1.000292$. This gives the following values :—

	He.	Ne.	A.	Kr.	X.
$\mu - 1$ for air = 1...	0.1238	0.2345	0.9665	1.450	2.364
$\mu \dots \dots \dots$	1.0000361	1.0000685	1.000282	1.000424	1.00069

For Neon, there are no better data at our disposal, so the value above given must stand. But recent determinations with the other gases show that the values obtained by Ramsay and Travers are a little too high.

Besides the determinations of Burton and of Cuthbertson and Metcalfe for He given above, we have also those of Scheel and Schmidt (*Ber. d. d. phys. Ges.* vi. p. 207 (1908)), and of Herrmann (*ib.* pp. 211, 467); but as Herrmann's results point to a dispersion much greater than that of air, and Scheel and Schmidt's results indicate anomalous dispersion, they are probably less trustworthy than those of the other mentioned observers.

For Kr and X, the dispersion has been measured by Mr. and Mrs. Cuthbertson (Proc. Roy. Soc. lxxxi. p. 440 (1908)), using large quantities of gas. They found for

$$\text{Kr, } \mu - 1 = 0.0004189 \left(1 + \frac{6.97}{10^{11} \lambda^2} \right),$$

and for

$$\text{X, } \mu - 1 = 0.0006823 \left(1 + \frac{10.14}{10^{11} \lambda^2} \right).$$

We can now calculate σ , using for He, A, Kr, and X the μ -values for $\lambda = \infty$, and for Neon the value obtained by Ramsay and Travers for white light. The results are tabulated in the appended table.

	μ for $\lambda = \infty$.	$\mu - 1$.	σ^3 .	σ .
He ...	1.0000347	3.47×10^{-5}	1.607×10^{-24}	1.17×10^{-8} cm.
Ne ...	1.0000685	6.85×10^{-5}	3.17×10^{-24}	1.47×10^{-8} cm.
A ...	1.0002792	27.92×10^{-5}	12.92×10^{-24}	2.35×10^{-8} cm.
Kr ...	1.0004189	41.89×10^{-5}	19.4×10^{-24}	2.69×10^{-8} cm.
X ...	1.0006823	68.23×10^{-5}	31.6×10^{-24}	3.16×10^{-8} cm.

From the two equations

$$b = \frac{2}{3} \cdot \pi N \sigma^3$$

and

$$\mu - 1 = \frac{\pi}{4} N \sigma^3,$$

we get

$$\mu - 1 = \frac{3}{8} \cdot b.$$

Let us see how far this relation holds good for the inactive gases. The table gives the comparison.

	$\mu - 1$.	$\frac{8}{3} \cdot \mu - 1$.	b .	$\frac{8}{3} \cdot \mu - 1$.
He ...	0.0000347	0.0000925	0.0007	7.57
Ne ...	0.0000685	0.0001826	?	?
A ...	0.0002792	0.0007445	0.001347	1.81
Kr ...	0.0004189	0.001115	0.001774	1.59
X ...	0.0006823	0.00182	0.002304	1.26

The agreement is not particularly good. Regarding some possible explanations of the anomalies reference may be made to Jeans, *l.c.* chap. xix.

3. Calculation of σ from the viscosity coefficient η .

We can also calculate the value of σ from the viscosity coefficient. Combining the two equations

$$\eta = 0.30967 \rho L \Omega$$

and

$$L = \frac{1}{\sqrt{2\pi} N \sigma^2},$$

we get

$$\eta = \frac{0.30967}{\sqrt{2\pi}} \cdot \frac{\rho \Omega}{N \sigma^2} \cdot \dots \dots \dots \quad (\text{I.})$$

Jeans (*l.c.* p. 250) deduced the relation

$$\eta = \frac{0.44}{\sqrt{2\pi}} \cdot \frac{\rho \Omega}{N \sigma^2} \cdot \dots \dots \dots \quad (\text{II.})$$

From I. we get for Helium $\sigma = 1.696 \times 10^{-8}$ cm.,
and for Argon $\sigma = 2.85 \times 10^{-8}$ cm.

From II. we get for Helium $\sigma = 2.02 \times 10^{-8}$ cm.,
and for Argon $\sigma = 3.39 \times 10^{-8}$ cm.

Jeans himself calculated σ from his formula, using Lord Rayleigh's values of η and taking $N = 4.0 \times 10^{19}$. He obtained for

Helium, $\sigma = 1.81 \times 10^{-8}$ cm.,
and for Argon, $\sigma = 2.786 \times 10^{-8}$ cm.

In concluding this section, we will recapitulate in tabular form all the results obtained above.

Molecular Velocities.

	He.	Ne.	A.	Kr.	X.
$G_0 \times 10^{-4}$ cm./sec. ...	13.11	5.845	4.13	2.86	2.28
$\Omega_0 \times 10^{-4}$ cm./sec. ...	12.08	5.385	3.806	2.64	2.10

Mean free paths.

He, $L_0 = 2.85 \times 10^{-5}$ cm.

A, $L_0 = 1.006 \times 10^{-5}$ cm.

Molecular Diameters.

	$\sigma \times 10^8$ cm.	He.	Ne.	A.	Kr.	X.
I.	From $\sigma = 6\sqrt{2}VL$.					
	(a) V from density of liquid ...	28.5	...	10.7
	(b) V from coefficient b	4.35	...	2.87
	(c) V from dielectric constant ...	0.597
	(d) V from refractive index	0.559	...	1.59
II.	From $\sigma = 4.58\sqrt{2}VL$.					
	(a) V from density of liquid ...	21.8	...	8.16
	(b) V from coefficient b	3.32	...	2.19
	(c) V from dielectric constant ...	0.455
	(d) V from refractive index	0.426	...	1.21
III.	From $V = \frac{\pi}{6} \cdot N\sigma^3$.					
	(a) V from density of liquid ...	4.34	...	4.43	4.93	4.88
	(b) V from coefficient b	2.3	...	2.86	3.14	3.42
	(c) V from dielectric constant ...	1.20
	(d) V from refractive index	1.17	1.47	2.35	2.69	3.16
IV.	From the viscosity coefficient.					
	a. Meyer's formula	1.696	...	2.85
	b. Jeans's formula	2.02	...	3.39

This miscellaneous collection of figures shows very plainly that even with the simple monatomic gases, the kinetic theory is not altogether satisfactory. A similar set of figures for helium and argon has been published by Jeans (*l. c.* and *Phil. Mag.* (6) viii. p. 692 (1904)), but, as far as my knowledge goes, the molecular constants of the other gases have not been calculated before.

PART II.—CALCULATION OF N—THE NUMBER OF MOLECULES IN UNIT GAS VOLUME.

From the equation $L = \frac{1}{\sqrt{2\pi N \sigma^2}}$, we can easily derive the relation $N = \frac{1}{320L^3V^2}$, where $V = \frac{\pi}{6} \cdot N\sigma^3$ as before.

Using the values of V obtained from the coefficient b of van der Waals's equation, from the dielectric constant K , and from the refractive index μ , we can obtain a series of values of N for helium and argon, for which gases L is known from the viscosity coefficient. The following table gives the results :—

	Values of $V \times 10^5$.			Values of $N \times 10^{-19}$.		
	From b .	From K .	From μ .	From b .	From K .	From μ .
He	1·8	2·47	2·31	0·415	22	25·2
A	33·7	...	18·61	2·7	...	8·87

The agreement is not good, but we must remember that any error in L or V becomes much magnified in the final result owing to the fact that these quantities occur as cube and square respectively. Still this alone will hardly explain the discrepancy. Sirk (*Ann. d. Phys.* xxv. p. 894 (1908)) calculated the values of $L^3 V^2$ for a large number of gases and vapours, including helium and argon, and found all sorts of values ranging between 1·7 and 28·2 ($\times 10^{-23}$), which would give values of N ranging from $18·4 \times 10^{19}$ to $1·10 \times 10^{19}$. The abnormally high values for helium Sirk ascribes to experimental errors in the determination of L and μ , but this I am not inclined to endorse, as both μ and η are now known with quite considerable accuracy. But I cannot see any other obvious explanation, especially as hydrogen gives a value of about 9×10^{19} calculated from μ .

PART III.—CALCULATION OF THE REFRACTIVE INDICES
OF THE LIQUEFIED GASES.

The formula proposed by Lorentz and Lorenz for the relation between the refractive index and the density is supposed to be independent of the state of aggregation. Indeed, this generally seems to be the case. Knowing then the refractive index of the gas and the densities of the gas and liquid, we can calculate the refractive index of the liquid. The table gives the results of the calculations.

	$\mu_{\text{gas.}}$	$\rho_{\text{gas.}}$	$\frac{\mu^2-1}{\mu^2+1} \cdot \frac{1}{\rho}$	$d_{\text{liquid.}}$	$\mu_{\text{liquid.}}$
He	1.0000347	0.0001769	0.131	0.15	1.03
Ne	1.0000685	0.00089	0.051	?	?
A	1.0002792	0.001782	0.104	1.423	1.23
Kr	1.0004189	0.003709	0.075	2.155	1.26
X.....	1.0006823	0.00584	0.078	3.52	1.46

Whether these figures will prove to be correct or not remains to be seen. As we have taken the values of μ for $\lambda = \infty$, the values of μ_{liquid} are also for $\lambda = \infty$. But as the dispersion formulæ are known, μ can be calculated for any value of λ . Nothing is yet known about the refractive indices or dispersion of these gases in the liquid state, except that the refractive index of helium is certainly very low. It ought not to be difficult to determine μ for liquid argon, as plenty of this substance is now easily procurable.

PART IV.—NOTE ON LORD KELVIN'S PAPER ON THE
SIZES OF ATOMS (Phil. Mag. [6] iv. p. 177 (1902)).

After the foregoing calculations were completed I found in W. P. Boynton's 'Kinetic Theory,' p. 278, a reference to a paper by Lord Kelvin, in which he had calculated the mean free path and molecular diameter of several gases, including argon. The result obtained by him for the value of L_0 : 3.89×10^{-6} cm. was so different from my own: 10.07×10^{-6} cm., although he had used practically the same experimental data, that I felt compelled to look for the cause of this discrepancy.

Lord Kelvin deduced the relation *

$$N\sigma^2 = 0.1629 \frac{G\rho}{\eta}$$

from the two equations for the diffusivity

$$D = \frac{1}{2} \frac{1}{\sqrt{3\pi}} \cdot \frac{G}{N\sigma^2},$$

$$\text{and} \quad D = \frac{\eta}{\rho},$$

and putting $\rho = 0.001781$, $\eta = 0.0002083$, and $G = 41400$, he found $N\sigma^2 = 57700$.

If we take the simple equations

$$\eta = \frac{1}{3} \rho L G \quad \text{and} \quad L = \frac{1}{\sqrt{2\pi N\sigma^2}},$$

$$\text{we get} \quad N\sigma^2 = \frac{1}{3\sqrt{2\pi}} \cdot \frac{G\rho}{\eta} = 0.075 \frac{G\rho}{\eta}.$$

The cause of the discrepancy is now obvious. Lord Kelvin, by a slip, used as the numerical factor $\frac{1}{2\sqrt{3\pi}}$ instead of $\frac{1}{3\pi\sqrt{2}}$, which makes all the figures given by him subject to correction, and gives an entirely different value for N from the one deduced by him: $N = 10^{20}$.

We will repeat his calculations, and compare them side by side with those deduced from the more correct equations.

In Part I. of this paper the relation $\eta = 0.30967 L\Omega\rho$ given by Meyer was used. Substituting in this

$$L = \frac{1}{\sqrt{2\pi N\sigma^2}} \quad \text{and} \quad \Omega = \sqrt{\frac{8}{3\pi}} G,$$

$$\text{we get} \quad N\sigma^2 = 0.0643 \frac{\rho G}{\eta}.$$

This formula we must also use in order to get the results

* I have altered Lord Kelvin's symbols to make them agree with those used in this paper.

obtained in Part I. The following table gives the comparison between the different methods of calculation :—

Lord Kelvin's formula. $N\sigma^2 = 0.1629 \frac{G\rho}{\eta}$.	$N\sigma^2 = 0.075 \frac{G\rho}{\eta}$.	$N\sigma^2 = 0.0643 \frac{G\rho}{\eta}$.
Taking $\rho = 0.001781$ $G = 4.14 \times 10^4$	Taking $\rho = 0.001782$ $G = 4.13 \times 10^4$	Taking $\rho = 0.001782$ $G = 4.13 \times 10^4$
Taking $\eta = 0.0002083$ $N\sigma^2 = 57700$	Taking $\eta = 0.0002114$ $N\sigma^2 = 26200$	Taking $\eta = 0.0002114$ $N\sigma^2 = 22500$
Taking $L = \frac{1}{\sqrt{2\pi N\sigma^2}}$ $L = 3.89 \times 10^{-6}$	Taking $L = \frac{1}{\sqrt{2\pi N\sigma^2}}$ $L = 8.66 \times 10^{-6}$	Taking $L = \frac{1}{\sqrt{2\pi N\sigma^2}}$ $L = 10.07 \times 10^{-6}$

In order to get at the value for σ Lord Kelvin formed an estimate of N from the density of liquid argon, taking for this Ramsay and Travers's value: 1.212. Assuming that the molecules are $q\sigma$ apart, where q is some numerical factor, he put

$$\frac{1}{N(q\sigma)^3} = \frac{d_{\text{liq.}}}{\rho_{\text{gas}}} = \frac{1.212}{0.001781} = 681,$$

and as $N\sigma^2 = 57700$, $N = 681^2 \cdot 57700^3 q^6 = 8.9 \times 10^{19} q^6$.

Seeing that q could not very well be less than unity, he took $q = 1.02$ to make $N = 10^{20}$, and substituting this value for N in $N\sigma^2 = 57700$, obtained for σ the value 2.40×10^{-8} cm.

Now this calculation, apart from the fact that the figures are inaccurate, also contains a fallacy, inasmuch as the factor $\frac{\pi}{6}$ is omitted. The relation should be

$$\frac{d_{\text{liq.}}}{\rho_{\text{gas}}} = \frac{1}{\frac{\pi}{6} N(q\sigma)^3}.$$

In conclusion, we will calculate the value of q , taking $N\sigma^2 = 22500$ and $d_{\text{liq.}} = 1.423$. We have

$$\frac{1}{\frac{\pi}{6} N(q\sigma)^3} = \frac{1.423}{0.001782} = 800,$$

and $N = \frac{(800)^2 \cdot (22500)^3}{\left(\frac{6}{\pi}\right)^2} q^6 = \frac{7.25 \times 10^{18}}{3.65} q^6 = 1.99 \times 10^{18} q^6$.

Taking $N = 2.75 \times 10^{19}$ as in Part I., we get $q^6 = 13.8$ and $q = 1.55$. Obviously this value of q must be the same as the ratio of σ obtained from $V = \frac{\pi}{6} N \sigma^3$, V being obtained from the density of the liquid, to σ obtained from the viscosity coefficient. These two values are (*cf.* Part I.) 4.43 and 2.85, and their ratio is 1.56.

PART V.—AN ESTIMATE OF THE DENSITY OF LIQUID NEON.

In Part I. we calculated the values of σ for the inactive gases from the formula $V = \frac{\pi}{6} N \sigma^3$, deriving V from the densities of the liquids and also from the refractive index. The results were as follows :—

$\sigma \times 10^8$.	He.	Ne.	A.	Kr.	X.
V from d ...	4.34	?	4.43	4.93	4.88
V from μ ...	1.17	1.47	2.35	2.69	3.16
Ratio	3.72	?	1.89	1.83	1.54

The last line in the table gives the ratio of the two values corresponding to q in Part IV. If we plot the ratios against the atomic weights and draw a smooth curve through the points thus obtained, we find for neon a ratio of about 2.5. This would give for σ from d a value of about 3.68. Hence

$$\frac{0.00089}{d} = \frac{\pi}{6} \cdot 2.75 \times 10^{19} \times (3.68)^3 \times 10^{-24},$$

whence $d = 1.24$.

This atomic volume would thus be about 16, which is not very different from the value—20.2—deduced by Ramsay and Travers (*Phil. Trans. l. c.*) from the periodic curve of the atomic volumes of all the elements.

London, Jan. 1909.

Added April 19th. To Part I.—Whilst the above was in the press, two papers dealing with the same subject have appeared, the one, by Sutherland (*Phil. Mag.* [6] xvii. p. 320), criticising the σ -values deduced by Jeans (*l. c.*), and the other, by Reinganum (*Ann. d. Phys.* xxviii. p. 142), criticising

the calculations of Sirk (*l.c.*). The gist of both papers is that the Clausius' equation $L = \frac{1}{\sqrt{2\pi N\sigma^2}}$ should be replaced by the equation $L = \frac{1}{\sqrt{2\pi N\sigma^2 F}}$, where the factor F has the value $e^{\frac{c}{T}}$ according to Reinganum and the value $1 + \frac{C}{T}$ according to Sutherland, c and C being constants and T the absolute temperature. These modified equations had been already deduced by their authors to account for the change of viscosity with temperature which, as is well known, does not take place in accordance with the relation $\eta = k\rho L\Omega$, *i.e.* $\eta \propto T^{\frac{1}{2}}$, L being considered constant. From Schierloh's measurements of η I have calculated c and C for A and He. The results are for He, $c=59$, $C=78.2$, and for A, $c=108$, $C=174.6$. The factor F has, therefore, the following values ($T=273$):—

a. $F = e^{\frac{c}{T}}$. Helium $F=1.217$. Argon $F=1.482$.

b. $F = 1 + \frac{C}{T}$. Helium $F=1.286$. Argon $F=1.64$.

The effect of F is to increase all values of σ calculated in Part I. § C1 in the ratio $F:1$, and to decrease those calculated in Part I. § C3 in the ratio $1:\sqrt{F}$. The values of σ calculated in Part I. § C2 are, of course, unaffected. The following table gives the new values compared with those calculated in Part I. of this paper:—

Values of $\sigma \times 10^8$ cm.

	$V = \frac{\pi}{6} N\sigma^3$	$\sigma = 6\sqrt{2}VL$			$\sigma = 4.58\sqrt{2}VL$			σ from η .			
		Rf.	S.	R.	Rf.	S.	R.		Rf.	S.	R.
<i>Helium.</i>											
V from b	2.3	4.35	5.6	5.3	3.32	4.27	4.04	Meyer...	1.696	1.49	1.53
V from μ	1.17	0.56	0.72	0.68	0.456	0.55	0.52	Jeans ...	2.02	1.78	1.83
<i>Argon.</i>											
V from b	2.86	2.87	4.62	4.26	2.19	3.6	3.25	Meyer...	2.85	2.23	2.34
V from b_H	3.36	4.57	7.5	6.78	3.49	5.73	5.18	Jeans ...	3.39	2.65	2.78
V from μ	2.35	1.59	2.6	2.35	1.21	1.99	1.79				

The columns *Rf* give the values calculated in Part I., *S* those using Sutherland's formula, and *R* those using Reinganum's formula. For b_H *vide infra*.

The agreement between the various sets is not better than it was before. For the other gases the values of c and C are unknown as the viscosities have not been measured.

Happel (*Ann. d. Phys.* xxi. p. 342, 1906) has calculated the values of b for A, Kr, and X from Boltzmann's modified form of van der Waals' equation. His results are given in c.c. per 1 g. Reduced to unit-volume they are for A $b_H=0.00214$, Kr $b_H=0.00283$, and X $b_H=0.00370$.

These values are greater than those calculated above from the original van der Waals' equation, and give for σ calculated from $V=\frac{\pi}{6}N\sigma^3$ the following results:—

	A.	Kr.	X.
$\sigma \times 10^8$ cm.	3.36	3.69	4.02

The agreement between b and $\frac{8}{3}(\mu-1)$ (*Cf.* Part I. § C2d) is now worse than before. Happel thinks, therefore, that either the Clausius-Mosotti relation or the Maxwell relation cannot hold for these gases. For He, at any rate, the latter seems to hold; the dielectric constants of the other gases are, however, unknown, so that for the present the discrepancy cannot be explained away.

To Part II.—Using Happel's value of b for Argon, N comes out to 1.08×10^{19} . I have also tried the effect of introducing Reinganum's correction on the values of N. Taking the values of V from μ as these are more comparable than those from b , we get for He $N=13.2 \times 10^{19}$, and for A $N=2.7 \times 10^{19}$. The latter value agrees well with the theoretical— 2.75×10^{19} , but the agreement is probably only a chance one.

LXXVII. *Thermionics*. By O. W. RICHARDSON, M.A.,
D.Sc., Professor of Physics, Princeton University*.

§ 1. WHILE fully alive to the desirability of restraining as much as possible the ever-increasing growth of our scientific nomenclature, the author has felt for some time that there is real need of a single word to denote the branch of physics which is alluded to in the literature with varying dignity as "the emission of ions by hot bodies," "the leak from hot wires," and a number of other phrases. After giving the matter due consideration, the author

* Communicated by the Author.

ventures to suggest that the word "Thermionics," which forms the title of this paper, is very suitable for the purpose. It suggests at once the thermal and electrical nature of the phenomena, while the derivation of the word ion also suggests the kinetic qualities of the ions.

The lack of such a word is found to be particularly aggravating in considering the phenomena attending a hot wire heated to a high temperature by an electric current. Here we have two currents: the current used to heat the wire and the thermionic current away from the surface of the latter. It is constantly necessary to distinguish between the two, and for this purpose the thermionic current is usually alluded to as "the leak from the wire" or simply "the leak." There are numerous objections to the use of the word "leak." It seems absurd to apply it to cases like that of a very highly refractory substance such as tungsten or carbon or of one of Wehnelt's lime-covered cathodes, in which the so-called "leak" may be as big or bigger than the rest of the heating current; there is also the moral objection that the leak analogy rather implies something undesirable and preventible; while finally we have the most weighty objection of all, that the term leak is not sufficiently adaptable. If we wish to refer to the particles which carry the leak we have to revert to some such phrase as "the ions emitted by the hot body," whereas the substantive Thermionics furnishes naturally the further substantive Thermion.

§ 2. So much by way of definition. The chief object of this paper is to point out a class of theoretical problems which become important in thermionics, to discuss, in as general a manner as possible, methods by which they can be solved, and to deduce the solutions for certain particular cases.

Suppose that in a region of space otherwise vacuous there are a number of hot surfaces A emitting ions and a number of conducting surfaces B. In general there will be an electric field in the region under consideration, so that any or all of the surfaces may be charged. The ions emitted by the surfaces A will move under the combined influence of their initial velocity and that of the electric field, and will ultimately reach one of the surfaces A or B or go off to an infinite distance. If the temperatures of the surfaces A are maintained constant the number and mode of distribution of velocity of the ions they emit will remain constant, and if in addition the potentials of the various surfaces are maintained constant it is clear that, whatever may happen at first, a steady state will ultimately be established in which the number and mode of distribution of velocity among the

ions received by any of the surfaces in a given time will be invariable. The problem is to find the number of ions which reach any of the surfaces B in a given time, together with their velocity components, when the steady state has been established. In the discussion it will be assumed that the motion of the ions is determined solely by their positional and velocity coordinates when they are emitted and by the electric field. The forces exerted on the ions by each other and by molecules of gas into whose sphere of action they may chance to penetrate are left out of account. These conditions are capable of being realized in practice with close approximation if thermionic currents of moderate size are experimented with in high vacua. In order to avoid complications arising out of recombination we shall also suppose the temperature conditions are such that ions of only one sign occur.

§ 3. General Discussion in Rectangular Coordinates.

Let the coordinates of any point on one of the surfaces A be $x_0 y_0 z_0$, and let an ion be projected with velocity components $u_0 v_0 w_0$ from $x_0 y_0 z_0$. Let us seek the condition that this shall strike one of the surfaces B whose equation is

$$\psi(x y z)=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

within an infinitesimal distance of the point $x_1 y_1 z_1$.

If V is the potential at any point of the field the equations of motion will be

$$m\ddot{x}=e\frac{\partial V}{\partial x}, \quad m\ddot{y}=e\frac{\partial V}{\partial y}, \quad m\ddot{z}=e\frac{\partial V}{\partial z}. \quad (2)$$

On integration these equations give three equations between $x y z$ and t involving six arbitrary constants which are determined by the values of $x_0 y_0 z_0 u_0 v_0 w_0$. After elimination of the time there result two equations which may be written

$$\phi_1(x y z \ x_0 y_0 z_0 \ u_0 v_0 w_0)=0, \quad . \quad . \quad . \quad (3)$$

$$\phi_2(x y z \ x_0 y_0 z_0 \ u_0 v_0 w_0)=0. \quad . \quad . \quad . \quad (4)$$

The curve in which the surfaces ϕ_1 and ϕ_2 intersect is the trajectory of the particle projected under the given initial conditions. The intersection of this curve with the surface $\psi(x y z)=0$ will give the point where the particle strikes the surface. The coordinates $x_1 y_1 z_1$ of such points will therefore be given by solving (1) and (4) for $x y$ and z , and the density of these points on the surface ψ will determine the thermionic current density into this surface in the steady state.

It is to be borne in mind that the equations for $x_1 y_1 z_1$ will not in general be of the first degree, so that there will be a number of roots corresponding to the successive real and imaginary intersections of the surfaces ψ , ϕ_1 and ϕ_2 . In any case the path of the particle will end as soon as it has reached the conducting surface B, and if this surface includes the whole of the analytical surface $\psi(x y z)=0$ the root to be chosen is that real root which corresponds to the shortest time of transit from $x_0 y_0 z_0$. The proper root can usually be easily picked out in simple cases. If the surface B is only a part of the analytical surface $\psi=0$ bounded by a curve or curves, it may in general be necessary to include roots corresponding to any number, less than that of the degree of the equations, of previous intersections of the trajectory and the surface $\psi=0$. The problem is then much more complicated.

The equations (3) and (4) may be solved for u_0 and v_0 giving

$$u_0 = \phi_3(x y z \ x_0 y_0 z_0 \ w_0), \quad . \ . \ . \ . \quad (5)$$

$$v_0 = \phi_4(x y z \ x_0 y_0 z_0 \ w_0). \quad . \ . \ . \ . \quad (6)$$

The equation $\phi_3=\text{constant}$ together with $\psi(x y z)=0$ will determine a curve lying in the surface ψ which contains the points of intersection of all trajectories for which u_0 and w_0 are constant. Similarly $\phi_4=\text{constant}$ determines a curve corresponding to constant values of v_0 and w_0 . If ξ and η denote lengths laid out along the normals to the level surfaces of u_0 and v_0 at any point, then

$$\left. \begin{aligned} \frac{\partial u_0}{\partial \xi} &= \sqrt{\left(\frac{\partial \phi_3}{\partial x}\right)^2 + \left(\frac{\partial \phi_3}{\partial y}\right)^2 + \left(\frac{\partial \phi_3}{\partial z}\right)^2} \\ \frac{\partial v_0}{\partial \eta} &= \sqrt{\left(\frac{\partial \phi_4}{\partial x}\right)^2 + \left(\frac{\partial \phi_4}{\partial y}\right)^2 + \left(\frac{\partial \phi_4}{\partial z}\right)^2} \end{aligned} \right\} \quad . \ . \ . \quad (7)$$

The number of particles which are emitted in unit time with velocity components between u_0 and $u_0 + du_0$ and v_0 and $v_0 + dv_0$ respectively may be denoted by $f_1(u_0)du_0$ and $f_2(v_0)dv_0$ where f_1 f_2 are functions which will be discussed later. For a constant value of w_0 the number which simultaneously have velocities between the above ranges will be proportional to $f_1(u_0) f_2(v_0) du_0 dv_0$, and these will fall on an area dS of the surface $\psi=0$, where

$$dS \cos (u_0 v_0 \wedge nS) = \frac{du_0 dv_0}{\frac{\partial u_0}{\partial \xi} \frac{\partial v_0}{\partial \eta}} \times \frac{1}{\sin u_0 \wedge v_0}$$

where $u_0 v_0 \wedge nS$ is the angle between the normal to the

surface $\psi=0$ and the tangent to the curve in which the surfaces $u_0=\phi_3$, $v_0=\phi_4$ intersect, and $u_0^\wedge v_0$ is the angle between the normals to the surfaces $u_0=\phi_3$ and $v_0=\phi_4$. Thus

$$\sin^2 u_0^\wedge v_0 =$$

$$\frac{\left(\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial z_1}-\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial y_1}\right)^2 + \left(\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial v_1}-\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial z_1}\right)^2 + \left(\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial y_1}-\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial v_1}\right)^2}{\left[\left(\frac{\partial\phi_3}{\partial v_1}\right)^2 + \left(\frac{\partial\phi_3}{\partial y_1}\right)^2 + \left(\frac{\partial\phi_3}{\partial z_1}\right)^2\right] \left[\left(\frac{\partial\phi_4}{\partial v_1}\right)^2 + \left(\frac{\partial\phi_4}{\partial y_1}\right)^2 + \left(\frac{\partial\phi_4}{\partial z_1}\right)^2\right]}$$

$$\cos(u_0^\wedge v_0^\wedge nS) = \frac{\left(\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial z_1}-\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial y_1}\right)\frac{\partial\psi}{\partial v_1} + \left(\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial v_1}-\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial z_1}\right)\frac{\partial\psi}{\partial x_1}}{\left[\left(\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial z_1}-\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial y_1}\right)^2 + \left(\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial v_1}-\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial z_1}\right)^2 - \frac{\partial\phi_3}{\partial x_1}\frac{\partial\phi_4}{\partial z_1}\frac{\partial\psi}{\partial y_1} + \left(\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial y_1}-\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial v_1}\right)\frac{\partial\psi}{\partial z_1} + \left(\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial y_1}-\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial v_1}\right)^2\right]^{\frac{1}{2}} \left[\left(\frac{\partial\psi}{\partial v_1}\right)^2 + \left(\frac{\partial\psi}{\partial y_1}\right)^2 + \left(\frac{\partial\psi}{\partial z_1}\right)^2\right]^{\frac{1}{2}}}$$

Hence

$$du_0 dv_0 = \frac{\left(\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial z_1}-\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial y_1}\right)\frac{\partial\psi}{\partial v_1} + \left(\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial v_1}-\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial z_1}\right)\frac{\partial\psi}{\partial x_1}}{\left[\left(\frac{\partial\psi}{\partial v_1}\right)^2 + \left(\frac{\partial\psi}{\partial y_1}\right)^2 - \frac{\partial\phi_3}{\partial x_1}\frac{\partial\phi_4}{\partial z_1}\frac{\partial\psi}{\partial y_1} + \left(\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial y_1}-\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial v_1}\right)\frac{\partial\psi}{\partial z_1} + \left(\frac{\partial\psi}{\partial z_1}\right)^2\right]^{\frac{1}{2}}} dS. \quad (8)$$

If the probability of the w_0 component of the velocity lying within the range w_0 and w_0+dw_0 is denoted by $f_3(w_0)dw_0$ the number of those reaching the surface $\psi=0$ with values of w_0 within this range will be proportional to

$$dw_0 \iint f_3(w_0) f_1(u_0) f_2(v_0) \frac{\left(\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial z_1}-\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial y_1}\right)\frac{\partial\psi}{\partial v_1}}{\left[\left(\frac{\partial\psi}{\partial v_1}\right)^2 + \left(\frac{\partial\phi_3}{\partial z_1}\frac{\partial\phi_4}{\partial v_1}-\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial z_1}\right)\frac{\partial\psi}{\partial x_1} + \left(\frac{\partial\phi_3}{\partial v_1}\frac{\partial\phi_4}{\partial y_1}-\frac{\partial\phi_3}{\partial y_1}\frac{\partial\phi_4}{\partial v_1}\right)\frac{\partial\psi}{\partial z_1} + \left(\frac{\partial\psi}{\partial y_1}\right)^2 + \left(\frac{\partial\psi}{\partial z_1}\right)^2\right]^{\frac{1}{2}}} dS.$$

To save space let us denote the fraction by

$$\chi(x_1 y_1 z_1 x_0 y_0 z_0 w_0) ;$$

then if n is the total number of ions emitted in unit time by unit area of the surface A the total number received by the surface ψ will be the real part of

$$n \iint dS_0 \int dw_0 \iint f(w_0) f_1(\phi_3) f_2(\phi_4) \chi dS. \quad . \quad . \quad (9)$$

Where dS_0 denotes an element of the surface A and the integral with respect to dw_0 is taken over all the values of w_0 which occur.

If we multiply the expression (9) by the charge e on an ion we obtain the current to the surface ψ . We can obtain the three components of the resultant pressure on this surface due to the impact of the ions if we multiply the integrand with respect to dS by $m \frac{dx_1}{dt}$, $m \frac{dy_1}{dt}$ and $m \frac{dz_1}{dt}$ respectively.

The values of the velocities are obtained from equations (2) and should be expressed as functions of $x_1 y_1 z_1 x_0 y_0 z_0$ and w_0 by means of the equations previously given. In a similar way we shall obtain the kinetic energy received by the surface if we multiply the integrand by

$$\frac{1}{2} m \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dy_1}{dt} \right)^2 + \left(\frac{dz_1}{dt} \right)^2 \right].$$

This must be identical with $ne(V_0 - V)$ + the value of the integral when

$$\frac{1}{2} m (u_0^2 + v_0^2 + w_0^2)$$

is substituted for

$$\frac{1}{2} m \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dy_1}{dt} \right)^2 + \left(\frac{dz_1}{dt} \right)^2 \right],$$

where V_0 is the potential of the surface A and V that of ψ .

Since χdS is equal to $du_0 dv_0$ we have

$$\iint f(w_0) f_1(\phi_3) f_2(\phi_4) \chi dS = \iint f(w_0) f_1(u_0) f_2(v_0) du_0 dv_0.$$

If the surface B forms the whole of the analytical surface $\psi(x y z) = 0$ the limits of integration for u_0 and v_0 will be determined, for any value of w_0 , by the values of u_0 and v_0 which correspond to the curve which is the locus of the points at which the trajectories having the given value of w_0 are tangential to the surface $\psi(x y z) = 0$. They will thus be certain functions of w_0 which are determined by the equation to the surface. If the surface B consists of the

portion of $\psi=0$ which is cut off by some closed curve the limits for $u_0 v_0$ will be determined partly by the bounding curve and partly by the locus of the tangents. With this understanding as to the meaning of the double integral we shall therefore have

$$\iota = n \iint dS_0 \int dw_0 \iint f(w_0) f_1(u_0) f_2(v_0) du_0 dv_0 \quad (10)$$

as an equivalent of the expression previously found. It will often be possible so to choose the direction of w_0 that $f(w_0)$ does not depend on u_0 and v_0 .

§ 4. *The Initial Boundary Conditions.*

Each element dS_0 of the surface A is at any instant shooting off a large number of ions in different directions and with all possible speeds. The researches of Richardson and Brown* and of the author† have shown that the number of ions leaving the element of surface dS_0 in unit time which have values of any velocity component u_0 within a given range, say between u_0 and $u_0 + du_0$, is approximately identical with the number given by Maxwell's Law of Distribution for the number of molecules of a gas having the same molecular weight crossing an equal and similar surface in equal time with the range of the velocity component u_0 within the same limits, the temperature of the gas being the same as the hot metal, and its pressure equal to the equilibrium pressure of the ions. It may be permissible to say here that more elaborate experiments now in progress have only tended to confirm this conclusion, and, so far as they have gone at present, point to the law being an exact one. It is to be remembered that we are dealing here, not with the distribution of velocity among the different molecules in a given volume, but with the distribution among those leaving a given surface in a given time, which is a different thing.

Assuming Maxwell's law to hold exactly we can write down the functions such as $f_1(u_0)$, $f_2(v_0)$ and $f_3(w_0)$ which determine the initial frequency of a velocity component within a given range. They will depend both on the kind of axes chosen and on their orientation relative to the surface. The following list, which embraces all the more important cases, may easily be verified. In each case n is the total number of ions emitted per unit area in the interval considered.

* Phil. Mag. [6] vol. xvi. p. 353 (1908).

† Phil. Mag. [6] vol. xvi. p. 890 (1908).

1. *Rectangular Coordinates.*

The axis of z is along the normal to the surface.

Number between \dot{z} and $\dot{z} + d\dot{z} = n\dot{z}F(\dot{z})d\dot{z} = 2nkm\dot{z}e^{-km\dot{z}^2}d\dot{z}$

$$,, \quad ,, \quad \dot{x} \text{ and } \dot{x} + d\dot{x} = nf(\dot{x})d\dot{x} = n\left(\frac{km}{\pi}\right)^{\frac{1}{2}}e^{-km\dot{x}^2}d\dot{x}.$$

$$,, \quad ,, \quad \dot{y} \text{ and } \dot{y} + d\dot{y} = nf(\dot{y})d\dot{y} = n\left(\frac{km}{\pi}\right)^{\frac{1}{2}}e^{-km\dot{y}^2}d\dot{y}.$$

where $3/4k$ is the mean kinetic energy and m is the mass of the ions.

2. *Spherical Coordinates.*

Let $\dot{\psi}$ be the resultant velocity, let θ be the angle it makes with the normal to the surface, and ϕ the angle the plane containing $\dot{\psi}$ and the normal makes with a fixed plane containing the normal. Then the number emitted per unit area per second which have $\dot{\psi}$ between $\dot{\psi}$ and $\dot{\psi} + d\dot{\psi}$, θ between θ and $\theta + d\theta$, and ϕ between ϕ and $\phi + d\phi$ is

$$\begin{aligned} & n\dot{\psi} \cos \theta F(\dot{\psi} \cos \theta) f(\dot{\psi} \sin \theta \cos \phi) f(\dot{\psi} \sin \theta \sin \phi) \dot{\psi}^2 d\dot{\psi} \sin \theta d\theta d\phi \\ &= n\dot{\psi}^3 F(\dot{\psi} \cos \theta) F_1(\dot{\psi} \sin \theta) \sin \theta \cos \theta d\dot{\psi} d\theta d\phi \\ &= \frac{2k^2m^2}{\pi} \dot{\psi}^3 e^{-km\dot{\psi}^2} \sin \theta \cos \theta d\dot{\psi} d\theta d\phi. \end{aligned}$$

3. *Cylindrical coordinates.*

- (α) The axis of z is along the normal to the surface,
 ρ = the radius perpendicular to the axis of z and
 θ = the angle ρ makes with a fixed plane passing through the z axis.

The number between

$$\dot{z} \text{ and } \dot{z} + d\dot{z} \text{ is } n\dot{z} F(\dot{z}) d\dot{z} = 2nkm\dot{z}e^{-km\dot{z}^2}d\dot{z}.$$

whilst the number for which $\dot{\rho}$ is between $\dot{\rho}$ and $\dot{\rho} + d\dot{\rho}$ and θ simultaneously between θ and $\theta + d\theta$ is

$$nf(\dot{\rho} \sin \theta) f(\dot{\rho} \cos \theta) d\dot{\rho} \dot{\rho} d\theta = n\frac{km}{\pi} \dot{\rho} e^{-km\dot{\rho}^2} d\dot{\rho} d\theta.$$

- (β) The axis of z lies in the tangent plane to the surface.
 $\dot{\phi}$ is the total component of velocity perpendicular to \dot{z} , i. e. the projection of the resultant velocity on a plane perpendicular to the z axis. θ is the angle $\dot{\phi}$ makes with the plane containing the axis of z and the normal to the surface.

Then the number whose velocity components lie between \dot{z} and $\dot{z} + d\dot{z}$ is

$$nf(\dot{z}) d\dot{z} = n \left(\frac{km}{\pi} \right)^{\frac{1}{2}} e^{-km\dot{z}^2} d\dot{z}.$$

The numbers which have components between $\dot{\phi}$ and $\dot{\phi} + d\dot{\phi}$, and for which at the same time θ lies between θ and $\theta + d\theta$ is

$$\begin{aligned} n\dot{\phi} \cos \theta F(\dot{\phi} \cos \theta) f(\dot{\phi} \sin \theta) \dot{\phi} d\dot{\phi} d\theta \\ = 2n \left(\frac{k^3 m^3}{\pi} \right)^{\frac{1}{2}} \dot{\phi}^2 e^{-km\dot{\phi}^2} \cos \theta d\dot{\phi} d\theta. \end{aligned}$$

The number for which $\dot{\phi}$ lies between $\dot{\phi}$ and $\dot{\phi} + d\dot{\phi}$ and for which θ has any value will therefore be

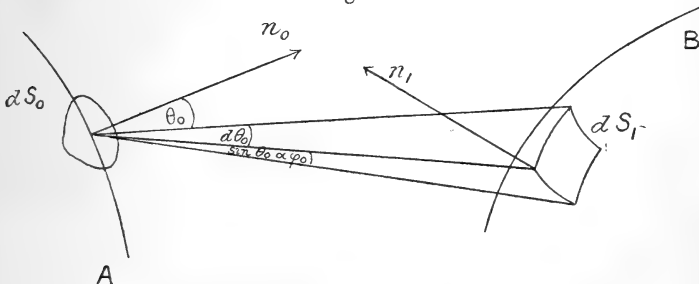
$$\begin{aligned} n\dot{\phi}^2 d\dot{\phi} \int_{-\pi/2}^{\pi/2} F(\dot{\phi} \cos \theta) f(\dot{\phi} \sin \theta) \cos \theta d\theta \\ = 4n \left(\frac{k^3 m^3}{\pi} \right)^{\frac{1}{2}} \dot{\phi}^2 e^{-km\dot{\phi}^2} d\dot{\phi}. \end{aligned}$$

We shall now consider the application of the considerations which have been brought forward to a number of particular problems.

§ 5. No Electric Forces.

When there are no forces in the field the general problem becomes comparatively simple, as the trajectories are straight lines, radiating from the point of origin, whose direction cosines are proportional to the initial velocity components.

Fig. 1.



Let dS_0 (fig. 1) represent an element of the surface A, n_0 being the normal to the element. Let dS_0 be the origin of polar coordinates of which n_0 is the polar axis. Let the element dS_1 be cut off from B by the planes ϕ_0 and $\phi_0 + d\phi_0$ containing n_0 and the cones whose semi-angles are θ_0 and

$\theta_0 + d\theta_0$. Let n_1 denote the normal to dS_1 and r the radius from dS_0 to dS_1 .

Then it follows from the equation on p. 820, that the number of ions received per second by dS_1 whose resultant velocity lies between $\dot{\psi}$ and $\dot{\psi} + d\dot{\psi}$ is

$$n\dot{\psi}^3 \cos \theta_0 F(\dot{\psi} \cos \theta_0) F_1(\dot{\psi} \sin \theta_0) d\dot{\psi} \sin \theta_0 d\theta_0 d\phi_0 dS_0.$$

But

$$dS_1 \cos n_1 \wedge r = r^2 \sin \theta_0 d\theta_0 d\phi_0.$$

The number received by unit area of the surface at dS_1 from dS_0 , and for which $\dot{\psi}$ lies between $\dot{\psi}$ and $\dot{\psi} + d\dot{\psi}$ is therefore

$$n\dot{\psi}^3 \cos \theta_0 F(\dot{\psi} \cos \theta_0) F_1(\dot{\psi} \sin \theta_0) d\dot{\psi} \frac{\cos n_1 \wedge r}{r^2} dS_0.$$

The total number received by the surface B from A and for which $\dot{\psi}$ lies between the assigned limits is thus

$$dn_{\dot{\psi}} = n\dot{\psi}^3 d\dot{\psi} \iint^A dS_0 \iint^B dS_1 F(\dot{\psi} \cos \theta_0) F_1(\dot{\psi} \sin \theta_0) \frac{\cos n_0 \wedge r \cos n_1 \wedge r}{r^2},$$

where the integral with respect to dS_1 is only to be extended over those portions of the surface B which are directly visible from dS_0 .

The pressure normal to dS_1 = momentum communicated per second

$$= dS_1 nm \int_0^{\infty} \dot{\psi}^4 d\dot{\psi} \iint^A F(\dot{\psi} \cos \theta_0) F_1(\dot{\psi} \sin \theta_0) \frac{\cos n_0 \wedge r \cos^2 n_1 \wedge r}{r^2} dS_0.$$

The kinetic energy received by the whole surface B per second is

$$T = n \iint^B dS_1 \int_0^{\infty} \frac{1}{2} m \dot{\psi}^5 d\dot{\psi} \iint^A F(\dot{\psi} \cos \theta_0) F_1(\dot{\psi} \sin \theta_0) \frac{\cos n_0 \wedge r \cos n_1 \wedge r}{r^2} dS_0.$$

The value of $F(\dot{\psi} \cos \theta_0) F_1(\dot{\psi} \sin \theta_0)$ given by Maxwell's law is $2 \frac{k^2 m^2}{\pi} e^{-km\dot{\psi}^2}$. The current to B is therefore

$$i = \frac{2}{\pi} ne \int_0^{\infty} k^2 m^2 \dot{\psi}^3 e^{-km\dot{\psi}^2} d\dot{\psi} \iint^A \iint^B \frac{\cos n_0 \wedge r \cos n_1 \wedge r}{r^2} dS_0 dS_1,$$

and the kinetic energy received per second is

$$T = \frac{n}{\pi} \int_0^\infty k^2 m^3 \dot{\psi}^5 e^{-km\dot{\psi}^2} d\dot{\psi} \iint_A \iint_B \frac{\cos n_0 \wedge r \cos n_1 \wedge r}{r^2} dS_0 dS_1.$$

The limits of integration over the surface B being restricted in the same way as before.

If one surface is the boundary of a small body of negligible dimensions then r is constant for any fixed point on the other and

$$\iint_S \iint_{S'} \frac{\cos n \wedge r \cos n' \wedge r}{r^2} dS dS' = \iint_S dS \frac{\cos n \wedge r}{r^2} \iint_{S'} \cos n' \wedge r dS'.$$

In particular if A is a small sphere of radius a ,

$$\iint_A \cos n_0 \wedge r dS_0 = \pi a^2,$$

and if ω is the solid angle subtended by B at A

$$\iint_A \iint_B \frac{\cos n_0 \wedge r \cos n_1 \wedge r}{r^2} dS_0 dS_1 = \pi a^2 \omega.$$

§ 6. Parallel Planes.

Suppose we have two parallel planes, A and B, whose equations are $z - z_0 = 0$ and $z - a = 0$, with a uniform electric field of intensity Z between them. Let an ion with velocity components $u_0 v_0 w_0$ start from $x_0 y_0 z_0$ in the plane A, x and y being measured along two mutually perpendicular axes parallel to the planes. The equations to the trajectories are

$$z - z_0 = \frac{1}{2} Z \frac{e}{m} \left(\frac{x - x_0}{u_0} \right)^2 + \frac{w_0}{u_0} (x - x_0),$$

and

$$\frac{x - x_0}{u_0} = \frac{y - y_0}{v_0}.$$

Solving for u_0 and v_0 we have (see p. 816)

$$\left. \begin{aligned} u_0 &= \frac{1}{2} w_0 \frac{x - x_0}{z - z_0} \left(1 \pm \sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}} \right) = \phi_3 \\ v_0 &= \frac{1}{2} w_0 \frac{y - y_0}{z - z_0} \left(1 \pm \sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}} \right) = \phi_4 \end{aligned} \right\} \quad (11)$$

It is clear that the positive signs correspond to the quickest times of transit to a given point, so that by taking the

positive roots we shall get the points where the trajectories cut the surface B for the first time. The positive roots will thus give us the number of ions striking the plane B from the side nearest to A. We shall for the present confine our attention to this case.

The equation to the plane B is

$$z - a = 0 = \psi.$$

We therefore have

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial z} = 1, \quad . \quad . \quad . \quad (12)$$

$$\left. \begin{aligned} \frac{\partial \phi_3}{\partial y} = \frac{\partial \phi_4}{\partial x} &= 0 \\ \frac{\partial \phi_3}{\partial x} = \frac{\partial \phi_4}{\partial y} &= \frac{1}{2} \frac{w_0}{z - z_0} \left(1 + \sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}} \right) \\ \frac{\partial \phi_3}{\partial z} &= -\frac{1}{2} \frac{x - x_0}{(z - z_0)^2} w_0 \frac{1 + Z \frac{e}{m} \frac{z - z_0}{w_0^2} + \sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}}}{\sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}}} \\ \frac{\partial \phi_4}{\partial z} &= -\frac{1}{2} \frac{y - y_0}{(z - z_0)^2} w_0 \frac{1 + Z \frac{e}{m} \frac{z - z_0}{w_0^2} + \sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}}}{\sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}}} \end{aligned} \right\} \quad (13)$$

Hence the value of χ (see p. 817) is

$$\chi = \frac{\partial \phi_3}{\partial x} \frac{\partial \phi_4}{\partial y} = \frac{1}{4} \frac{w_0^2}{(z - z_0)^2} \left(1 + \sqrt{1 + 2Z \frac{e}{m} \frac{z - z_0}{w_0^2}} \right)^2.$$

Hence, for example, the current to a rectangle in the plane B bounded by the lines $x = x_2$, $x = x_1$, $y = y_2$, and $y = y_1$ will be

$$\begin{aligned} i &= n \iint dS_0 \int_0^\infty 2km w_0 e^{-km w_0^2} dw_0 \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{km}{4\pi} \frac{w_0^2}{(a - z_0)^2} \left(1 + \sqrt{1 + \frac{2Ze(a - z_0)}{mw_0^2}} \right)^2 \\ &\quad \times e^{-\frac{km}{4} w_0^2 \frac{(x - x_0)^2 + (y - y_0)^2}{(a - z_0)^2} \left(1 + \sqrt{1 + \frac{2Ze(a - z_0)}{mw_0^2}} \right)^2} dx dy \end{aligned}$$

where the integral with respect to dS_0 is extended over the heated part of the plane A. Changing the variables to

$$\begin{aligned} u &= \frac{1}{2} \sqrt{km} \frac{x - x_0}{a - z_0} \left(1 + \sqrt{1 + 2Z \frac{e}{m} \frac{a - z_0}{w_0^2}} \right) \\ v &= \frac{1}{2} \sqrt{km} \frac{y - y_0}{a - z_0} \left(1 + \sqrt{1 + 2Z \frac{e}{m} \frac{a - z_0}{w_0^2}} \right) \end{aligned}$$

this may be written

$$i = n \iint dS_0 \int_0^{\infty} \frac{2km}{\pi} w_0 e^{-km w_0^2} dw_0 \int_{\frac{1}{2}\sqrt{km} \frac{y_2 - y_0}{a - z_0} (1 + \sqrt{1 + 2Z \frac{e(a - z_0)}{m w_0^2}})}^{\frac{1}{2}\sqrt{km} \frac{y_1 - y_0}{a - z_0} (1 + \sqrt{1 + 2Z \frac{e(a - z_0)}{m w_0^2}})} e^{-v^2} dv \\ \times \int_{\frac{1}{2}\sqrt{km} \frac{x_2 - x_0}{a - z_0} (1 + \sqrt{1 + 2Z \frac{e(a - z_0)}{m w_0^2}})}^{\frac{1}{2}\sqrt{km} \frac{x_1 - x_0}{a - z_0} (1 + \sqrt{1 + 2Z \frac{e(a - z_0)}{m w_0^2}})} e^{-u^2} du.$$

To get the current from a hot strip of the plane A of infinite length, bounded by the lines $x = \xi_2$ and $x = \xi_1$, which is received by unit length of a strip of B bounded by the lines $x = x_2$ and $x = x_1$, we may integrate with respect to both y and y_0 from $+\frac{l}{2}$ to $-\frac{l}{2}$, where l is any very great length, and divide the result by l . In this case we get

$$i = \frac{n}{l} \int_{-\frac{l}{2}}^{+\frac{l}{2}} dy_0 \int_{\xi_1}^{\xi_2} dx_0 \int_0^{\infty} \frac{2km}{\pi} w_0 e^{-km w_0^2} dw_0 \int_{-\infty}^{\infty} e^{-v^2} dv \\ \times \int_{\frac{1}{2}\sqrt{km} \frac{x_2 - x_0}{a - z_0} (1 + \sqrt{1 + \frac{2Ze(a - z_0)}{m w_0^2}})}^{\frac{1}{2}\sqrt{km} \frac{x_1 - x_0}{a - z_0} (1 + \sqrt{1 + \frac{2Ze(a - z_0)}{m w_0^2}})} e^{-u^2} du.$$

This, as it should be, is equal to the expression (4) on p. 894 of the author's paper on the "Kinetic Energy of the Ions emitted by Hot Bodies" (Phil. Mag. [6] vol. xvi. 1908), which was obtained by a slightly different method.

If the area of B to which it is desired to find the current is not rectangular, the limits of integration with respect to x and y will have to be suitably adjusted. For example, if the area is bounded by the circle in the plane $z = a$, whose equation is $(x - x_3)^2 + (y - y_3)^2 = b^2$, the limits for x will be $x_3 \pm \sqrt{b^2 - (y - y_3)^2}$ and for y , $y_3 \pm b$.

If we take the negative signs in equations (11) we shall obtain the points where the trajectories cut the plane B the second time. In this case the value of χ is

$$\chi = \frac{1}{4} \frac{w_0^2}{(a - z_0)^2} \left(1 - \sqrt{1 + \frac{2Ze(a - z_0)}{m w_0^2}} \right)^2.$$

The most interesting application of this is obtained when $a - z_0$ is made to approach the limit zero. The planes A and

B then coincide and we have

$$\chi = \frac{1}{4} \frac{Z^2 e^2}{m^2 w_0^2},$$

$$u_0 = -\frac{1}{2} \frac{Ze(x-x_0)}{mw_0},$$

$$v_0 = -\frac{1}{2} \frac{Ze(y-y_0)}{mw_0}.$$

The thermionic current starting from any region A of a plane which flows to another region B of the same plane in a uniform electric field perpendicular to the plane is therefore given by

$$\begin{aligned} i = n \iint_A dx_0 dy_0 \int_0^\infty \frac{2km}{\pi} w_0 e^{-kmw_0^2} dw_0 \iint_B \frac{km}{4} \frac{z^2 e^2}{m^2 w_0^2} \\ \times e^{-\frac{km}{4} \frac{z^2 e^2}{m^2 w_0^2} [(x-x_0)^2 + (y-y_0)^2]} dx dy. \end{aligned}$$

When Ze is negative this solution corresponds to the physical case, but when Ze is positive we see that if u_0 is positive $u-u_0$ is negative. In this case the points of intersection correspond to negative values of the time and have no physical existence. They are, however, identical with the points of intersection of the ions starting from the back of the strip which corresponds to the case when the limits for w_0 are from 0 to $-\infty$.

§ 7. *Inclined Planes.*

If the plane B is inclined to the plane A, the electric force being still uniform and perpendicular to the plane A, the trajectories will be unaltered and $\frac{\partial \phi_3}{\partial x}$ &c. will still be given by equations (13). Let the equation to B be

$$z-z_0-a(x-x_0')=0,$$

the line of intersection of the two planes being thus parallel to the axis of y . Then

$$\frac{\partial \psi}{\partial x} = -a, \quad \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial z} = 1,$$

and

$$\chi = \frac{\left(\frac{\partial \phi_3}{\partial x} + a \frac{\partial \phi_3}{\partial z}\right) \frac{\partial \phi_4}{\partial y}}{\sqrt{1+a^2}}$$

$$= \frac{w_0^2 \left(1 + \sqrt{1 + 2Z \frac{e a (x-x_0')^2}{m w_0^2}}\right)^2}{4a^2 \sqrt{1+a^2} (x-x_0')^2} \left(1 - \frac{x-x_0}{x-x_0'}\right)$$

$$\times \frac{1 + Z \frac{e a (x-x_0')}{m w_0^2} + \sqrt{1 + 2Z \frac{e a (x-x_0')}{m w_0^2}}}{\left(1 + \sqrt{1 + 2Z \frac{e a (x-x_0')}{m w_0^2}}\right) \sqrt{1 + 2Z \frac{e a (x-x_0')}{m w_0^2}}}$$

The positive sign in front of the square root has been taken so that this value of χ corresponds to the ions reaching the plane B from the side making an acute angle with A.

If $x_0' = x_0$ the solution will correspond to a narrow strip of hot metal lying in the plane A along the line of intersection of the two planes. In this case

$$\chi = \frac{Z \frac{e}{m} \left(1 + \sqrt{1 + 2Z \frac{e a (x-x_0)}{m w_0^2}}\right)}{4a \sqrt{1+a^2} (x-x_0) \sqrt{1 + 2Z \frac{e a (x-x_0)}{m w_0^2}}}$$

for the current to the inner surface of B. For the current to the outer surface

$$\chi = - \frac{Z \frac{e}{m} \left(1 - \sqrt{1 + 2Z \frac{e a (x-x_0)}{m w_0^2}}\right)}{4a \sqrt{1+a^2} (x-x_0) \sqrt{1 + 2Z \frac{e a (x-x_0)}{m w_0^2}}}$$

If we make $a(x-x_0) = z-z_0$ approach zero this becomes

$$\chi = \frac{Z^2 e^2}{4m^2 w_0^2}$$

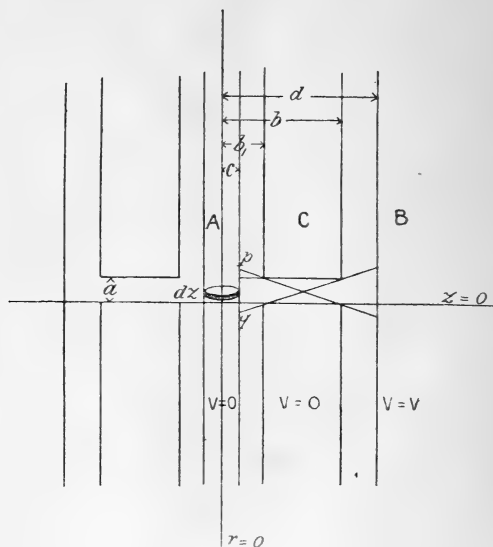
The plane B then coincides with the plane A and we get the appropriate expression for calculating the number of ions returned to the emitting plane itself. This value agrees with that obtained previously.

§ 8. A Cylindrical Problem.

We shall conclude this paper with the discussion of a problem, which has presented itself in work in the laboratory,

in which the surface A is a circular cylinder. Let the radius of the hot cylinder A be c . This is surrounded by two concentric solid conducting cylinders C and B. The internal and external radii of C are b_1 and b respectively,

Fig. 2.



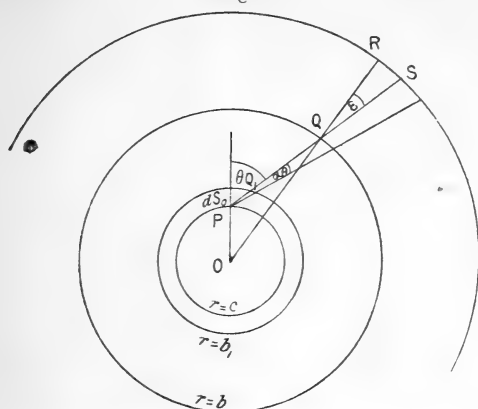
the internal radius of B being d . All the cylinders are of indefinite length, but whereas A and B are continuous, the part of C which lies between two planes perpendicular to the axis of the cylinders is missing, leaving a gap whose width is a . Let the common axis be taken as the axis of z , the equations of these two planes being $z=0$ and $z=a$. a is supposed to be small compared with b or d . A and C are maintained at the common potential zero, while B is maintained at a different potential v . The problem is to determine the thermionic current which reaches the cylinder B.

If a is sufficiently small the electric intensity will be inappreciable, except in the region bounded by the cylinders whose radii are b and d . When r is less than b we may therefore regard the trajectories as rectilinear; when r lies between b and d they will be determined by the field of force between two charged concentric cylinders together with the initial velocity components.

Consider a particle starting from any point P on A. Its initial velocity may be resolved into two components, one (\dot{z}) parallel to the axis of the cylinders, and the other ($\dot{\phi}$) in

a plane perpendicular to this axis. Let $\dot{\phi}$ make an angle θ with the normal OP, then PQ (fig. 3) will be the resolved

Fig. 3.



part of the straight part of the trajectory on the plane perpendicular to z . It is clear from fig. 2 that ions will only escape through the gap in the cylinder of radius b provided they start from A within the limits comprised by the points p and q . A comparison of figs. 2 and 3 shows that the conditions for escape are

- (1) when P lies between $z=0$ and $z=a$, $\dot{z}/\dot{\phi}$ must lie between $-z/PQ$ and $(a-z)/PQ$,
- (2) when P lies between $z=0$ and $z=-a \frac{b_1-c}{b-b_1}$,
 $\dot{z}/\dot{\phi}$ must lie between $-z/PQ_1$ and $(a-z)/PQ$.
- (3) when P lies between $z=a$ and $z=a \left(1 + \frac{b_1-c}{b-b_1}\right)$,
 $\dot{z}/\dot{\phi}$ must lie between $-z/PQ$ and $(a-z)/PQ_1$.

Q_1 Q and R are the points in which PQ intersects the cylinders of radii b_1 b and d respectively. So that

$$PQ = \sqrt{b^2 - c^2 \sin^2 \theta} - c \cos \theta,$$

$$PQ_1 = \sqrt{b_1^2 - c^2 \sin^2 \theta} - c \cos \theta.$$

The region on the surface of A between $z=0$ and $z=a$ may be referred to as the umbra; the wings between $z=-a \frac{b_1-c}{b-b_1}$ and $z=0$, and between $z=a$ and $z=a \left(1 + \frac{b_1-c}{b-b_1}\right)$ respectively as the penumbrae.

Referring to p. 821 we see that the number of ions emitted per second by any element dS_0 of A for which

\dot{z} lies between \dot{z} and $\dot{z} + d\dot{z}$

$\dot{\phi}$ lies between $\dot{\phi}$ and $\dot{\phi} + d\dot{\phi}$

and θ lies between θ and $\theta + d\theta$,

simultaneously, is

$$2n \frac{k^2 m^2}{\pi} \dot{\phi}^2 \cos \theta e^{-km(\dot{\phi}^2 + \dot{z}^2)} d\dot{z} d\dot{\phi} d\theta dS_0.$$

The problem is unaltered as the vertical element dz on the surface of A is rotated round the axis of the cylinders, so that we may take $2\pi c dz$ for the element of surface dS_0 . The number of ions for which $\dot{\phi}$ lies between $\dot{\phi}$ and $\dot{\phi} + d\dot{\phi}$ and θ lies between θ and $\theta + d\theta$ which escape from the gap of width a in the cylinder of radius b is therefore

$$du = 4nc k^2 m^2 \dot{\phi}^2 e^{-km\dot{\phi}^2} d\dot{\phi} \cos \theta d\theta \\ \times \left[\int_a^{a(1+\frac{b_1-c}{b-b_1})} \int_{-\frac{z}{PQ}\dot{\phi}}^{\frac{a-z}{PQ_1}\dot{\phi}} + \int_0^a \int_{-\frac{z}{PQ}\dot{\phi}}^{\frac{a-z}{PQ}\dot{\phi}} + \int_{-a\frac{b_1-c}{b-b_1}}^0 \int_{-\frac{z}{PQ_1}\dot{\phi}}^{\frac{a-z}{PQ}\dot{\phi}} \right] dz d\dot{z} e^{-km\dot{z}^2}$$

If $\frac{d-b}{b}$ is small any region such as RS (fig. 3) may be regarded as part of a plane parallel to the tangent plane at Q. In that case the condition that the ions escaping through the gap should reach the outer cylinder assumes the simple form

$$\dot{\phi} \cos \epsilon \geq \sqrt{2 \frac{e}{m} V},$$

where V is the difference of potential between B and C, and

$$\cos \epsilon = \cos \angle RQS = \sqrt{1 - \frac{c^2}{b^2} \sin^2 \theta}.$$

The total number of ions which reach the cylinder B per second will therefore be obtained by integrating du with respect to $d\dot{\phi}$ and $d\theta$, the limits being :

$$\text{for } \dot{\phi} : \text{from } \infty \text{ to } \sqrt{\frac{2Ve b^2}{m(b^2 - c^2 \sin^2 \theta)}}, \text{ and}$$

$$\text{for } \theta : \text{from } \pi/2 \text{ to } -\pi/2.$$

§ 9. Case of a Thin Hot Wire.

Under certain restrictions the preceding expressions simplify very considerably. In the first place we notice that if $\frac{b_1 - c}{b - b_1}$ is small compared with unity the penumbrae may be left out of account. This leaves us with only the central one of the three integrals in square brackets and corresponds to the case when the inner surface of the cylinder B is very close to the outer surface of the hot cylinder A. Furthermore if c is small compared with b the projections PQ (fig. 3) of the trajectories on a plane perpendicular to the axis all become approximately radial, and we have, to this approximation, $PQ = b$. Under these circumstances the current to the outer cylinder will therefore be equal to

$$i = 4\pi nec \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \, d\theta \int_{\sqrt{\frac{2e}{m}V}}^{\infty} k^2 m^2 \dot{\phi}^2 e^{-km\dot{\phi}^2} d\dot{\phi} \int_0^a dz \int_{-\frac{z}{b\dot{\phi}}}^{\frac{a-z}{b\dot{\phi}}} e^{-kmz^2} dz.$$

If a/b is reasonably small only small values of z will be important, since large values of $\dot{\phi}$ occur only rarely and much more rarely therefore will large values of z occur. It is therefore appropriate to expand the exponential in powers of z , giving

$$\begin{aligned} i &= 8\pi nec \int_{\sqrt{\frac{2e}{m}V}}^{\infty} k^2 m^2 \dot{\phi}^2 e^{-km\dot{\phi}^2} d\dot{\phi} \int_0^a dz \int_{-\frac{z}{b\dot{\phi}}}^{\frac{a-z}{b\dot{\phi}}} \sum_{n=0}^{\infty} (-1)^n \frac{k^n m^n z^{2n}}{n!} dz \\ &= 4\pi nec \int_{\sqrt{\frac{2e}{m}V}}^{\infty} 2k^2 m^2 \dot{\phi}^2 e^{-km\dot{\phi}^2} d\dot{\phi} \sum_{n=0}^{\infty} (-1)^n \frac{k^n m^n a^{2(n+1)} \dot{\phi}^{2n+1}}{(2n+1)! (n+1) b^{2n+1}}. \end{aligned}$$

By continued integration by parts we see that

$$\int_a^{\infty} 2k^n m^n \dot{\phi}^{2n-1} e^{-km\dot{\phi}^2} d\dot{\phi} = \frac{1}{(n-1)! (km\alpha^2)^{n-1}} e^{-km\alpha^2} \sum_{s=0}^{s=n-1} \frac{1}{(km\alpha^2)^s (n-1-s)!}$$

Hence

$$i = 4\pi nec \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n a^{2(n+1)}}{(2n+1)! b^{2n+1}} (2keV)^{n+1} e^{-2keV} \left[\sum_{s=0}^{s=n+1} \frac{1}{(2keV)^s (n+1-s)!} \right] \right\}. \quad (14)$$

The number of integral values of n which it will be necessary to take in a practical case depends on the smallness of a/b .

We get the first-order terms by putting $n=0$. To this approximation

$$\iota = 4ne c \frac{a^2}{b} (1 + 2keV) e^{-2keV}.$$

To the third order in a/b we have to add the terms corresponding to $n=1$, which give

$$- \frac{4}{3} ne c \frac{a^4}{b^3} (1 + 2keV + 2k^2e^2V^2) e^{-2keV}$$

and so on.

The value of the current when $V=0$, *i.e.* when all the cylinders are at the same potential, will be given by putting $V=0$ in the doubly infinite series for ι . This gives

$$\begin{aligned} \iota_0 &= 4ne ca \sum_{n=0}^{n=\infty} \frac{(-1)^n a^{2n+1}}{(2n+1) b^{2n+1}} \\ &= 4ne ca \tan^{-1}(a/b). \quad . \quad . \quad . \quad . \quad (15) \end{aligned}$$

In this case there are no forces in the field so that this result should be the same as that given by the general formula (p. 822)

$$\iota_0 = \frac{2}{\pi} ne \int_0^\infty k^2 m^2 \psi^3 e^{-km\psi^2} d\psi \iint^A \iint^B \frac{\cos n_0 \wedge r \cos n \wedge r}{r^2} dS_0 dS,$$

where the surfaces A and B are the portions of two concentric circular cylinders of radii c and b which are cut off by planes, perpendicular to the common axis, distant a from each other, and where c is small compared with b . We then have

$$\begin{aligned} dS_0 &= cd\phi_0 dz_0, & dS &= bd\phi dz, & \cos n \wedge r &= b/r, \\ r &= \sqrt{b^2 + (z - z_0)^2} & \text{and} & & dz_0 \int d\phi_0 \frac{c \cos n_0 \wedge r}{r^3} &= dz_0 \frac{2bc}{r^4}, \end{aligned}$$

Hence

$$\begin{aligned} \iota_0 &= 8ne b^3 c \int_0^\infty k^2 m^2 \psi^3 e^{-km\psi^2} d\psi \int_0^a dz \int_0^a \frac{dz_0}{(b^2 + (z - z_0)^2)^2} \\ &= 4ne ac \tan^{-1} \frac{a}{b} \end{aligned}$$

in agreement with the previous result.

Formulae (14) and (15) are subject to the restrictions that c/b and $\frac{b_1 - c}{b - b_1}$ are small. If these conditions do not hold the series become much more cumbersome.

If in addition $\frac{a^2}{b^2}$ can be neglected compared with unity we have

$$\iota_0 = 4\pi ec a^2/b$$

and

$$\iota/\iota_0 = (1 + 2keV) e^{-2keV}.$$

The equation between the current and the opposing potential-difference in this case therefore differs from the similar equation for infinite parallel planes by the inclusion of the factor $(1 + 2keV)$.

LXXVIII. *On the Action between Metals and Acids and the Conditions under which Mercury causes Evolution of Hydrogen.* By S. W. J. SMITH, M.A., D.Sc., Lecturer on Physics, Imperial College of Science and Technology*.

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- § 6. The problem for experiment.
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- § 14. Possible effects at the Jet.
- § 15. A kinetic representation of § 3.
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§ 1. **INTRODUCTORY.**—Under ordinary conditions metals like mercury, silver and copper are unable to displace hydrogen from solutions of acids (dilute or concentrated) with appreciable and easily demonstrable evolution of the gas.

An attempt is here made to show how, in the case of mercury, the fluidity of the metal at the ordinary temperatures may be utilized to exhibit the cause of the inability and to supply a method by which it may be overcome.

The effects with hydrochloric and sulphuric acids only have been examined, but there is no apparent reason why

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solutions of other acids should fail to yield results of the same kind.

§ 2. *Electrolytic Solution Pressure*.—A method of regarding the question whether a metal will dissolve in an acid solution has been suggested by Nernst and developed by himself, Ostwald and others. The general idea of this method has become familiar, and provides a means of forming a more suggestive picture of what actually happens than is obtained from purely thermochemical considerations (*Cf. e. g.* Thomsen, 'Thermochemistry,' pp. 349 to 356, &c.), unsatisfactory in other ways as well.

In order to present a consecutive account of the present experiments, this view is restated below in a form as far as possible free from hypotheses not absolutely necessary*.

§ 3. *A conception of the Interaction of Metals and Acids*.—Imagine that a metal M comes suddenly into contact with an air-free solution of an acid HX . It is known that in general equilibrium will be impossible; a certain quantity of H will be precipitated upon the surface of M and an equivalent quantity of M will dissolve.

It may be that this interchange will take place only to an indefinitely small amount, but we can safely say that no salt MX is absolutely insoluble and that every metal has at least some tendency to go into solution. It is the relative magnitude of this tendency which is to be regarded as the characteristic variable distinguishing Hg , for example, from metals like Fe and Zn .

On account of the electrostatic resisting forces which would arise, ions of H cannot escape spontaneously from a solution of HX , nor can ions of M escape from the metal M ; but with M and HX in contact the conditions alter. The assumed tendencies of M and H to spread beyond their original boundaries can now become effective without development of electrostatic resisting forces; because equivalents of H and M can pass across the common surface of acid and metal without change in the electric charge on either side. Neutral molecules of H can escape from solution while ions of M enter.

It is a necessary conclusion from a consideration of this kind (abstract thermodynamics furnishes many other examples) that solution of the metal and precipitation of hydrogen must begin at the interface (assuming no other change possible), whether accompanied by loss or gain of heat from the

* *Cf.* Nernst, 'Theoretical Chemistry,' 2nd Engl. edit. p. 724.

rest of the system. The familiar parallel is the fact that a gas will always expand spontaneously from higher pressure to lower if the external constraints permit—the work which it does being performed at the expense, if necessary, of its own internal energy.

It would appear, therefore, that every metal must be able to displace to some extent the hydrogen of an acid solution. If in any case the action is imperceptible, it must be because the tendency of the metal to enter solution is very small. The entry of an insignificant amount is then sufficient to balance the tendency. After this, further solution of the metal with escape of hydrogen will be a process like the compressing of one quantity of gas by the expansion of another—the constraints of the two quantities of gas being such that mutual expansion and contraction is the only change possible.

In such a case equilibrium is reached when the work which could be done by any further expansion of the second gas would be less than the work required to increase the compression of the first. Similarly, the replacement of hydrogen by a metal will cease at a point defined by the condition that further escape of hydrogen would produce less available work than would be required to cause the equivalent quantity of metal to enter solution.

§ 4. *Symbolic expression of the argument of § 3.*— Assuming as a rough approximation that the process is reversible and takes place isothermally, the conception may be expressed symbolically as follows.

The work done (diminution of available energy) when one equivalent weight of hydrogen escapes from solution may be written in the form $\mu_h^s - \mu_h^n$, in which μ_h^s is a function of the strength of the solution (increasing with the concentration), and μ_h^n depends upon the pressure at which the hydrogen escapes (increasing with the pressure),

Similarly the work done (gain of available energy) when an equivalent of the metal enters solution may be expressed as $\mu_m^s - \mu_m^n$, in which μ_m^s is a function of the amount of metal already dissolved per ccm. near the interface, and μ_m^n is a physical constant of M (at the temperature of the interaction and for a given curvature of surface if the metal is fluid), which may be large or small according to the nature of the metal.

The relation necessary for equilibrium is

$$\mu_h^s - \mu_h^n = \mu_m^s - \mu_m^n \quad . \quad . \quad . \quad (i.*)$$

The amount of M which enters solution (and of H which escapes) before equilibrium is attained will be determined by the value of

$$c = f(\mu_m^s) = f(\mu_h^s - \mu_h^n + \mu_m^n),$$

where c is the concentration of the salt MX near the surface which is necessary in order that μ_m^s may acquire the value required to satisfy (i.). If no such value of c can arise equilibrium is impossible (unless owing to secondary effects other conditions supervene), and interaction will continue until the supply of acid or of metal is exhausted.

§ 5. *Possible effect of Surface Tension.*—The above statement of the conditions of equilibrium neglects possible variation of the surface energy during the interaction. In many cases there may be relatively little variation; but in one at least, where the metal is mercury and the interface consequently separates two liquids, changes of surface tension are easy to detect. In this case the surface energy can be seen to diminish with increase in the concentration of the mercury salt in solution.

Consequently there is now a greater diminution of available energy than is represented by the left-hand member of (i.), when an equivalent of H is replaced in solution by an equivalent of M. Hence μ_m^s must attain a greater value in the solution before equilibrium is reached than would be required to satisfy (i.).

Thus if $-\gamma$ is the decrease of surface tension which would accompany the exchange of equivalents of H and M across unit surface when equilibrium is attained, we should have

$$\mu_h^s - \mu_h^n + \gamma = \mu_m^s - \mu_m^n, \quad . \quad . \quad . \quad (i. a)$$

and the equilibrium concentration of the salt of M in solution would now be

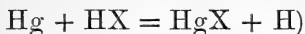
$$c = f(\mu_h^s - \mu_h^n + \mu_m^n + \gamma).$$

* In the strictly reversible system $M : MX : HX : H_{Pt}$, the corresponding equation of equilibrium (no electromotive force) is approximately of the form

$$aT \log c_{HX} - E_h = bT \log c_{MX} - E_m,$$

where E_h and E_m are 'electromotive constants' of hydrogen (at given pressure) and the metal, at the temperature of equilibrium T .

§ 6. *The Problem for Experiment.*—If the equation (i. a) represents the condition of equilibrium between mercury and an acid (assuming that the only reaction possible is of the type



it is clear that when the substances come into contact a certain amount of hydrogen must be displaced. Otherwise the right-hand member of the equation can never become equal to the left.

If (as is likely in the case of Hg) the quantity μ_m^n is very small, the quantity $\mu_m^s - \mu_m^n$, (being at constant temperature of the form $k \log c/c_0$, where c_0 is very small), may acquire a considerable value, even when c is small, *i. e.* when only a small quantity of hydrogen has been displaced.

Thus equilibrium may be reached and displacement of hydrogen cease (neglecting diffusion effects) before the amount separated per unit surface has become perceptible.

If, however, some means could be found of removing the mercury salt as fast as it was formed the reaction would continue, and thus the displacement of hydrogen might be rendered evident.

§ 7. *Detrimental effect of Oxygen in the Surface-layer.*—The simplest way of obtaining an experimental answer to the question whether the direct displacement of hydrogen by mercury ever occurs is not immediately obvious. The purest mercury in contact with the air will become coated with a film of condensed oxygen—possibly a minute layer of oxide. Hence, even if the acid with which it may be brought into contact is free from dissolved oxygen, the interaction contemplated in the equations above may be prevented.

When an equivalent of hydrogen forsakes the acid solution in the presence of oxygen, the loss of available energy can be greater than before because, instead of separating as gas, the hydrogen can now become part of a molecule of water. Hence the amount of metal which must dissolve before equilibrium is reached will much exceed that required to give μ_m^s the value sufficient to satisfy (i. a) above. The equilibrium value of μ_m^s in the present case may be written

$$\mu_m^s = \mu_h^s - \mu_h^{\text{OH}_2} + \gamma + \mu_m^n \quad \dots \quad (\text{ii.})$$

In which the quantity $\mu_h^{\text{OH}_2}$ is much less than the corresponding term μ_h^n of equation (i. a).

It is known from electrical measurements that the earlier
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stages of this reaction proceed with great rapidity. In the presence of a sufficient quantity of oxygen, μ_m^s may thus almost at once reach a greater value than that required to prevent the evolution of hydrogen*.

It is therefore essential to experiment with a mercury surface as far as possible oxygen-free, or to devise a means of removing from near any mercury surface as much as is desired of the mercury salt in solution.

§ 8. *Possible methods of eliminating this effect.*—Imagine two masses of mercury A and B immersed in the same acid. Let the surface of A be one not originally oxygen-free and surrounded in consequence by solution containing dissolved mercury salt. Suppose that the state of the solution around A has become practically steady without appreciable diffusion of the mercury salt into the region round B. Let the surface of B be one originally free from oxygen, and suppose that no interaction has yet taken place between it and the acid. Left to itself this mercury might interact with the acid, some of it displacing some of the hydrogen; but suppose that, instead, it is brought into contact with the mass A at one point or more without considerable change in the extent of either surface in contact with the acid.

Well-known electrochemical phenomena leave no doubt as to what will happen. Mercury will immediately begin to deposit on A and to enter the solution round B. Mercury-salt will in fact disappear and appear in equal quantities round A and B respectively until the concentration of the salt in solution round both is the same. This change will be effected by a displacement of anionic 'chains' in the solution from A towards B and by some analogous process (shift of electrons) in the mercury from B to A. The time taken by the process to complete itself will depend upon the length of the ionic chains. If these are short this time will be very small compared with the time taken by A to reach the steady state acquired before the contact.

From the point of view already described this process occurs in a way analogous to the expansion of a gas when the external constraints permit. Here the constraints virtually permit the expansion of the mercury salt from the space round A into the space round B until the concentration is the same in both. The electrical phenomena are incidents, not causes, of the flow of matter which takes place.

* The effect of the presence of oxygen can also be presented in a thermochemical form, omitted here for the sake of space.

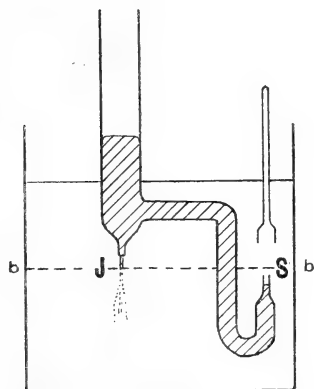
From whatever point of view the result may be regarded there is no doubt that metallic contact between A and B would reduce the amount of mercury salt in solution round the former. If B were very large compared with A, the amount of mercury left in solution round A would be very small. Similarly if A were connected with a succession of masses initially like B but small (each being removed before another took its place), the amount of salt in solution round A could be continuously lowered.

Although it appears possible to obtain oxygen-free surfaces of mercury (like B), they are usually in rapid motion and very difficult to observe. It is easy, however, to study a surface (like A) subjected to metallic contact with a succession of surfaces much more nearly oxygen-free than itself.

§ 9. *Experimental realization.*—Mercury poured into a vertical glass tube drawn to a fine capillary at the lower end escapes in a narrow stream. Since the surface of the mercury per unit mass is much greater in this than in the tube, a considerable quantity of the mercury below the surface in the tube must enter the surface in the jet. Thus a surface film, upon the mercury originally, must become much thinner in the jet.

Suppose the stream to enter an acid solution as at J in the figure. It will possess much less oxygen per unit surface

Fig. 1.



than mercury at rest. The concentration of oxygen may fall below the amount sufficient to raise μ_m^s to the value required to satisfy (i. a), and therefore a small quantity of hydrogen may form.

The chance of formation of hydrogen will be greatest at the end of the jet where it breaks into drops and suddenly presents a new surface to the solution. If the jet is completely immersed this chance will be only a little greater than at the sides; but it will be much enhanced if the end of the jet just touches the surface of the solution (level of liquid at b in fig. 1).

Thus suppose the length of the completely immersed jet to be l and the velocity of efflux v . The "electrochemical" forces already described will tend to equalize the distribution of mercury salt in solution round the jet, although the various elements of the surface have been in contact with the solution for times varying between zero and l/v . The end of the jet after rupture is a surface bounded by that portion of the rest of the jet which has been longest in contact with the solution, and the concentration of mercury salt round the end will therefore be raised practically instantaneously to a considerable value.

If the jet breaks in the surface of the solution, however, the electrochemical short-circuit is reduced to a minimum because the sides of the jet are now practically out of contact with the solution.

Very little hydrogen can be produced on any element of the surface even when the jet has its greatest efficiency, and the difficulty of formation of extremely small bubbles (owing to surface tension effects) may prevent evolution of gas otherwise possible in accordance with (i. a). Suppose, however, that the jet is connected to a small mercury surface at rest in contact with the solution as at S (fig. 1). If there is less mercury in solution round J than round S, the metal will precipitate at S and enter solution at J. This action will continue as long as the concentration of Hg in solution round S exceeds the practically steady value which would exist round J if S were absent.

Consequently the concentration of the mercury salt round S must tend to diminish continuously until it is as small as that round J. But if, before this can happen, the amount of Hg in solution at S becomes smaller than that required to satisfy (i. a), a new reaction will begin. Direct action between the mercury of S and the acid will occur with evolution of hydrogen and formation of a new supply of mercury salt in solution.

This action will be continuous, for, in virtue of the continuous effect of J, Hg will be continuously removed at S. A steady state (neglecting secondary actions such as described below, § 12) will be reached when the rate of evolution of

hydrogen at S is exactly equivalent to the rate of removal of mercury from solution at S by J. For every equivalent of acid that disappears at S, an equivalent of the mercury salt will appear in solution at J.

§ 10. *Results.*—In the above way it is possible to conceive that mercury will decompose sulphuric or hydrochloric acid with evolution of hydrogen. Experiment justifies the conception, for, if either acid (of sufficient strength) is used in the vessel of fig. 1, a stream of hydrogen is evolved at S.

If the point of the capillary (at J) is below the surface of the acid the effect occurs most rapidly when the head of mercury just suffices to produce a short, approximately cylindrical, jet at J. (The ratio l/v for the jet has then a minimum value because, with increase of head, the jet length increases more rapidly than the velocity.)

The effect occurred with the strongest sulphuric acid available ('pure redistilled' s. g. about 1.84). If the acid is diluted the same evolution of gas can be obtained, but, when the density of the acid falls below about 1.25 the rate of evolution is very slow and occurs only when the jet breaks in the surface of the solution. Finally, when the density of the acid is below about 1.19, there is no perceptible evolution of gas in any position of the jet*.

The experience with hydrochloric acid is similar. With the most concentrated acid used (s. g. about 1.16) there was a fairly rapid evolution of gas when the jet broke in the surface. Gas ceased to come off when the density of the acid fell below about 1.09. The molecular concentrations of these limiting solutions are about the same—roughly 6 equivalent gram mols. per litre—although that of the HCl solution is slightly the smaller†. (Cf. § 13 below.)

§ 11. *Proof of Evolution of Hydrogen.*—The gas was not proved to be hydrogen by a direct test in every case; but only in the case most liable to suspicion, viz. when concentrated sulphuric acid was employed; and in one other, viz. when equal volumes of this acid and water were mixed. It was proved to be hydrogen in the following way:—Collected in a small tube over the concentrated acid, it did not dissolve appreciably in the latter, nor, subsequently, in recently boiled distilled water by which the acid was displaced. It was therefore neither H_2S nor SO_2 .

It was difficult to test the gas positively since with the

* In practice it is easier to adjust J and S if separate columns of mercury connected by a wire are used instead of the apparatus of fig. 1.

† Mr. J. S. G. Thomas has since made a more exact determination of the limiting concentrations. He finds them to be 6.25 gram equivalents per litre for H_2SO_4 and 5.75 gram equivalents per litre for HCl.

arrangement used it took a considerable time to collect a few cubic millimetres. It was possible, however, to show that it underwent contraction on explosion with oxygen in the following way. A bubble of the gas, 4 mms. long, was collected at the top of a tube which ended in a capillary through which a fine platinum wire had been sealed. About 0.8 mm. of oxygen (prepared electrolytically) was added and a second fine wire was pushed into the collecting tube from below until it reached almost to the first. A spark was then passed, and the remaining gas was found to occupy about 2.5 mm. of the tube*. Thus the gas evolved behaved like hydrogen, the only constituent common to the two acids employed.

§ 12. *Secondary effects with Sulphuric Acid.*—Although, in the absence of oxygen, the simplest direct interaction between mercury and sulphuric acid is



there are other possible interactions of which the next† in simplicity would be



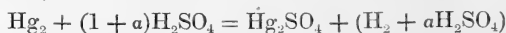
Here every fifth molecule of the acid may be supposed to be reduced by the hydrogen resulting from the direct action between four molecules of the acid and mercury.

According to the view adopted in this paper the reaction I. can go on only so long as the concentration of mercury salt in solution does not exceed the value given by (i. a); but according to the same view the reaction II. can occur before and after this limit to reaction I. is passed. It probably does not occur to the exclusion of I. because it involves a greater rearrangement of the constituents of the reacting molecules than is involved in the displacement of hydrogen (*cf.* Thomsen, *l. c.* p. 354 *et passim*).

To take the case of concentrated sulphuric acid, which was carefully examined. The H_2S of reaction II. will interact with a further quantity of the acid precipitating sulphur. It will also in part precipitate the very nearly insoluble

* Mr. W. F. Higgins kindly attempted to make a spectroscopic test of the gas, but various difficulties were encountered which it did not seem profitable to attempt to overcome since the gas had already been proved to be neither H_2S nor SO_2 nor oxygen.

† In the reaction representing the secondary reducing effect of hydrogen, viz.,



the minimum value of a is $1/4$.

sulphide of mercury by interaction with the sulphate in solution. The former reaction may be represented by the equation



In this case, neglecting for simplicity the energy variation due to the decomposition of the SO_4 ion, the equilibrium value of μ_m^s may be represented qualitatively by

$$\mu_m^s = \frac{4}{3}(\mu_h^s - \mu_h^{\text{H}_2\text{O}}) + \gamma + \mu_m^n, \quad \dots \quad (\text{iii.})$$

which shows that more mercury must now enter solution before equilibrium is attained than when hydrogen ceases to be evolved in accordance with (i. *a*). The reaction with precipitation of mercury sulphide leads to a similar result.

Thus we are led to infer that the production of sulphur and of sulphide of mercury may continue after the evolution of hydrogen has ceased.

This inference is fully confirmed by experiment:—

(*a*) When the diameter of S is small—less or not much greater than that of J—the mercury in solution round S is removed almost at once by the action of J and hydrogen simultaneously appears. Very soon, however, a yellowish-white cloud begins to form (particularly round the portions of S where the curvature is least and where the evolution of hydrogen is most noticeable). This cloud probably arises mainly from the decomposition of H_2S . In fact when the effectiveness of J was decreased by reducing the head some of the last bubbles to escape seemed (when observed through a microscope) to be surrounded by a film exactly like that which is produced when bubbles of H_2S are passed into concentrated H_2SO_4 . The formation of sulphide of mercury can also be detected after the jet has been in action for some time.

These reactions rapidly reduce the rate of evolution of hydrogen at S. The production of sulphur is a process which the action of J cannot reverse and the amount increases continuously, raising the electric resistance of the solution in the capillary and hence diminishing the effectiveness of J. The insoluble sulphide precipitated on the surface of the mercury further increases the circuit resistance. In addition this sulphide seems to prevent the evolution of hydrogen directly although the amount of mercury salt in solution may be very small. For instead of escaping, the hydrogen must now apparently interact with the sulphide in the surface layer producing H_2S and eventually sulphur. Thus after a

time the only effect of J at S is to produce sulphur. If, however, the capillary at S is rinsed out by expelling a little of the mercury, the evolution of hydrogen begins again at the fresh surface of acid and mercury.

(b) When the diameter of S is large—for example, twenty times that of J—sulphur and sulphide only are formed, however efficient the jet may be. In this case the jet only slowly reduces the amount of mercury salt in solution round S, and the reactions depending on the formation of sulphuretted hydrogen begin before the evolution of hydrogen is possible and continue in such a way that it can never occur. In a qualitative sense it may be said that in this case the slower reaction (requiring greater molecular rearrangements) has time to prevent the first.

It is only when the acid is concentrated that these effects occur, for diluted sulphuric acid is not reduced to or by sulphuretted hydrogen. Thus even in the case (b) just mentioned, hydrogen is evolved freely when sulphuric acid solution of s.g. 1.5 is substituted for the concentrated.

§ 13. *Effects of Dilution of the Acids.*—The loss of available energy when an equivalent of hydrogen leaves the acid solution and is evolved as gas at atmospheric pressure falls continuously* as the solution is diluted. Consequently the amount of mercury which can be in solution round the electrode without preventing the evolution of hydrogen becomes continuously less.

Before the hydrogen can escape it must reach a certain concentration (in the neutral state) in the mercury and in the solution at the surface layer. Disregarding the difficulty of formation of very minute bubbles, evolution just fails to occur when, electrode and solution being saturated, there would be no loss of available energy if an infinitesimally small quantity of hydrogen left the solution and the infinitesimal equivalent of mercury entered.

Thus if there is a limit below which J (however effective) cannot reduce the concentration of the mercury salt round S (§ 9), there is also a limit to the concentration of the acid which can be decomposed at S with evolution of gas. The

* In the case of concentrated sulphuric acid, dilution seems first to increase the rate of evolution of hydrogen. This result might be anticipated for two reasons. The first effect of dilution is to increase the conductivity and, probably, the ionic concentration of the acid. Thus the effectiveness of the jet and the value of μ_h^s will simultaneously rise. Again, some of the hydrogen which would otherwise escape will react with the concentrated acid while in the nascent state (§ 12). There should thus be a particular strength of sulphuric acid for which the rate of evolution of gas is a maximum.

experiments of § 10 show that this limit is reached at a concentration of about 6 equivalents per litre in the case of each acid.

Further, assuming that equally concentrated solutions of the acids and of their corresponding mercury salts are approximately equally dissociated, one would expect in virtue of equation (i.) that the limiting concentrations of the two acids would be approximately equal. This also agrees with experiment.

It happens, however, that the quantity γ in (i. a) is greater for HCl than for H_2SO_4 . It might therefore be anticipated that if the corresponding dissociations were exactly equal, the limiting concentration for HCl would be rather less than for H_2SO_4 ; but although the results suggest the fulfilment of this anticipation, the data are not sufficiently accurate to make it worth while to attempt a quantitative proof—especially as there are other possible explanations.

§ 14. *Possible effects at the Jet.*—The explanation which has been given of the behaviour of the still mercury surface S supposes that the concentration of mercury salt round the jet J is small. In fact the behaviour of S is (by hypothesis) controlled by the rate of independent formation of mercury salt at J. Consequently when hydrogen is evolved at S the amount of mercury in solution round J should be insufficient to prevent the evolution of hydrogen at J.

If any such evolution takes place it is very difficult to detect. Mercury was allowed to run for a long time from a capillary tube fused into the top of a glass bulb which was filled with concentrated sulphuric acid and terminated below in a tube dipping into mercury. No trace of gas could be seen. The level of the acid was lowered until the jet broke in the surface, the space above being filled with air. In this case, the first impression is that there is a copious evolution of gas (*cf.* Paschen, *Wied. Ann.* vol. xli. p. 56, 1890). But this is an illusion. The greater part, if not all, of the gas which appears to form at the surface of the mercury drops is air dragged in from above. This effect is very pronounced in liquids of great viscosity. It is less conspicuous in sulphuric acid than it is in glycerine.

In consequence of this phenomenon it is impossible to tell by inspection whether any gas is evolved as the result of chemical action between the acid and the mercury. In a further experiment benzene (which had previously been shaken up with another sample of concentrated H_2SO_4 and then decanted) was substituted for the air above the acid. There was now a continuous circulation of drops of benzene

within the acid like that of the air bubbles before. A considerable quantity of gas accumulated at the top of the apparatus; but it may have resulted from some chemical reaction in which the benzene took part. The evolution was even more noticeable when the benzene was replaced by pentane.

It has already been pointed out that the escape of hydrogen at the jet may be impossible although it takes place at S (§ 9). At the latter the effect is cumulative and all the hydrogen is evolved at the same surface. Here minute bubbles can coalesce into larger ones in which the pressure is not greatly above that of the air. At the jet, however, the hydrogen which mercury replaces must first reach a certain concentration in every fresh element of the jet and of the solution surrounding it. Then it must overcome the resistance to the formation of a minute bubble before it can escape as gas.

That direct action between the jet and the acid might result in visible production of hydrogen, but for the counteractions just described, can be seen by allowing the jet to break in the surface of a concentrated solution of sulphate of copper. The surface of the mercury which collects on the bottom of the vessel containing the sulphate presents a tarnished appearance like that produced by the addition of copper. This result, it will be seen, is in close accord with the present point of view and suggests the need for qualification of the familiar statement: "The more electropositive metals, Cu, precipitate the less electropositive metal Hg." It is probable that other metals, such as Pb and even Cd, could be precipitated by mercury, from solutions of their salts, in a similar way.

§ 15. *A kinetic representation of § 3.*—The action between an acid and a metal is formulated in § 3 in a way which avoids the necessity of dealing with the kinetics of the process by which equilibrium is attained. The following is perhaps the simplest picture of what actually happens. The metal is assumed to be monovalent, but it is easy to see what change must be made when the valency is n .

Into the space (the "border layer") which separates acid and metal, Faraday tubes can stretch from the acid and from the metal respectively. Of these, a possible pair forming side by side will be (1) a tube stretching from the acid nearly across the border layer, with its negative end (an anion of the acid) in the solution and its positive end (an ion of hydrogen) near the metal, (2) a tube stretching from the metal, with its negative end on the metal and its positive end (a metal ion) almost reaching the solution.

These tubes forming simultaneously and close together will interact, yielding a molecule of the salt MX in solution and a neutral equivalent of H deposited on M.

The number of interactions of this kind taking place per sq. cm. of surface in the unit of time will depend upon the frequency with which such tubes form side by side, *i. e.*, upon the number of molecules of HX in solution per cc. on the one hand and upon some specific property of the metal (determining the rate at which tubes of the second kind form) on the other. For the "velocity" of the reaction

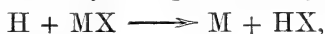


we may therefore write

$$v = k c_h C_m,$$

where k is a constant at given temperature, c_h is the concentration of the hydrogen ions in solution, and C_m is a specific constant of the metal M.

As the result of this action, hydrogen will accumulate upon the metal and MX will be formed in solution. A reverse action, similar in kind to the first, now becomes conceivable. This reaction, which may be represented by



will proceed with a velocity

$$v' = k' c_m C_h,$$

where c_m represents the ionic concentration of the metal in solution and C_h is a specific constant of hydrogen deposited at given pressure upon M.

If a steady state is reached, after a certain quantity of hydrogen has been displaced and the equivalent quantity of M has dissolved, it will be defined by the condition $v = v'$ or

$$k c_h / C_h = k' c_m / C_m.$$

We can thus deduce kinetically a result identical with that obtainable by application of the logarithmic formula of Nernst and contained as a particular case in the general equation of § 4.

Since there is no effective transfer of electricity across the border layer, a possible contact difference of potential between metal and solution would not affect the available work equation of § 4. Similarly it would not affect the final equation of equilibrium deduced kinetically. For if a potential-difference existed it would have the same relative effect upon v' as upon v .

§ 16. *Summary of Conclusions.*—Pure mercury reacts with acid solutions with displacement of hydrogen in the same way as metals like zinc.

The reaction stops before a perceptible quantity of hydrogen is evolved because a very small quantity of mercury salt in solution is sufficient to cause it to cease.

The surface film of mercury which has been in contact with the air probably contains more than enough oxygen to oxidise to water all the hydrogen that would be displaced before the direct action ceased.

The amount of oxygen per sq. cm. of the surface film can be reduced to a very small quantity by allowing the mercury to escape from a containing tube in the form of a narrow jet.

But for certain counteracting influences due to the fact that the substance displaced is a gas, this jet might be used to obtain hydrogen from acid solutions of sufficient strength.

By the aid of the jet the direct action between mercury and the acid, with displacement of hydrogen, can be made continuous. The jet, when in direct communication with a mercury surface at rest in the same solution, prevents the concentration of the mercury salt, formed by the displacement of hydrogen, remaining or becoming large enough at the still surface to stop the evolution of gas. In consequence, hydrogen escapes freely and can be collected and analysed.

Owing to direct action at its own surface, the jet cannot reduce the concentration of mercury salt round the still surface below a certain limit. Further, the amount of mercury salt which is sufficient to stop the direct action at the still surface diminishes with the concentration of the acid solution used.

In consequence it will be impossible to decompose the acid by means of the jet when the strength of the solution falls below a certain limit. This limit might be expected to be about the same for different acids. It was found to be about the same for hydrochloric and sulphuric acids (roughly 6 gram equivalents per litre), although rather lower for the former than for the latter.

The various reactions which occur when metals are placed in contact with concentrated sulphuric acid are elucidated by the experiments described (§ 12).

I am very much indebted to Mr. J. S. G. Thomas, B.Sc., for frequent and valued help while performing the experiments I have described.

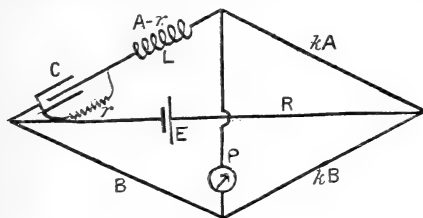
LXXIX. *On Pirani's Method of Measuring the Self-Inductance of a Coil.* By E. C. SNOW, B.A.*

IN Pirani's method of measuring the self-inductance of a coil, it is assumed that the self-inductance of the galvanometer used may be neglected. That this assumption is legitimate in the case in which the discharge of the condenser employed (which, it will be shown, is of the same nature as the discharge through the galvanometer) is continuous has been proved†.

The object of the present analysis is to investigate the case in which the discharge of the condenser is oscillatory.

A condenser of capacity C , the "internal" resistance of which can be neglected, is placed in series with the coil whose self-inductance is to be measured. The condenser is shunted by a non-inductive resistance r . This combination forms one arm of a Wheatstone's bridge, the resistances of the other arms of which are adjusted so that a steady balance holds between the four arms.

One or other of two methods can now be used: (i.) keeping r constant, C can be varied until there is no throw on closing the galvanometer-key before the battery-key; or (ii.) keeping C constant r can be varied, the total resistance of the arm being kept constant, until the same result is arrived at.



The following symbols will be used:—

I , the current in the battery.

x , the current in the galvanometer or (in the case of an alternating current) the telephone.

a , the current in the coil whose self-inductance is required.

α , the current in the shunt resistance r .

c , the current flowing into the condenser.

b , the current in the resistance B .

a' , the current in the resistance kA .

b' , the current in the resistance kB .

* Communicated by the Physical Society: read March 12, 1909.

† O. de A. Silva, *L'Eclairage Electrique*, vol. I. pp. 113-116.

R , the resistance of the battery.

ρ , the resistance of the galvanometer or telephone.

L , the self-inductance of the coil.

L' , the self-inductance of the galvanometer or telephone.

E , the applied E.M.F., not necessarily constant.

The resistances of the arms of the bridge are adjusted as indicated.

The following seven equations are derived immediately from Kirchhoff's laws applied to the various circuits:—

$$a = C + \alpha \quad . \quad . \quad . \quad (1) \quad b = b' - x \quad . \quad . \quad . \quad (2)$$

$$a = a' + x \quad . \quad . \quad . \quad (3) \quad I = a + b \quad . \quad . \quad . \quad (4)$$

$$RI + Bb + kBb' = E. \quad . \quad . \quad . \quad (5)$$

$$L \frac{da}{dt} + rx + (A - r)a + \rho x + L' \frac{dx}{dt} - Bb = 0. \quad . \quad . \quad (6)$$

$$\rho x + kBb' - kAa' + L' \frac{dx}{dt} = 0. \quad . \quad . \quad (7)$$

$$\text{Also} \quad c = Cr \frac{d\alpha}{dt} \quad . \quad . \quad . \quad (8)$$

From (1), (2), (3), (4), and (7) we find that

$$a = \frac{fx + kBE}{h} + p \frac{dx}{dt},$$

and

$$b = \frac{kAE - gx}{h} - q \frac{dx}{dt},$$

where

$$f = \{\rho + kB + kA\} \{R + B + kB\} - k^2 B^2.$$

$$g = R \{\rho + kA + kB\} + k^2 AB.$$

$$h = kA \{R + B + kB\} + kRB.$$

$$ph = L' \{R + B + kB\}.$$

$$qh = L'R.$$

Substituting these values of a and b in (6) we have

$$\begin{aligned} pL \frac{d^2x}{dt^2} + \left(L \frac{f}{h} + M \right) \frac{dx}{dt} + \left(m - \frac{f}{h} r \right) x \\ = -rx + \frac{kBr}{h} E - \frac{kBL}{h} \frac{dE}{dt}, \quad . \quad . \quad (9) \end{aligned}$$

where

$$M = L' + Ap + Bq - pr,$$

and

$$hm = Af + Bg + ph.$$

Differentiating (9), and eliminating α and $\frac{d\alpha}{dt}$ between the resulting equation together with (8) and (9), we finally obtain

$$\begin{aligned} & LCP \frac{d^3x}{dt^3} + \left\{ MC + LC \frac{f}{h} + \frac{Lp}{r} \right\} \frac{d^2x}{dt^2} \\ & + \left\{ p + mC + \frac{M}{r} + \frac{f}{hr} (L - Cr^2) \right\} \frac{dx}{dt} + \frac{m}{r} x \\ & = \frac{kB}{hr} (Cr^2 - L) \frac{dE}{dt} - \frac{kBCL}{h} \frac{d^2E}{dt^2}. \quad \dots (10) \end{aligned}$$

The solution of this differential equation will consist of two parts, a complementary function and a particular integral. The former is the solution obtained by putting the right-hand side of (10) equal to zero. This part will, consequently, be the complete solution in the case for which E is constant.

In this case the value of x is, therefore,

$$x = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + A_3 e^{-\alpha_3 t},$$

where α_1 , α_2 , and α_3 are the roots of the cubic equation

$$\begin{aligned} & LCPy^3 - \left\{ MC + LC \frac{f}{h} + L \frac{p}{r} \right\} y^2 \\ & + \left\{ p + mC + \frac{M}{r} + \frac{f}{hr} (L - Cr^2) \right\} y - \frac{m}{r} = 0, \end{aligned}$$

and A_1 , A_2 , and A_3 are constants determined by the initial conditions.

From the above value of x —the current through the galvanometer—the value of α , the current through the shunted resistance, can be obtained.

For, from (1) and (8) above, we have

$$Cr \frac{d\alpha}{dt} + \alpha = a = \frac{f}{h} x + \frac{kBE}{h} + p \frac{dx}{dt}.$$

Putting in the values of x and $\frac{dx}{dt}$, and solving for α , we find

$$\begin{aligned} \alpha = & A_5 e^{-\frac{t}{Cr}} + \frac{kBE}{h} + A_1 \left(\frac{f}{h} - p\alpha_1 \right) \frac{e^{-\alpha_1 t}}{1 - Cr\alpha_1} \\ & + A_2 \left(\frac{f}{h} - p\alpha_2 \right) \frac{e^{-\alpha_2 t}}{1 - Cr\alpha_2} + A_3 \left(\frac{f}{h} - p\alpha_3 \right) \frac{e^{-\alpha_3 t}}{1 - Cr\alpha_3}, \end{aligned}$$

A_5 being another constant determined by the initial conditions.

From this we see that the discharge of the condenser will be continuous or oscillatory according as α_1 , α_2 , and α_3 are all real or one real and two imaginary (these are the only cases which have to be considered for a cubic equation). But from the above value of x , if α_1 , α_2 , and α_3 are all real it is seen that the discharge of the galvanometer is continuous, while if two roots are imaginary it is oscillatory. Hence the discharge of the condenser is of the same nature as that through the galvanometer.

To consider the case of the oscillatory discharge of the condenser, therefore, we must put

$$\alpha_2 = k_2 + ik_3,$$

and consequently

$$\alpha_3 = k_2 - ik_3, \quad i \text{ being } \sqrt{-1}.$$

The terms $A_2 e^{-\alpha_2 t} + A_3 e^{-\alpha_3 t}$ now become $A_4 e^{-k_2 t} \cos(k_3 t - \epsilon)$. A_4 and ϵ being other constants.

The complete value of x now is

$$x = A_1 e^{-\alpha_1 t} + A_4 e^{-k_2 t} \cos(k_3 t - \epsilon).$$

The initial conditions will be of the form

$$(x)_0 = 0, \quad \left(\frac{dx}{dt}\right)_0 = u, \quad \text{and} \quad \left(\frac{d^2 x}{dt^2}\right)_0 = v,$$

u and v being constants.

Differentiating the expression for x and putting $t=0$, we shall have the following equations to determine the constants A_1 , A_4 , and ϵ .

$$A_1 + A_4 \cos \epsilon = 0,$$

$$A_1 \alpha_1 + A_4 k_2 \cos \epsilon - A_4 k_3 \sin \epsilon = -u,$$

$$A_1 \alpha_1^2 + A_4 (k_2^2 - k_3^2) \cos \epsilon - 2A_4 k_2 k_3 \sin \epsilon = v.$$

These give

$$\Delta A_1 = v + 2uk_2,$$

$$\Delta A_4 \cos \epsilon = -v - 2uk_2,$$

$$\Delta A_4 \sin \epsilon = \frac{1}{k_3} \left\{ (\alpha_1 - k_2)(v + 2uk_2) + \Delta u \right\}.$$

where $\Delta \equiv (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$, α_2 and α_3 having the values used above.

In the case of a constant E.M.F. a ballistic galvanometer is used. This measures the total quantity of electricity

flowing from the time $t=0$. The conditions of the experiment require that the galvanometer should give no deflexion after the current has flowed for an appreciable time. This

condition* can be expressed by $\int_0^{\infty} x dt = 0$, the upper limit of integration being taken as infinity, since the current will always be zero after a small interval has elapsed. This condition will give

$$0 = A_1 \int_0^{\infty} e^{-a_1 t} dt + A_4 \cos \epsilon \int_0^{\infty} e^{-k_2 t} \cos(k_3 t) \cdot dt \\ + A_4 \sin \epsilon \int_0^{\infty} e^{-k_2 t} \sin(k_3 t) \cdot dt ; \\ i. e. \quad \frac{A_1}{a_1} + A_4 \cos \epsilon \frac{k_2}{k_2^2 + k_3^2} + A_4 \sin \epsilon \frac{k_3}{k_2^2 + k_3^2} = 0.$$

Also $2k_2 = a_2 + a_3$, and $k_2^2 + k_3^2 = a_2 a_3$.

Putting in the values of the constants, therefore, we have

$$\{v + u(a_2 + a_3)\} \{a_2 a_3 + a_1(a_1 - a_2 - a_3)\} + \Delta a_1 u = 0 ;$$

$$i. e. \{v + u(a_2 + a_3)\} \{(a_1 - a_2)(a_1 - a_3)\} + \Delta a_1 u = 0,$$

and, therefore, $v + u(a_1 + a_2 + a_3) = 0$ (11)

Now from equation (9) above we have, when $t=0$,

$$pL\left(\frac{d^2x}{dt^2}\right)_0 + \left(L\frac{f}{h} + M\right)\left(\frac{dx}{dt}\right)_0 = \frac{kBr}{h}E,$$

the term $\frac{dE}{dt}$ being always zero in the case of a constant E.M.F.

Also, since $(a)_0=0$, we shall have, on putting $t=0$ in the expression for a , $\frac{kBE}{h} + p\left(\frac{dx}{dt}\right)_0 = 0$.

Hence

$$pL\left(\frac{d^2x}{dt^2}\right)_0 + \left\{L\frac{f}{h} + M + pr\right\}\left(\frac{dx}{dt}\right)_0 = 0.$$

$$i. e. \quad pLv + \left\{L\frac{f}{h} + M + pr\right\}u = 0.$$

* Lord Rayleigh, B. A. Report, 1883, p. 444, has drawn attention to the imperfections of a galvanometer as an instrument for indicating whether the integral sum of the transient currents through it is zero or not. See also a paper by Alex. Russell, M.A., M.I.E.E., Phil. Mag. 6th series, vol. xii. No. 69, 1906, where a full investigation and explanation of the necessary corrections is given.

Also

$$\alpha_1 + \alpha_2 + \alpha_3 = \left\{ MC + LC \frac{f}{h} + L \frac{p}{r} \right\} \div LCp.$$

Using these last expressions, (11) becomes

$$C \left\{ L \frac{f}{h} + M + pr \right\} - \left\{ MC + LC \frac{f}{h} + L \frac{p}{r} \right\} = 0.$$

$$\text{i. e. } \frac{p}{r} (L - Cr^2) = 0, \quad \text{or} \quad L = Cr^2.$$

Therefore, in the case of the oscillatory discharge of the condenser, with a constant E.M.F., the relation $L = Cr^2$ holds. Hence the expression for the inductance of the coil is independent of the inductance of the galvanometer.

Cass Institute, E.C.,
Feb. 23, 1909.

LXXX. *The Secondary Spectrum of Hydrogen.*

By A. DUFOUR*.

IN a recent note† Mr. C. F. Hogley has studied the secondary spectrum of hydrogen with the object of ascertaining whether it is really due to this gas. With this view, he works with vacuum-tubes having a bulb containing charcoal that can be cooled by liquid air; it is known that in these conditions the charcoal absorbs all gases except hydrogen, neon, and helium. Mr. Hogley finds that the stellar spectrum and the secondary spectrum of hydrogen have, in his tubes, relative intensities that remain unchanged, when the charcoal is cooled by liquid air; this would not take place if the secondary spectrum of hydrogen was due to an absorbable gas different from hydrogen. He is led to the following conclusion:—"So far, then, as this method can decide, the evidence points to the conclusion that hydrogen is really the source of the secondary spectrum."

This conclusion is, then, the same as the conclusion I had pointed out formerly in an investigation dealing with the same subject‡. In this work I relate at first the numerous researches previously made on this subject, and next various experiments that I have made either with vacuum-tubes constituted of different materials, or with tubes filled

* Communicated by the Author.

† C. F. Hogley, *Phil. Mag.* s. 6, vol. xvii. p. 581 (1909).

‡ A. Dufour, *Ann. de Chimie et de Physique*, 8^e sér. t. ix. pp. 361-432 (1907).

with pure hydrogen under a pressure nearly equal to the atmospheric pressure, and in which is a voltaic arc of high voltage between two metallic electrodes. In all the cases, taking minute precautions for avoiding the impurities that may come from the walls or from the electrodes, and using hydrogen as pure as possible, I have obtained the two spectra always together. Particularly I have also seen that if I cool with liquid air the wall of one of these last tubes, the relative intensities of the two spectra remain unchanged. These experiments, made in tubes at high pressure, do not seem to be exposed to the criticisms that may be urged against the use of vacuum-tubes, because the possible impurities can then be neglected.

The new work of Mr. Hogley confirms, then, the conclusion of my former researches, which is the following :—The secondary spectrum of hydrogen is due to hydrogen itself, and not to impurities, as has been believed a long time. The stellar spectrum is supposed to be due to a relatively simpler vibrating system, for instance to the atom of hydrogen, while the secondary spectrum might be attributed to the molecule of hydrogen, more complicated than the atom.

LXXXI. *On a Want of Symmetry shown by Secondary X-Rays.* By W. H. BRAGG, M.A., F.R.S., Elder Professor of Mathematics and Physics in the University of Adelaide, and J. L. GLASSON*.

[From "Transactions of the Royal Society of South Australia," vol. xxxii., 1908.]

ON the assumption that the Röntgen rays consist of æther pulses it has been shown by J. J. Thomson ('Conduct. of Electr. through Gases,' p. 323) that it is possible to account for the existence of secondary Röntgen rays by assuming that the primary pulses set in motion electrons over which they pass, and cause them to become new centres of radiation. If the electron easily follows the guiding force of the primary pulse, then the secondary radiation resembles the primary in quality. But if the electron is hampered by attachments to other portions of the atom to which it belongs, then the new pulse has not the same quality as the old; the time of motion of the electron is dragged out, and the pulse produced is softer.

Now, if an electron becomes in this way a centre of radiation the intensity of the secondary effect must be

* Communicated by the Physical Society: read April 23, 1909.

symmetrical about the line of motion of the electron. In particular, the intensity of the secondary radiation must be symmetrical about a plane passing through the electron perpendicular to the primary ray, since this ray contains the line of motion referred to. This deduction forms an integral part of Thomson's theory of secondary Röntgen radiation, and its truth has been assumed in calculations intended to show that experimental results are in agreement with theory. Barkla proves the same deduction in a paper published in the *Philosophical Magazine* of February 1908.

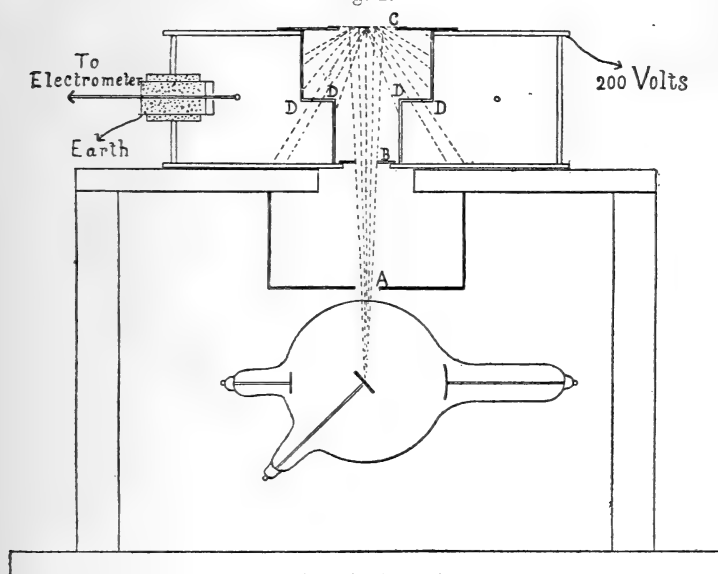
Now it has recently been shown (Bragg and Madsen, *Trans. Roy. Soc. S.A.*, May 1908) that the cathode radiations excited by γ rays show a very marked want of symmetry about the plane normal to the exciting ray; and again (Madsen, *Trans. Roy. Soc. S.A.*, July 1908) that there is a similar want of symmetry in respect to the secondary γ rays. The γ rays and X-rays resemble one another so closely in all their known properties, that it is fairly safe to assume any effect found to be true of the one kind to be true also of the other kind, though perhaps to a different degree. In this case, indeed, Cooksey ('*Nature*,' April 2, 1908) has already shown that the secondary cathode radiations excited by X-rays are not at all symmetrical about the normal plane, the emergence rays being greater than the incidence, as in the case of the γ rays.

It remained, therefore, to examine the secondary X-rays excited by primary X-rays; and the experiments described in this paper were made with that object. We find that in general want of symmetry does exist, that it is sometimes very pronounced, and that is in keeping with expectation based on Madsen's study of the secondary γ rays. Hard γ rays show a very large difference between the quantities of emergence and incidence radiation; for soft γ rays the difference is smaller. Since X-rays are to be looked on as a very soft form of γ rays, the difference should be smaller still; and this is what we have found to be the case.

The general form of the apparatus which we have used is shown in fig. 1. Variations of the upper portion of it are shown in figs. 2 and 3. A small pencil of X-rays passed upwards through apertures in lead plates at A and B, and then along the axis of the ionization-chamber and out into the open. In our first experiments the upper part of the apparatus was arranged as in fig. 3. The primary rays did not pass through the effective part of the ionization-chamber, being separated therefrom by the cylindrical screen SS, which could be made of various thicknesses and various

materials. But if a thin sheet of any substance was laid over the hole at B, secondary X-rays spread out therefrom, and some passed through the screen SS, and caused a deflexion in the electrometer. The difference between the

Fig. 1.



4 in.

Fig. 2.

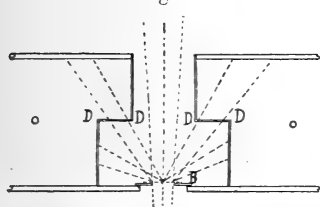
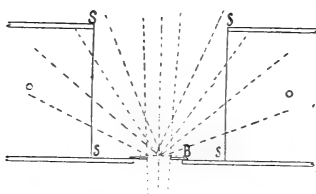


Fig. 3.

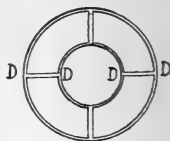


deflexions (*a*) without and (*b*) with the sheet at B was taken as a measure of the emergence secondary X-ray radiation. When the sheet was removed from B, and the same or a similar sheet placed in the plane of the top of the screen so as to be struck from below by the primary rays, then the measure of the incidence secondary radiation was obtained as the difference between the deflexions (*a*) without and (*c*) with the sheet so placed.

In this way it was easy to show that the expected want of symmetry actually existed, particularly with aluminium, celluloid, or paper as the radiators, substances of small atomic weight. But the experiments were open to some extent to the objection that a was too large compared with $b-c$, and that possibly the excess of emergence over incidence was an apparent effect due to actual variations of a under different circumstances. The current a was, in fact, due to several causes. There was a small natural ionization leak even when the X-rays were not acting; there was an effect due to primary X-rays which had penetrated the walls of the chamber, though they were made of zinc one-eighth of an inch thick. But the greatest part of a was due to a diffusion of soft rays about the primary beam, much of which came through the hole at B at such an angle as to penetrate the screen SS; it could be largely cut out by thickening the screen. Again, part of a was due to radiation returned from the open air above the ionization-chamber. Some of these radiations might be appreciably interfered with by placing the radiating sheet at B or at the top of the chamber. We were, however, able to satisfy ourselves by special experiments that the want of symmetry was quite real, and that as a matter of fact no valid objection could be made. But we abandoned the first arrangement for a second which, as we expected, would show the want of symmetry more clearly, and which proved better than the first in every way. The first method was exactly the same as that used by Madsen in examining the secondary γ rays; but it was clear that the enormous difference which these rays showed was not going to be repeated in the case of the X-rays.

Our new arrangement was, as shown in fig. 1, or, inverted, in fig. 2. Two cylinders of brass, each 2 in. long, but of different diameters—4 in. and 2 in.—were fixed to a connecting piece DD, shown in plan in fig. 4. The latter resembled a light brass wheel with four spokes, and various thin screens cut in the form of flat rings could be attached to it, filling up all the spaces between the spokes. In fig. 1 the double cylinder is shown as arranged for the measurement of incidence secondary radiations; the radiating sheet was placed at C, supported by a sheet of celluloid lying flat on the top of the cylinder. A hole was cut in the centre of the celluloid sheet big enough to allow the primary beam to pass through without touching the edges; and a fluorescent screen was used to make sure

Fig. 4.



that this was the case. The radiating sheets were of thin metal, about $1\frac{1}{2}$ in. square. In fig. 2 the cylinder is shown as arranged for the measurement of emergence secondary radiations: it hardly requires further explanation.

We expected that this arrangement would show up the want of symmetry better than the former, because the portions of the emergence and incidence beams under comparison would be more nearly normal to the plate. Looking upon the radiations as material, we should naturally expect the intensity of the secondary radiation to decrease gradually as its direction increased in inclination to the forward direction of the primary ray. The emergence rays lie, in inclination, between 0° and 90° ; the incidence between 90° and 180° . In our first arrangement we compared the emergence rays between about 40° and 90° , with the incidence rays between about 90° and 140° . There should be a larger ratio of emergence to incidence with the newer arrangement, since the emergence rays between about 30° and 50° would be compared with the incidence between about 130° and 150° . This proved to be the case; the improvement was considerable. Again, with the new arrangement, the current with no radiator in position became relatively far smaller. For example, when the radiator was Al, .4 mm. thick, and the absorbing screen DD of tinfoil (two thin sheets), the currents with and without the radiator at B in fig. 1 caused deflexions of 86 and 26 mm. in ten seconds respectively; the currents with and without the radiator at B in fig. 2 were 220 and 35 respectively. There could be very little error, therefore, in taking the incidence and emergence radiations as 60 and 185 respectively; and the want of symmetry is beyond doubt.

It should be observed that the emergence radiation can never be shown to an unfair advantage in these experiments, and is often at a disadvantage, for the radiator, when placed as in fig. 2, cuts down the very primary rays to which the secondary radiation is due. It is not difficult to show that if the thickness of the radiator is so adjusted as to give the maximum emergence current (it can of course be too thick or too thin), then the ratio of this maximum to the maximum incidence current (which can be obtained simply by making the radiator thick enough) is only $2/e$ of the true ratio of emergence to incidence; provided that the secondary rays are as penetrating as the primary, and that we are considering homogeneous radiations. But if, other conditions being the same, the secondary rays are less penetrating than the primary, then the ratio, as found, is more nearly

correct, and is very nearly so when the secondary rays are much less penetrating than the primary, as, for example, when we are considering secondary cathode rays due to X- or γ rays.

We have made a large number of measurements by the method described above, using the following metal sheets as radiators :—Pt, weight per square cm., $\cdot 0150$ gr. : Sn, $\cdot 0096$ gr. ; Cu, $\cdot 0083$ gr. ; Fe, $\cdot 0077$ gr. ; Al, $\cdot 105$ gr. ; celluloid, $\cdot 20$ gr. As screens we have used various thicknesses of Sn, Cu, and Al.

The proportion of emergence to incidence radiation differs considerably for the different radiators, but is much the same for different screens or different thicknesses of screen, except that the proportion tends to increase slightly as the screen is made thicker ; and the tendency is most pronounced in the case of those metals which give out a quantity of soft secondary radiation. For example, Fe and Cu show little difference between incidence and emergence radiations until the screen is so thick that only a small fraction of either of the radiations can pass through. The results vary somewhat with the state of the bulb ; and since these variations are comparable with those which are met with on changing the nature of the screens, we are not now in a position to discuss smaller variations in detail. We must content ourselves with quoting a few results in order to show the want of symmetry, which is a persistent effect. When, for example, two tinfoils were used as screen (weight per square cm. of each, $\cdot 0056$), we obtained the following figures, which represent movements of the scale in mm. during 10 secs. :—

Radiator	Sn.	Cu.	Fe.	Al.
Emergence Current	176	140	39	185
Incidence Current	122	119	15	60

With four tinfoils the figures were :—

Radiator	Sn.	Cu.	Fe.	Al.
Emergence Current.....	143	24	23	116
Incidence Current	87	1	0	34

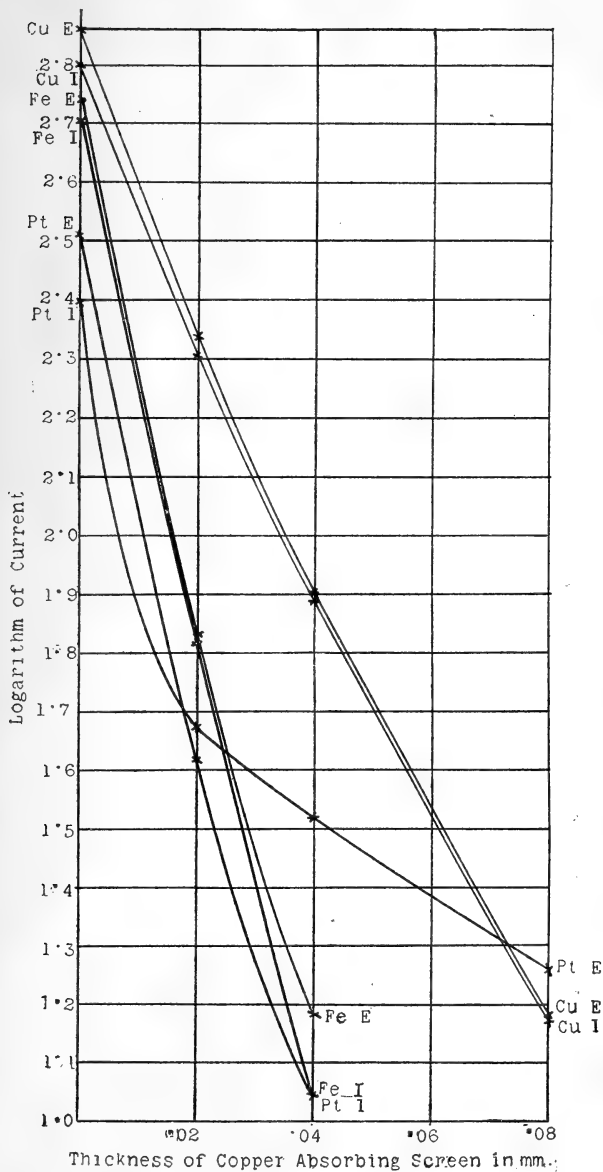
Again, using a copper screen $\cdot 002$ cm. thick, we found :—

Radiator.....	Pt.	Sn.	Cu.	Fe.	Al.	Cellu- loid.
Emergence Current...	86	140	361	118	80	138
Incidence Current ...	65	104	364	118	32	93

Putting together a number of results for Cu screens of

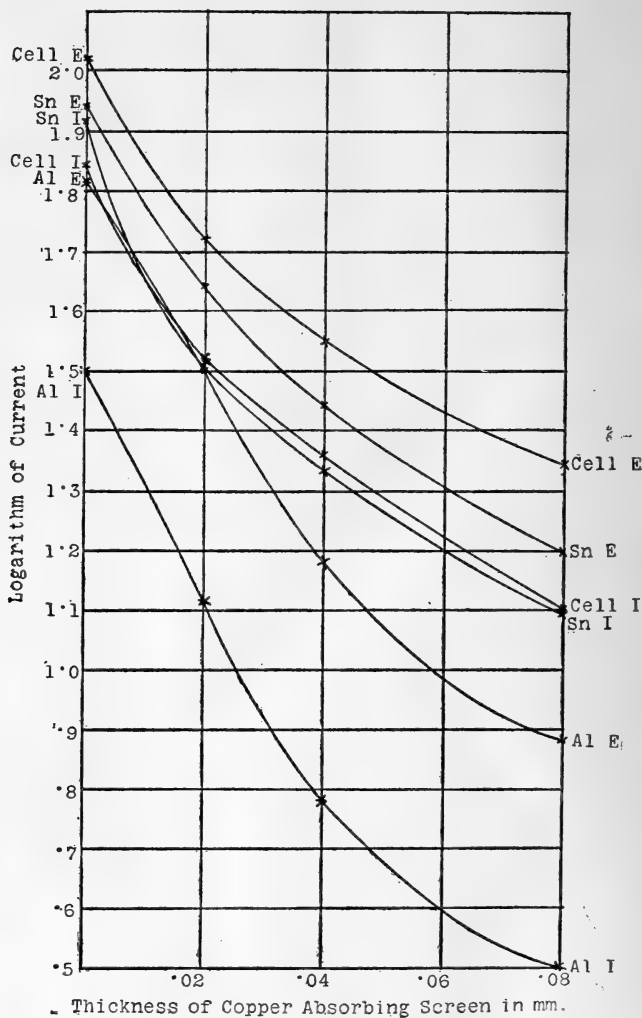
different thicknesses we obtain the logarithmic curves of absorption shown in the accompanying figures (figs. 5 and 6).

Fig. 5.



It should be observed that some of the results thus shown were obtained at different times, so that too much must not be built upon a comparison between them ; only the relative

Fig. 6.



positions of the emergence and incidence curves of each substance are sufficiently correct, and the form of each curve as showing the homogeneity or otherwise of the various radiations. One figure shows the emergence (E) and

incidence (I) curves for Pt, Cu, and Fe; the other the corresponding curves for Sn, Al, and celluloid.

The experiments described in this paper show that a very marked want of symmetry occurs in the case of secondary X-rays, the emergence rays being generally greater than the incidence. This is another instance of the close parallelism between X- and γ rays. On a material theory of X- and γ rays the effect is easily explained, and is to be classed with the scattering to which β , and also, as lately shown clearly by Geiger, α rays are subject. But if the X- and γ rays consist of energy bundles of very small volume, as suggested by J. J. Thomson, then these bundles must be capable of deflexions in going through atoms—that is to say, swung out of their paths by the electrical forces to be found within the atoms, just as neutral pairs would be in virtue of their electrical fields. It seems hard to understand the distinction between such bundles and entities generally classed as material.

In the course of this investigation we have made a number of experiments on the quantities and qualities of the secondary radiations. This subject has been fully treated by Barkla, some of whose recent papers have not yet reached us, and any discussion we gave might be merely a duplication of part of his inquiry. There is, however, one point to which we should like to refer.

Very hard γ rays follow a density law of absorption, treating all atoms alike, except in respect to weight. Soft γ rays are not independent of atomic groupings of matter, and are far more strongly absorbed by heavy atoms than by light, after allowance has been made for weight. The same is generally true of X-rays; but in the case of very soft X-rays there is a tendency to revert to the density law again. For instance, X-rays that have passed through the glass of the bulb are soft to copper, silver, tin, and so on, but hard to aluminium, carbon, and low atomic weight generally. No doubt those rays which are soft to such light atoms have already been absorbed by the glass. But secondary X-rays from most substances are softer than anything emerging from the bulb and contained in the primary ray. The difference is not very great when the absorption is measured with the aid of screens made of substances of the higher atomic weights, because to these the primary rays are soft already. But if the screens are made of aluminium, still more of filter-paper, the difference now seems to be very great, for the secondary rays are soft even to low atomic weights. For example, in one experiment, a sheet of copper

weighing $\cdot 018$ gr. per square cm. caused a drop of $\cdot 401$ in the logarithm (to base 10) of the primary rays, and only of $\cdot 447$ in the case of the emergence secondary rays from copper, of $\cdot 645$ in the case of platinum rays, and $\cdot 805$ of iron rays. But when four filter-papers weighing $\cdot 02$ gr. per square cm. were used as screen, the drop in the case of the primary rays was $\cdot 010$ —only one-fortieth of the drop caused by a copper screen of nearly equal weight. In the case of the secondary rays, however, the same screen caused a drop in the case of copper rays of $\cdot 100$, platinum rays $\cdot 053$, and iron rays of $\cdot 188$ —that is to say, for these soft rays the filter-papers are much more nearly on an equality with copper, weight for weight, than they were for hard rays. It is interesting to bear this in mind when considering the very large quantities of secondary ionization which some substances seem to give. The ionization is always measured in air, which of course consists of atoms not very different in weight from those contained in filter-papers. Consequently primary rays, and secondary rays which differ very little from the primary, are very penetrating to air, and cause relatively small ionizations therein. But secondary rays from Cu and Fe are softened so much as to bring them within reach, so to speak, of air, which rapidly converts them into cathode rays, so that there is a very large ionization. For the cathode rays produced from these secondary rays have probably but little less energy than those produced from the primary; the speed of the cathode ray does not differ very greatly with the penetration of the primary X-ray, so far as experiments have shown. The very large secondary radiations, which some substances appear to give, therefore, owe their magnitude largely to the fact that the air in which they are measured is sometimes ten to twenty times as favourable to them as to the primary rays which produced them. In this way we may account to some extent for the startling results obtained by Crowther in the case of arsenic and bromine (Phil. Mag. Nov. 1907).

LXXXII. *Notices respecting New Books.*

Lehrbuch der Thermodynamik. Based on Lectures of Dr. J. D. v. D. WAALS by Dr. PH. KOHNSTAMM. First part. Leipzig and Amsterdam: Maas & van Suchtelen.

THE present volume deals in general with the theoretic side of thermodynamics from the point of view of the physical chemist, and most especially with the portion concerned with the phenomena

of equilibrium. The treatment throughout is exceedingly simple and yet thorough; indeed we are not acquainted with any book in which the theoretic side is more lucidly and satisfactorily treated. If we may express any regret, it is that experimental data are not co-mingled with the theoretic. No doubt there are certain advantages in the method adopted. The thread of theory may easily be broken if loaded with too great a weight of fact. Moreover, we have not the slightest doubt that Dr. van der Waals would be the last man in the world to recommend that a student's attention should be confined exclusively to the work before us. Be this as it may, this treatise (of which the first part only is yet published) promises to be one of the most important elementary accounts of the subject. We may draw attention in particular to the concluding section on Laplace's Theory of Capillarity and on the capillary layer.

Magneto- und Electrooptik. By Dr. WOLDEMAR VOIGT. With figures in text. Leipzig: B. G. Teubner, 1908. Price 14 marks.

WORKERS in electro-optics and others will welcome this volume, which represents the lectures given by the author for some years past. There is probably no subject which at the present time is more revelatory of the structure of the atom than the one with which it deals, and there is no one more qualified than Dr. Voigt to deal with it. A general idea of the scope of the book can be conveyed by the titles of the chapters. These are:—1. The Faraday effect. 2. The Zeeman effect. 3. The Theory of magneto-optic effect for normal isotropic bodies. 4. Investigation of the theory of more-complicated types of Zeeman effect. 5. Magneto-optic effect in absorbing crystals. 6. The magneto-optic Kerr effect. 7. The electron theory of the magnetic Kerr effect. 8. Electro-optic action in isotropic and anisotropic bodies. 9. Oscillations of "bound" electrons under the influence of electric fields. 10. The electron theory of electro-optic effects. It is surprising how successful the few assumptions underlying the electron hypothesis are in accounting for the complicated phenomena concerned. It is true that there are at present relations which are inexplicable at present on this hypothesis. Considering, however, that we are obliged to assume a particular character of the electronic system—and the chances against assuming the right one must be amazingly great—it is remarkable that any success at all is achieved. In agreement with this statement is the fact that the phenomena which are most fully explained are those which are independent of the precise structure of the atom. Those who desire the most complete presentation of this subject both from the theoretic and the experimental side will find here what they wish.

The Norwegian Aurora Polar expedition 1902-1903. Volume I. On the Cause of Magnetic Storms and the Origin of Terrestrial Magnetism. By KR. BIRKELAND. First Section. Quarto. 315 pp. & 21 Plates. Christiania, 1908. London: Longmans, Green & Co. (Price 22s. net.)

THE volume discusses records derived during 1902-3 from magnetographs at four temporary stations situated respectively in the extreme north of Norway, in Nova Zembla, in Spitzbergen, and in Iceland, and simultaneous records from a number of magnetic observatories in various parts of the world. The 21 large plates at the end of the volume reproduce the photographic curves taken during times of special magnetic disturbance from October 6, 1902, to March 31, 1903. In the body of the work are numerous charts illustrating the disturbing forces at the several arctic and cooperating stations at various stages of each disturbance. The Expedition was due to Prof. Birkeland, who personally defrayed nearly half the total expense, the remainder of the funds coming about equally from the Norwegian Government and from three private individuals. The main object was the further development of the theory, which Prof. Birkeland propounded a good many years ago, that magnetic disturbances are due to electrical discharges in the upper atmosphere, which appeal to our visual organs as aurora. To elucidate the nature of the phenomena, Prof. Birkeland has made many experiments with a terella, or miniature earth, of iron, highly magnetized by currents in a surrounding coil, which is exposed to discharges from a cathode in a highly exhausted glass bulb. With this apparatus he has produced a number of visual effects resembling aurora, photographs of some of which appear in the work. The cathode is supposed to represent the sun, the cathode rays from which reach the earth's atmosphere, where their further course is dependent on the earth's magnetic field. The mathematics answering to this hypothesis has been developed by Prof. Störmer, to whose calculations there are various references. Prof. Birkeland practically defines *disturbance* as the difference between the value of a magnetic element at the instant considered and the value at the same hour on the representative undisturbed day. This disturbance he regards as measuring the magnetic force at the time and place due to the hypothetical auroral current, and from the way in which the amplitude and direction of the resultant disturbance vary from station to station he draws conclusions as to the nature, locus, and intensity of the current. The simplest type of magnetic disturbance he believes to be accounted for by a current which approaches to and recedes from the earth along radii, the intermediate—or, as the author terms it, the “horizontal”—portion being a straight line of length $2l$ whose extremities are at the same height, and whose centre is at a distance h above the ground.

Formulae for the magnetic force due to such a current are developed on p. 102, and numerical results for special values of h and l are given on p. 103, the current being taken to be one

million amperes. Other simple cases are considered later in the volume. The assumptions on p. 102 that the current is linear and that the central portion is a straight line, and not the arc of a circle, are probably concessions to the mathematical difficulties of the case, but the peculiar sense in which the author employs the term "horizontal" should be noticed. For instance, in one of the cases on p. 103 we have $h=300$ and $2l=5000$ kilometres. But the height of the "horizontal" portion above the ground will vary in this case from 300 to over 750 kilometres. From statements in the volume, one infers that it is to be supplemented by a subsequent discussion on Earth currents, Auroras and Störmer's mathematical work. Meantime it is difficult to express an opinion on its theoretical results. What now appear to be discontinuities in the reasoning may have a different aspect given to them by the subsequent work indicated. The author has a way of passing from observation to theory, and from theory to experiment, which makes it difficult for the ordinary man to follow the argument or recognize its goal. It has long been recognized that an intimate connexion exists between magnetic storms and auroras when the latter are visible in temperate latitudes. It has also long been known that in the northern hemisphere auroras are most numerous in high latitudes, and that there is much more magnetic disturbance there than further south. The disturbances at Prof. Birkeland's Arctic stations were usually much larger than at the observatories in temperate latitudes, and considering the distance apart of his stations, the differences between the disturbances recorded at them were exceptionally large. The results are interesting and important in themselves, and are obviously favourable to the view that if—as most people suppose—magnetic storms are due to atmospheric electric currents, their principal seat in the northern hemisphere is in the Arctic. But beyond this it does not seem to the writer that anything is *proved*. It seems to him that what is mainly wanted at the present stage is a more profound examination of the phenomena presented by magnetic storms. Until the laws of distribution of disturbances have been so far ascertained that observation can check theory, a predisposition in favour of a particular theory may be a positive disadvantage to an investigator.

Towards the end of the volume there seems a fuller recognition of the fact that the phenomena of magnetic storms are often so complicated that the electric currents capable of producing them must have a distribution whose investigation would require the use of the most advanced mathematics. Considering the extension of view—or of imagination—which the discovery of radium has produced in physical circles, the size of the currents which Prof. Birkeland postulates and his conclusions on pp. 311–315 as to the expenditure of solar energy will doubtless appear much less startling now than they would a dozen years ago. One cannot, however, but wonder what figures he would have reached if instead of the relatively trifling disturbances of the winter of 1902–3, he had had to deal with those of 1857, or 1870, or even 1892.

C. CHREE.

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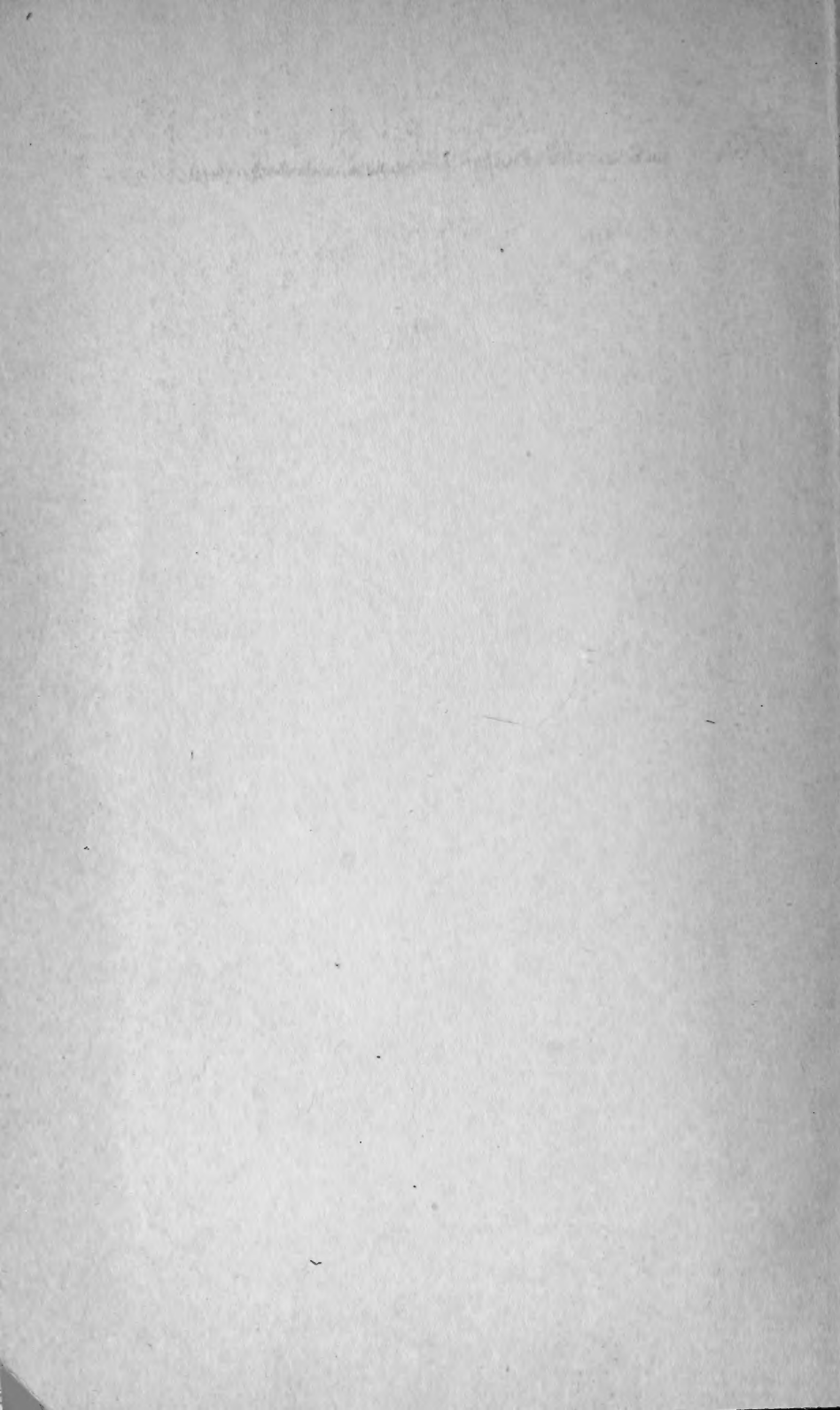
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